ESTIMATION AND CONTROL WITH QUANTIZED MEASUREMENTS

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To Sue
Katie
Smith
Susan

Foreword

This is the sixtieth volume in the M.I.T. Research Monograph Series published by the M.I.T. Press. The objective of this series is to contribute to the professional literature a number of significant pieces of research, larger in scope than journal articles but normally less ambitious than finished books. We believe that such studies deserve a wider circulation than can be accomplished by informal channels, and we hope that this form of publication will make them readily accessible to research organizations, libraries, and independent workers.

Howard W. Johnson

Preface

The demand on digital facilities such as communication systems and data-storage systems is constantly increasing. In the past the pressures have been relieved by upgrading their capacities; primarily through the advances being made in digital hardware, attention has turned to the problem of using these facilities more efficiently. Many of the efforts are described in a general way as "data-compression," "redundancy-reduction," and "bandwidth-compression," and most of them rely on the quantization and subsequent reconstruction of data.

This monograph presents the results of some research pertaining to the distinct but related tasks of efficient estimation and control based on quantized measurements. It has been published in the hope that both researchers and engineers will find some useful ideas to expand and adapt to their own needs. The reader is assumed to have a familiarity with probability theory and random processes (at the level of Lanning and Battin, *Random Processes in Automatic Control*, or Papoulis, *Probability, Random Variables, and Stochastic Processes*), and a basic understanding of estimation theory.

Discrete-time problems are considered and the emphasis is placed on coarsely quantized measurements and on linear and, when pertinent, time-varying systems. The heart of the material is a new interpretation and outlook on the problem of generating nonlinear estimates from quantized measurements. The development of the minimum variance, or conditional mean estimate is quite fundamental, since it lays the groundwork for other types of estimates. Approximate and more easily implemented nonlinear filters are examined in some detail, especially in conjunction with three communication systems, so that the subject matter is not limited to theory alone. The design of optimal linear estimators is re-examined, and their performance is compared with that of nonlinear filters. Not surprisingly, consideration of the control of stochastic systems that have quantized measurements leads to insights into, and control strategies for, systems with other types of nonlinearities.

The major portion, but by no means all, of this research was submitted as a Ph.D. thesis to the Department of Aeronautics and Astronautics at the Massachusetts Institute of Technology. This monograph is an extensive revision of the original work, and it incorporates new analytical and numerical results and the welcome comments of many anonymous reviewers.

I wish to thank the members of my thesis committee, Professor Wallace E. Vander Velde, chairman, Professor Arthur E. Bryson, Jr., and Professor James E. Potter for their pertinent questions and fruitful suggestions. My association with them has been extremely valuable, for not only are they outstanding teachers, but their approach to technical problems is one that I particularly admire. Mr. Charles F. Price acted as a sounding board for many ideas and provided comments and suggestions about the initial draft; his help is greatly appreciated. I should also like to acknowledge the profitable discussions with Professor Terrence Fine, Dr. Herbert Gish and Dr. Donald Fraser. Finally I would like to thank my wife, Susan, for typing the initial draft and its revisions and for her patience and encouragement.

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R. E. Curry

Ithaca, New York
June, 1969

Contents

1 Introduction

1.1 Background

The mathematical operation of quantization exists in many communication and control systems. Quantizing elements may appear as digital transducers, analog-to-digital converters, or digital-to-analog converters; even the digital computer itself is a quantizing device, because of its finite word length. Measurements will be quantized if they are produced by digital sensors, transmitted over a digital communication link, or processed by a digital computer.

Quantization of measurements is the irreversible process of rounding arithmetical numbers, and information is lost by this operation. The word *quantizer* is reminiscent of a nonlinear input-output staircase graph. Although this is a valid representation in many instances, it is overly restrictive, and we shall interpret the information in a quantizer's output in a different manner.

A quantizer is any zero-memory input-output device that designates the interval or intervals in which its input lies.

This definition enlarges the class of nonlinear devices that may be considered quantizers. For now we see that elements containing dead-zone and saturation regions may be viewed as quantizers. Thus, regardless of the input-output graph, a quantized measurement will mean that the measured quantity lies in a known region.

The study of coarsely quantized measurements is important at this time because in many cases it is less expensive to add equipment for a more complicated processing of coarsely quantized data than to expend time, money, and effort on reducing the size of the quantum intervals. This possibility is a direct result of the advances and improvements that are being made in digital hardware.

For instance, the ever-increasing demand to transmit and store more information is being felt in almost all areas. The usual approach has been to upgrade the transmission and storage capacity of a system, but now it appears that further increases in performance will be attained most economically by a more efficient use of existing digital facilities (*IEEE*, 1967). This can be done by representing the same amount of information with fewer bits, but more sophisticated data-processing is required to recover the information.

