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Algebraic Topology—Homotopy and Homology

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Introduction

This book is the result of lecture courses on algebraic topology given by the author at the University of Manchester in 1967–1970, at Cornell University in 1970–1971 and at the Georg August University, Göttingen, in 1971–1972. The level of the material is more advanced than that of a first-year graduate course in algebraic topology; it is assumed that the student has already had a course on basic algebraic topology which included singular homology, the fundamental group and covering spaces. Moreover, a student who has never encountered differentiable manifolds will probably have difficulty with Chapter 12. On the other hand no knowledge of homotopy theory beyond the fundamental group is assumed.

The last few years have seen the publication of several excellent textbooks on basic algebraic topology, most notably the book by Spanier [80], which I suggest as a companion volume to this one. There is a certain overlap between Spanier's book and this text—particularly in Chapters 0–6, 14 and 15—but the present book goes considerably further and has as its goal that the reader should be brought to a point from which he could begin research in certain areas of algebraic topology: stable homotopy theory, *K*-theory, cobordism theories.

Despite the title "Algebraic Topology" this book does not (and could not) pretend to achieve the same very advanced level in all areas of this subject. The choice of topics to be emphasized is, of course, heavily influenced by the research interests of the author. Thus, for example, unstable homotopy theory is only developed to the point at which it really begins to be interesting and is then dropped in favor of stable homotopy theory. The reader who finds that his appetite for unstable homotopy theory has been whetted is advised to follow the signposts set up by Adams in [10]. Another important branch of algebraic topology which is omitted is obstruction theory—partly due to lack of space, partly because one could scarcely give a better introduction than Thomas' *Seminar on fibre spaces* [86].

The following basic idea occurs repeatedly as a leitmotiv in this text: the majority of the problems which have been solved by means of algebraic topology have first been reduced to the question of the existence or non-existence of a continuous function $f: X \to Y$ between two given topological spaces X, Y. One tries to prove the non-existence, for example, by finding an appropriate functor F from the category of topological spaces to some algebraic category—that is for every space X we are given $F(X)$, a group, ring, module, ..., and for every continuous function $f: X \to Y$ we have $F(f): F(X) \to F(Y)$, a homomorphism preserving the given algebraic structure. Then one seeks to demonstrate that the algebraic map $F(f): F(X) \to F(Y)$ cannot possibly exist. (Proofs of existence are in general more difficult to handle). From this point of view the richer the natural algebraic structure on $F(X)$ the better: if $F(X)$, $F(Y)$ have a very complex algebraic structure, then there will not be many homomorphisms $\phi: F(X) \to F(Y)$ preserving this structure, and thus the chances of showing $F(f)$ cannot exist are good. At several points, then, (cf. Chapters 2, 13, 17, 18) we shall strive to enrich the natural algebraic structure available on our functors. In Chapter 19 we make the happy discovery that we have a sufficiently complex natural algebraic structure on $F(X)$ that we can (under favorable circumstances) say precisely which algebraic maps $\phi: F(X) \to F(Y)$ are of the form $\phi = F(f)$ for some continuous function $f: X \to Y$. At this point existence proofs become possible.

Chapters 0 and 1 contain respectively certain results from set-theoretic topology which are repeatedly used in the text and the basic definitions of category theory; both chapters should be in the nature of a review for the reader. Chapter 2 takes up the sets $[X, Y]$ of homotopy classes of maps $f: X \to Y$ and deals with such questions as: under what conditions on X or Y is $[X, Y]$ a group, when is a sequence

$$[X, W] \xleftarrow{f^*} [Y, W] \xleftarrow{g^*} [Z, W]$$

exact, etc. (enrichment of structure!). In Chapter 3 we then specialize to $X = S^n$ and consider $\pi_n(Y, y_0) = [S^n, s_0; Y, y_0]$, which is always a group of $n \geqslant 1$—the *nth homotopy group* of Y. The more elementary properties of these groups are demonstrated in this chapter. In Chapter 4 we define the notions of *fibration* and *weak fibration* and show that for a weak fibration $p: E \to B$ with fibre $F = p^{-1}(b_0)$ there is an exact sequence

$$\cdots \longrightarrow \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \longrightarrow \cdots.$$