For an example of this approach suppose that the word length in a digital telemetry system could be shortened and that the data-processing could be so employed as to keep the information loss at an acceptable value. Then the designers of the system would have the following tradeoffs, or a combination of them, at their disposal:

1. reduce the transmitter's power, size, and weight and send the same information in the same amount of time;
2. use the same transmitter to send the same information in a shorter time; this would allow more instruments to be installed and monitored.

Coarsely quantized measurements occur in other situations, as in the taking of sonar bearings (Korsak, 1967) or during the alignment of some inertial platforms. In the latter, as the platform approaches a horizontal position, the horizontal component of the specific force of the platform support becomes very small. Its detection by pulse-rebalanced accelerometers, which provide quantized information about velocity, is a lengthy, if not difficult, problem: the pulses occur at a frequency that is proportional to the platform deviation (which is small) and inversely proportional to the velocity increment of each pulse (which may be large).

One solution is to change the velocity increment of each pulse, but this may have an adverse effect on the accelerometer's accuracy in other modes of operation. An alternative approach during the alignment is to use the information that a pulse has *not* occurred to improve the knowledge of the platform angle. This is equivalent to a quantized measurement.

The example of platform alignment leads to the question of control with quantized measurements. Control and estimation are interrelated through their connection with the system as a whole, and the measurement device (in this case a quantizer) may be expected to influence the choice of control laws that give satisfactory results. In some cases, especially in those of optimal stochastic control, the interdependence of estimation and control is so strong that the two functions merge and must be designed as a single entity.

1.2 Objectives and Scope

The objective of this work is to examine the two distinct but related problems of optimal estimation and control with arbitrarily quantized measurements. Consideration is limited to discrete-time problems, and emphasis is placed on coarsely quantized measurements and linear, possibly time-varying, systems. A quadratic criterion is used in the optimal control analyses.

1.3 Estimation with Quantized Measurements

Wiener (1966) was among the first to deal with the optimal estimation of stochastic processes. He derived an integral equation for the weighting function of the optimal, linear, realizable filter for minimizing a mean-square-error criterion and solved the integral equation by spectral factorization. Later attempts (see Davenport, 1958) were made to remove such restrictions as the need for an infinite amount of data. Both Kalman (1960) and Swerling (1959), using the state-space approach, showed that the solution to the problem of optimal linear estimation could be generated by difference equations (and, later, differential equations, Kalman and Bucy, 1961). This formulation allows for nonstationary processes and a finite amount of measurement data and can be implemented in a straightforward manner with digital and analog equipment.

The advances in the area of nonlinear filtering have not been as spectacular, because the probability-density function of the state cannot be represented by a finite set of parameters (as it can in the case of Gaussian random variables). In continuous-time nonlinear estimation (problems with a nonlinear system or measurement equation) the probability-density function of the state conditioned on the available measurements must be found by solving a partial differential equation (Bucy, 1965; Kushner, 1964; Wonham, 1964). Quadratures are used to find the conditional mean and other moments.

No partial differential equation for problems in discrete-time nonlinear filtering exists, but the conditional probability-density function of the state must be updated by Bayes' rule and is usually a multidimensional integration (Ho and Lee, 1964). Some success has been obtained by linearizing the equations and then applying the theory of linear filters (Mowery, 1965). We note that this technique has little, if any, value for quantized measurements, because the slope of the nonlinearity is either zero or infinite.

With specific reference to quantized signals Bennett (1948) investigated the spectrum of the quantizer output signal when the input signal had a flat spectrum with sharp cutoff. Like Widrow (1960), he assumed the quantum intervals to be uniform and of infinite extent. Under these assumptions Widrow obtains interesting results by using the Nyquist sampling theory on probability-density functions and characteristic functions. Max (1960) investigated the optimal choice of quantizer parameters (for example, width and placement of quantum intervals), to minimize the mean square error between the quantizer input and output. Ruchkin (1961), Steiglitz (1966), and Kellog (1967) have all investigated the linear filtering of quantized signals according to various criteria. Only Kellog has considered coarse quantization.

To the best of the writer's knowledge few attempts have been made at nonlinear estimation with quantized measurements. Balakrishnan (1962) derives some results concerning an adaptive, nonlinear predictor for quantized data. Meier and his associates (1967, a), taking the Bayesian approach, use a uniform quantizer of infinite extent and derive the equations for the conditional mean and conditional covariance for a scalar state variable and only one (scalar) quantized measurement.

Estimation with quantized measurements is of prime interest to designers of digital communication systems. Three systems that have received particular attention in the past are the pulse code modulation (PCM), predictive quantization, and predictive-comparison data-compression. Studies of the quantization and reconstruction problem (PCM) have been mentioned above in connection with Ruchkin (1961), Steiglitz (1966), and Kellog (1967). Fine (1964) gives a theoretical and general treatment of optimal digital systems with an example of predictive quantization (feedback around a binary quantizer). Bello and his associates (1967) have computed Fine's nonlinear feedback function by Monte Carlo techniques and give some simulation results. Gish (1967), O'Neal (1966), and Irwin and O'Neal (1968) consider the design of linear feedback functions for the predictive-quantization system; predictive-comparison data-compression systems are in this class. Davisson (1967) treats the optimal linear feedback operation and examines an adaptive system (1966). Other approaches to data-compression have been described (*IEEE*, 1967; Davisson, 1968).

The topics on estimation with quantized measurements that are covered in this monograph are briefly summarized as follows. Chapter 2 treats the nonlinear estimation of parameter and state vectors, based on quantized measurements. The primary emphasis is placed on the determination of the minimum variance (conditional mean) estimate. Chapter 3 deals with the design of the three digital communication systems mentioned above and presents results of Monte Carlo simulation. Chapter 4 is devoted to optimal linear estimators for quantized, stationary, random processes.

1.4 Optimal Control with Quantized Measurements

The efforts of previous investigators have been in the area of linear, time-invariant, closed-loop systems that include a quantizer somewhere within the loop. Their work may be divided into two categories: deterministic and stochastic.

Typical of deterministic approaches are those taken by Bertram (1958) and by Johnson (1965), in which bounds for the system's behavior are found.

Stochastic approaches are taken by Widrow (1960), Kosyakin (1966), Graham and McRuer (1961), Smith (1966), and Gelb and Vander Velde (1968), in which the quantizer is approximated by a gain element or a noise source or both. Generally speaking, the quantum intervals are assumed to be small enough for simplifying assumptions to be made about the quantization noise as, for example, that it is white noise. System design and compensation are then carried out by using the conventional linear-design tools.

The control of systems with quantized measurements may also be pursued via the state-space techniques. A cost criterion is established, and the control actions are chosen so as to minimize the expected value of the cost. This is the problem of *optimal stochastic control*, or *combined estimation and control*. The general statement of the problem for nonlinear systems (including nonlinear measurements) and the method of solution (dynamic programming) are outlined by Fel'dbaum (1960), Dreyfus (1965, 1964) and Aoki (1967).

In only one case has the optimal stochastic control been shown to be at all practical to use or even to find. If the dynamic system and measurements are linear, the cost quadratic, the noises additive and Gaussian, and the initial conditions of the state Gaussian, then, as has been shown by Joseph and Tou (1961), the optimal stochastic control sequence does not have to be determined by dynamic programming but may be found much more easily with the aid of the separation theorem. The separation theorem states that the computation of the optimal stochastic control may be carried out in two parts: first with a Kalman filter for generating the conditional mean of the state and next with the optimal (linear) controller that is derived if all disturbances are neglected.

The topics on optimal stochastic control are as follows. Chapter 5 contains the statement of the problem and presents results for the optimal control of a two-stage process with quantized measurements; it also gives the derivation of a separation theorem for nonlinear measurements. Chapter 6 considers suboptimal stochastic control algorithms and offers a new algorithm, which is widely applicable and compares favorably with other methods in simulations; it also discusses the design of optimal linear controllers (including linear feedback for the predictive-quantization systems). Chapter 7 summarizes the results and conclusions and suggests topics for further research.

1.5 Notation

The notation follows general practice: lower case and upper case bold face letters denote vectors and matrices, respectively, e.g. $\boldsymbol{a}$ and $\boldsymbol{A}$. E is the expectation operator. Subscripts refer to the time index ($\boldsymbol{x}_n = \boldsymbol{x}(t_n)$), and superscripts refer to elements within an array ($\boldsymbol{x} = \{x^i\}$). In places where confusion might arise, the dummy arguments of probability density functions are explicitly shown with a mnemonic correlation between the dummy arguments and the random variables. For example, $p_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ is the joint probability density function of the random vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$; the dummy arguments $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$ refer to $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$, respectively. When the context is clear, $p(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is used to represent the joint probability density function.

2 Nonlinear Estimation with Quantized Measurements

2.1 Introduction

This chapter contains some results in the nonlinear estimation of parameter and state vectors when the measurements are quantized. Many different criteria may be used to derive estimates of random variables (see, for example, Lee, 1964, Chapter 3), the techniques vary according to the amount of probabilistic structure assumed a priori and the amount of computation that can be performed. The next section of this chapter introduces the concept of maximum-likelihood estimation with quantized measurements. The remainder of the chapter is devoted to Bayesian estimates with emphasis on the minimum variance, or conditional mean, estimate. Estimates of parameters are considered first and lead to apparently new formulas for the conditional mean and covariance of random variables whose a priori distribution is normal. These results are extended to Gaussian, linear systems, and the Kalman filter plays a surprising role in the determination of the conditional mean. The last section describes several nonlinear approximations for calculating the conditional mean of both parameter vectors and state vectors.

2.2 Maximum-Likelihood Estimates of Parameters

In this section we derive the necessary conditions for the maximum-likelihood estimate of parameters when the observations have been quantized. This method of estimation can be used when no probabilistic