We define the geometrically very important notion of a *fibre bundle* and show that every fibre bundle is a weak fibration. There follow some important examples of fibre bundles:

$$O(n) \to O(n)/O(n-k), \quad O(n)/O(n-k) \to O(n)/O(n-k) \times O(k)$$

and others. The chapter is concluded by remarking that a covering $p: X' \to X$ is a fibre bundle with discrete fibre and by using this remark to compute the homotopy groups of S^1 and $T^n = S^1 \times S^1 \times \cdots \times S^1$.

Proofs of deeper results for arbitrary topological spaces X, Y are difficult; it is not easy to demonstrate, for example, the existence of continuous functions $f: X \to Y$. We turn therefore to the smaller category of CW-complexes. Since CW-complexes are built up by glueing cells D^n together, it is possible to construct maps and homotopies cell by cell. This property permits strong statements about $[X, Y]$ when X or Y is a CW-complex. In Chapter 5 we define CW-complexes and prove some straightforward properties. Chapter 6 contains some deeper homotopy results, such as: $\pi_n(X, x_0)$ depends only on the cells of dimension at most $n+1$; the suspension homomorphism

$$\Sigma: \pi_q(X, x_0) \to \pi_{q+1}(SX, *) \qquad ([f] \mapsto [1 \wedge f])$$

is an isomorphism for $q < 2n + 1$ if X is an n-connected CW-complex; a map $f: X \to Y$ between CW-complexes induces an isomorphism $f_*: \pi_q(X, x_0) \to \pi_q(Y, y_0)$, $q \geq 0$, if and only if f is a homotopy equivalence.

At this point we turn from homotopy theory to homology and cohomology theories. A *generalized homology theory* is a family of functors $\{h_n : n \in \mathbb{Z}\}$ from the category of topological spaces to the category of abelian groups which satisfies the first six of the seven Eilenberg-Steenrod axioms (i.e. the "Dimension Axiom" is not necessarily satisfied). Chapter 7 is an investigation of the properties of such theories which follow directly from the axioms; this amounts to carrying out such parts of the program of Eilenberg and Steenrod [40] as still go through without the seventh axiom.

Chapter 8 contains a construction of Boardman's stable category of spectra and demonstrations of some of its most important properties—in particular that $\Sigma: [E, F] \to [E \wedge S^1, F \wedge S^1]$ is a bijection for all spectra E, F. We show how to construct a homology theory E_* and a cohomology theory E^* for every spectrum E. Then in Chapter 9 we prove that we have already constructed all possible cohomology theories in Chapter 8 (Brown's representation theorem). Then come the three most important known examples of homology and cohomology theories: ordinary homology (Ch. 10), K-theory (Ch. 11) and bordism (Ch. 12). In Chapter 10 we also show how to compute the singular homology groups of an arbitrary CW-complex and prove the Hurewicz isomorphism theorem. Chapter 11 contains the computation of the homotopy groups of the stable groups O, U, Sp.

Next comes a chapter on products in homology and cohomology. Chapter 13 begins with the universal coefficient theorems (how does one express $H_*(X; G)$ and $H^*(X; G)$ in terms of $H_*(X; \mathbb{Z})$ and G?) and the

Künneth theorem (how does one express $H_*(X \times Y)$ in terms of $H_*(X)$, $H_*(Y)$?). Next the $\times$-, $\cup$- and $\cap$-products for singular homology and cohomology are briefly discussed. Then we make a digression in order to construct the smash product of two spectra, whereupon we can describe products in the generalized homology and cohomology theories E_*, E^* associated to a *ring spectrum* **E**. We describe explicitly the products for ordinary homology, *K*-theory and bordism.

Chapter 14 then applies what we have learned about products to investigations of duality (Alexander, Lefschetz, Poincaré for manifolds; Spanier-Whitehead for finite spectra) and to questions of orientability of manifolds with respect to generalized cohomology theories. Spanier-Whitehead duality also permits a proof of a representation theorem for homology theories similar to the one for cohomology theories in Chapter 9.

In Chapter 15 the level of difficulty increases with the introduction of spectral sequences. Everyone finds spectral sequences baffling at the first encounter. Experience, however, shows that spectral sequences are among the topologist's most effective tools, so that the effort required to master their use is well worth while. We develop the Atiyah–Hirzebruch–Whitehead and Leray–Serre spectral sequences and make some important applications of them: Gysin, Wang and Serre exact sequences, Leray–Hirsch theorem, Thom isomorphism theorem.

Chapter 16 is concerned with the calculation of the homology and cohomology rings $E_*(BG)$, $E^*(BG)$ of the stable classifying spaces BG, $G = O$, U and Sp. Here the Atiyah–Hirzebruch–Whitehead spectral sequence proves very useful. In the process we construct certain classes $c_i \in E^*(BG)$ with whose help we can form invariants $c_i(\xi) \in E^*(X)$ of isomorphism classes of G-vector bundles $\xi \to X$—the so-called *characteristic* classes. The chapter ends with a proof of the Bott periodicity theorem $BU \simeq \Omega_0^2 BU$.

Chapter 17 represents a large step in our natural-algebraic-structure-enrichment program: we show how to make $E^*(X)$ into a module over an algebra $A^*(E) = E^*(E)$ and $E_*(X)$ into a comodule over a Hopf algebra $A_*(E) = E_*(E)$ in a natural way (under favorable conditions on E). We compute the Hopf algebras $A_*(E)$ for $E = MU$, MSp, K and KO and find they have satisfyingly complex algebraic structures (recall our leitmotiv). As an application we find bounds on the image of the Hurewicz homomorphisms

$$h_K : \pi_q(MU) \to K_q(MU)$$
$$h_{KO} : \pi_q(MSp) \to KO_q(MSp)$$

for small values of q.

Chapter 18 contains a determination of the algebras $A^*(H(\mathbb{Z}_2))$ and $A_*(H(\mathbb{Z}_2))$ including the construction of certain elements $Sq^i \in A^*(H(\mathbb{Z}_2))$. The analogous results for $A_*(H(\mathbb{Z}_p))$, $A^*(H(\mathbb{Z}_p))$ are stated without proof.

In Chapter 19 we enjoy the triumph of our natural-algebraic-structure-enrichment program: using the comodule structure of $E_*(X)$ over $E_*(E)$ constructed in Chapter 17 we can build a spectral sequence (the *Adams spectral sequence*) $\{E_r^{st}, d_r\}$ whose E_2-term is the purely algebraic construct

$$\mathrm{Ext}^{s,t}_{E_*(E)}\,(\tilde{E}_*(S^0),\, E_*(X))$$

($\mathrm{Ext}_C^{**}(-,-)$ is a functor derived from $\mathrm{Hom}_C^*(-,-)$, C a coalgebra, and thus has to do with algebraic maps) and which converges (for connected E) to $\pi_*^S(X)/D_*$, the stable homotopy of X modulo a certain subgroup D_* which depends on E. For $E = MU$ or MSp we show that $D_* = 0$; for $E = H(\mathbb{Z}_p)$ D_* is the subgroup of elements of finite order prime to p. Thus in cases where the spectral sequence proves manageable one can start from a knowledge of $E_*(X)$ as $E_*(E)$-comodule and compute $\pi_*^S(X)/D_*$. We show how the Adams spectral sequence has permitted the determination of $\pi_q^S(S^0)$ for small q. We then turn to a consideration of the spectral sequence for $E = K$, KO. Here E is not connected, so the spectral sequence may not converge, but it still provides homomorphisms

$$e_C\colon \ker k_K \to \mathrm{Ext}^{1,q+1}_{K_*(K)}\,(\tilde{K}_*(S^0), K_*(X))$$
$$e_R\colon \ker h_{KO} \to \mathrm{Ext}^{1,q+1}_{KO_*(KO)}\,(\widetilde{KO}_*(S^0), KO_*(X)).$$

The Ext-groups for $X = S^0$ are computed and the result is used to localize a non-trivial direct summand in $\pi_*^S(S^0)$ whose order is related to the Bernoulli numbers.

Chapter 20 then represents an extended application of the Adams spectral sequence. The cases $E = H(\mathbb{Z}_p)$ and $X = MG$, $G = O$, SO, or U are, fortunately, of the sort in which the Adams spectral sequence is manageable and permits a complete determination of $\pi_*(X) = \pi_*(MG) \cong \Omega_*^G$, the G-cobordism ring. We also prove the theorem of Hattori and Stong, which describes the image of

$$h_K\colon \pi_*(MU) \to K_*(MU).$$

After reading Chapter 20 the student should be able to understand without undue difficulty the papers [13, 14] of Anderson, Brown and Peterson in which Ω_*^{SU} and Ω_*^{Spin} are determined.

This summary undoubtedly makes clear that the level of mathematical expertise demanded of the reader rises rather markedly from Chapter 2, say, to Chapter 20. The student who begins with the minimal prerequisites described in the first paragraph will not acquire the facility needed for understanding the later chapters merely by reading straight through. He must try to master the material in each chapter to such an extent that he can apply it to the solution of problems other than those worked out in the text. In some cases he will find it valuable to seek further applications in the books and articles listed at the end of each chapter.

The bibliography included here does not attempt to be comprehensive. Steenrod's valuable compendium of all mathematical reviews having to do with topology makes such a comprehensive bibliography unnecessary. Instead this bibliography has two goals: (1) to suggest to the student where he might begin to pursue a given topic further and (2) to acknowledge the sources from which much of the material in this text is drawn.

In addition, however, I wish to acknowledge quite explicitly and with gratitude my debt to Frank Adams. It is no exaggeration to say that most of what I know about algebraic topology I learned from him. Anyone familiar with his work will recognize the influence of his way of looking at algebraic topology on the presentation of the subject given here. Moreover, on certain topics (e.g. Chapter 8, Chapter 9 in the case of finite CW-complexes, the construction of $E \wedge F$) I have largely reproduced his presentation of the topic with only small alterations (which may not have been for the better).

I further wish to express my deep gratitude to Egbert Brieskorn for his encouragement and helpful suggestions. My thanks also go to Fräulein Ingrid Sochaczewsky and Frau Christiane Preywisch for help in typing the manuscript.

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Göttingen, November, 1973 ROBERT M. SWITZER

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Chapter 0

Some Facts from General Topology

It is assumed that the reader is familiar with the elements of general topology—e.g. the most important properties of continuous functions, compact sets, connected sets, etc. Nevertheless, certain general topological results which will be used repeatedly in this book are assembled here for the reader's convenience.

0.1. Let X be a topological space and $A_1, A_2, \ldots, A_n$ be closed subspaces such that $X = \bigcup_{i=1}^n A_i$.

Suppose $f_i: A_i \to Y$ is a function, $1 \leqslant i \leqslant n$; there is a function $f: X \to Y$ such that $f|A_i = f_i$, $1 \leqslant i \leqslant n$, if and only if $f_i|A_i \cap A_j = f_j|A_i \cap A_j$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. In this case f is continuous if and only if each f_i is.

We shall often set about defining a continuous function $f: X \to Y$ by cutting up X into closed subsets A_i and defining f on each A_i separately in such a way that $f|A_i$ is obviously continuous; we then have only to check that the different definitions agree on the "overlaps" $A_i \cap A_j$.

0.2. *The universal property of the cartesian product:* let $p_X: X \times Y \to X$, $p_Y: X \times Y \to Y$ be the projections onto the first and second factors respectively. Given any pair of functions $f: Z \to X$ and $g: Z \to Y$ there is a unique function $h: Z \to X \times Y$ such that $p_X \circ h = f$, $p_Y \circ h = g$. h is continuous if and only if both f and g are. This property characterizes $X \times Y$ up to homeomorphism. The unique h will often be denoted by (f, g).

In particular, to check that a given function $h: Z \to X \times Y$ is continuous it will suffice to check that $p_X \circ h$ and $p_Y \circ h$ are continuous.

0.3. *The universal property of the quotient:* let α be an equivalence relation on a topological space X, let X/α denote the space of equivalence classes and $p_\alpha: X \to X/\alpha$ the natural projection. Given a function $f: X \to Y$, there is a function $f': X/\alpha \to Y$ with $f' \circ p_\alpha = f$ if and only if $x\alpha x'$ implies $f(x) = f(x')$. In this case f' is continuous if and only if f is. This property characterizes X/α up to homeomorphism.

An important example of a quotient which we frequently encounter is that of a space X with a closed subspace A collapsed to a point. Explicitly, if $A \subset X$ is a closed non-empty subspace, we take the relation

$$\alpha = A \times A \cup \{(x,x) : x \in X\} \subset X \times X$$

and let $X/A = X/\alpha$. For $A = \varnothing$ we adopt the convention that $X/A = X/\varnothing = X^+ = X \cup \{\text{a disjoint point}\}$. This convention will be seen to be justified by the fact that every theorem we state about X/A will hold equally well for $X/\varnothing$.

0.4. *Product and quotient combined:* if α is an equivalence relation on a topological space X and β is an equivalence relation on Y, then there is an obvious equivalence relation $\alpha \times \beta$ on $X \times Y$:

$$(x,y)\, \alpha \times \beta\, (x',y') \Leftrightarrow x\, \alpha\, x' \quad \text{and} \quad y\, \beta\, y'.$$

There is a unique continuous function

$$\phi : \frac{X \times Y}{\alpha \times \beta} \to (X/\alpha) \times (Y/\beta),$$

such that $\phi \circ p_{\alpha \times \beta} = p_\alpha \times p_\beta$ and which is even bijective—but not necessarily a homeomorphism. One important case in which ϕ is a homeomorphism is that where β is the identity relation 1 and Y is locally compact.

Proof: We must prove that if $T \subset \dfrac{X \times Y}{\alpha \times 1}$ is open then $\phi(T)$ is open in $(X/\alpha) \times Y$. T open means $T' = p_{\alpha \times 1}^{-1}(T)$ is open in $X \times Y$. Suppose $(x_0, y_0) \in \phi(T)$ and choose $x_0 \in X$ with $p_\alpha(x_0') = x_0$. By the definitions of ϕ and T', we have $(x_0', y_0) \in T'$; and since T' is open, there exist neighborhoods U of x_0' and K of y_0 in X, Y respectively such that $U \times K \subset T'$. We may assume K is compact.

Let $J' = \{x' \in X : x' \times K \subset T'\}$. We first show that J' is open. For every $x' \in J'$ we can, because K is compact, find an open neighborhood $U_{x'}$ of x' such that $U_{x'} \times K \subset T'$, which proves J' is open. If $J = \{x \in X/\alpha : x \times K \subset \phi(T)\}$, then clearly $p_\alpha^{-1}(J) = J'$, so J is open in X/α. Also $(x_0, y_0) \in J \times K \subset \phi(T)$, so $\phi(T)$ is open. $\square$

Example. The unit interval $I = [0,1]$ is compact, so $\dfrac{X \times I}{\alpha \times 1} \cong (X/\alpha) \times I$.

0.5. A *homotopy* from X to Y is a continuous function $F : X \times I \to Y$. For each $t \in I$ one has $F_t : X \to Y$ defined by $F_t(x) = F(x,t)$ for all $x \in X$. The functions F_t are called the "stages" of the homotopy. If $f, g : X \to Y$ are two maps, we say f is *homotopic* to g (and write $f \simeq g$) if there is a homotopy $F : X \times I \to Y$ such that $F_0 = f$ and $F_1 = g$. In other words, f can be continuously deformed into g through the stages F_t. If $A \subset X$ is a subspace, then F is a homotopy *relative to A* if $F(a,t) = F(a,0)$, all $a \in A$, $t \in I$.

0.6. *The relation $\simeq$ is an equivalence relation.*

Proof: $f \simeq f$ is obvious; take $F(x,t) = f(x)$, all $x \in X$, $t \in I$. If $f \simeq g$ and F is a homotopy from f to g, then $G: X \times I \to Y$ defined by $G(x,t) = F(x, 1-t)$, $x \in X$, $t \in I$, is a homotopy from g to f—i.e. $g \simeq f$. If $f \simeq g$ with homotopy F and $g \simeq h$ with homotopy G, then $f \simeq h$ with homotopy H defined by

$$H(x,t) = \begin{cases} F(x, 2t) & 0 \leqslant t \leqslant 1/2 \\ G(x, 2t-1) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Note that here we use 0.1 to show H is continuous. $\square$

0.7. Thus the set of all continuous functions $f: X \to Y$ is partitioned into equivalence classes under the relation $\simeq$. The equivalence classes are called *homotopy classes*, and the set of all homotopy classes is denoted by $[X; Y]$. If $f: X \to Y$, then the homotopy class of f is denoted by $[f]$.

As an application of 0.4 we get the following proposition.

0.8. Proposition. *If α is an equivalence relation on a topological space X and $F: X \times I \to Y$ is a homotopy such that each stage F_t factors through X/α—i.e. $x\alpha x' \Rightarrow F_t(x) = F_t(x')$—then F induces a homotopy $F': (X/\alpha) \times I \to Y$ such that $F' \circ (p_\alpha \times 1) = F$.*

Proof: The hypothesis on F is precisely that F factors through $\dfrac{X \times I}{\alpha \times 1}$—i.e. there exists a continuous function $F'': \dfrac{X \times I}{\alpha \times 1} \to Y$ such that $F'' \circ p_{\alpha \times 1} = F$. By 0.4 $\phi: \dfrac{X \times I}{\alpha \times 1} \to (X/\alpha) \times I$ is a homeomorphism. Then $F' = F'' \circ \phi^{-1}$: $(X/\alpha) \times I \to Y$ is the required homotopy. $\square$

Example. If A is a closed subspace of X and $F: X \times I \to Y$ is a homotopy such that $F(a,t) = F(a',t)$ for all $a, a' \in A$, $t \in I$, then F induces a homotopy $F': (X/A) \times I \to Y$.

0.9. *Function spaces:* if X and Y are topological spaces, we let Y^X denote the set of all continuous functions $f: X \to Y$. We give this set a topology, called the *compact-open topology*, by taking as a subbase for the topology all sets of the form $N_{K,U} = \{f: f(K) \subset U\}$, $K \subset X$ compact, $U \subset Y$ open.

0.10. The *evaluation function* $e: Y^X \times X \to Y$ defined by $e(f,x) = f(x)$ is continuous if X is locally compact. In all the cases we shall consider X will be $I = [0,1]$.

0.11. *The exponential law:* if X, Z are Hausdorff spaces and Z is locally compact, then the natural function

$$E: Y^{Z \times X} \to (Y^Z)^X$$

defined by $(Ef(x))(z) = f(z,x)$ is a homeomorphism.

0.12. *Base points:* in what follows we shall often have to consider not just a topological space X but rather a space X together with a distinguished point $x_0 \in X$ called the *base point*. The pair (X, x_0) is called a *pointed space* (one also speaks of pointed sets). When we are concerned with pointed spaces (X, x_0), (Y, y_0), etc. we always require that all functions $f: X \to Y$ shall preserve base points—i.e. $f(x_0) = y_0$—and that all homotopies $F: X \times I \to Y$ be relative to the base point—i.e. $F(x_0, t) = y_0$, all $t \in I$—unless an explicit disclaimer to the contrary is made. We shall use the notation $[X, x_0;\ Y, y_0]$ to denote the homotopy classes of base point-preserving functions—where homotopies are rel x_0, of course. $[X, x_0;\ Y, y_0]$ is a pointed set with base point f_0, the constant function: $f_0(x) = y_0$, all $x \in X$.

Let us use the notation $(Y, B)^{(X, A)}$ to denote the subspace of Y^X consisting of those functions such that $f(A) \subset B$, where $A \subset X$, $B \subset Y$ are subspaces. There are obvious generalizations of this notation, such as $(Y, B, B')^{(X, A, A')}$, etc. In particular, if (X, x_0), (Y, y_0) are pointed spaces, then we have the space $(Y, y_0)^{(X, x_0)}$ of base point-preserving functions. It has as base point f_0, where $f_0(x) = y_0$, all x.

If (X, x_0), (Y, y_0) and (Z, z_0) are three pointed spaces, we can form

$$((Y, y_0)^{(Z, z_0)}, f_0)^{(X, x_0)}, \quad f_0(z) = y_0, \quad \text{all } z \in Z,$$

which is a subspace of $(Y^Z)^X$. Thus we may ask what subspace of $Y^{Z \times X}$ corresponds to it under the exponential function of 0.11. One readily sees that the answer is

$$(Y, y_0)^{(Z \times X,\ Z \times \{x_0\} \cup \{z_0\} \times X)}$$

We use the notation $Z \vee X$ for the subspace $Z \times \{x_0\} \cup \{z_0\} \times X$ of $Z \times X$. It can be thought of as the result of taking the disjoint union $Z \cup X$ and identifying z_0 with x_0. $Z \vee X$ is again a pointed space with base point (z_0, x_0). Then we have

0.13. Proposition. *The exponential function* $E: Y^{Z \times X} \to (Y^Z)^X$ *induces a continuous function*

$$E: (Y, y_0)^{(Z \times X,\ Z \vee X)} \to ((Y, y_0)^{(Z, z_0)}, f_0)^{(X, x_0)}$$

which is a homeomorphism if Z *and* X *are Hausdorff and* Z *is locally compact.*

Remark. The subspace $Z \times \{x_0\} \cup \{z_0\} \times X \subset Z \times X$ is called the *wedge sum* of Z and X and is characterized by the property that for any continuous functions $f: (Z, z_0) \to (W, w_0)$, $g: (X, x_0) \to (W, w_0)$ there is a unique continuous function $h: (Z \vee X, *) \to (W, w_0)$ such that $h|Z = f$, $h|X = g$. (Compare this with 0.2.) The unique function h will be denoted by

$(Z \vee X, *) \xrightarrow{(f,g)} (W, w_0)$. Of course, given continuous functions $f:(X, x_0) \to (X', x_0')$ and $g:(Y, y_0) \to (Y', y_0')$ there is a continuous function $f \vee g:(X \vee Y, (x_0, y_0)) \to (X' \vee Y', (x_0', y_0'))$ which is f on X, g on Y. We use $*$ to denote the base point $(x_0, y_0) \in X \vee Y$.

Exercise. Use 0.11 to give an independent proof of 0.4 in the case where X and Y are Hausdorff.

Chapter 1

Categories, Functors and Natural Transformations

In modern mathematics whenever one defines a new class of mathematical objects one proceeds almost in the next breath to say what kinds of functions between objects will be considered; thus, for example, topological spaces and continuous functions, groups and homomorphisms, rings and ring homomorphisms. If we formalize this observation, we are led to the notion of a category.

1.1. Definition. A *category* is

a) a class of *objects* (e.g. spaces, groups, etc.);

b) for every ordered pair (X, Y) of objects a set $\hom(X, Y)$ of *morphisms* with *domain* X and *range* Y; for $f \in \hom(X, Y)$ we write $f: X \to Y$ or $X \xrightarrow{f} Y$;

c) for every ordered triple (X, Y, Z) a function $\hom(Y,Z) \times \hom(X, Y) \to \hom(X,Z)$ called *composition*. If $f \in \hom(X, Y)$ and $g \in \hom(Y,Z)$ then the image of (g,f) in $\hom(X,Z)$ will be denoted by $g \circ f$.

These objects and morphisms are required to satisfy two axioms:

C1) If $f \in \hom(X, Y)$, $g \in \hom(Y,Z)$, $h \in \hom(Z, W)$, then $h \circ (g \circ f) = (h \circ g) \circ f$ in $\hom(X, W)$.

C2) For every object Y there is a $1_Y \in \hom(Y, Y)$ such that $1_Y \circ g = g$ for every $g \in \hom(X, Y)$ and $h \circ 1_Y = h$ for every $h \in \hom(Y,Z)$, all X, Z.

One can show that 1_Y is unique.

1.2. Definition. Two objects X, Y are called *equivalent* if there are morphisms $f \in \hom(X, Y)$ and $g \in \hom(Y, X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. f and g are called *equivalences*.

1.3. Examples. i) The category $\mathcal{S}$ of all sets and all functions.

ii) The category $\mathcal{T}$ of all topological spaces and all continuous functions.

iii) The category $\mathcal{PS}$ of pointed sets and functions preserving base point.

iv) The category $\mathscr{PT}$ of pointed topological spaces and continuous functions preserving base point.

v) The category $\mathscr{G}$ of groups and homomorphisms.

vi) The category $\mathscr{A}$ of abelian groups and homomorphisms.

vii) The category $\mathscr{M}_R$ of left R-modules (R some fixed ring) and R-homomorphisms.

viii) The category $\mathscr{T}'$ in which the objects are topological spaces but $\hom(X, Y) = [X, Y]$. Given $[f] \in \hom(X, Y)$, $[g] \in \hom(Y,Z)$, we define $[g] \circ [f]$ to be $[g \circ f]$. One readily checks that $[g] \circ [f]$ is well defined and that C1), C2) are satisfied. In like fashion we have $\mathscr{PT}'$.

Note that X and Y are equivalent in $\mathscr{T}$ if and only if they are homeomorphic, whereas in $\mathscr{T}'$ they are equivalent if and only if they are *homotopy equivalent*.

In algebraic topology one attempts to assign to every topological space X some algebraic object $F(X)$ in such a way that to every continuous function $f: X \to Y$ there is assigned a homomorphism $F(f): F(X) \to F(Y)$. One advantage of this procedure is, for example, that if one is trying to prove the non-existence of a continuous function $f: X \to Y$ with certain properties, one may find it relatively easy to prove the non-existence of the corresponding algebraic function $F(f)$ and hence deduce that f could not exist. In other words, F is to be a "homomorphism" from one category (e.g. $\mathscr{T}$) to another (e.g. $\mathscr{G}$ or $\mathscr{A}$). If we formalize this notion, we are led to define a functor.

1.4. Definition. A *functor* from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a function which

a) to each object $X \in \mathscr{C}$ assigns an object $F(X) \in \mathscr{D}$;

b) to each $f \in \hom_{\mathscr{C}}(X, Y)$ assigns a morphism

$$F(f) \in \hom_{\mathscr{D}}(F(X), F(Y)).$$

F is required to satisfy the two axioms:

F1) For each object $X \in \mathscr{C}$ we have $F(1_X) = 1_{F(X)}$.
F2) For $f \in \hom_{\mathscr{C}}(X, Y)$, $g \in \hom_{\mathscr{C}}(Y,Z)$ we have

$$F(g \circ f) = F(g) \circ F(f) \in \hom_{\mathscr{D}}(F(X), F(Z)).$$

In the arrow notation we have that if $f: X \to Y$ then $F(f): F(X) \to F(Y)$. We also have the notion of cofunctor; cofunctors "reverse the arrow".

1.5. Definition. A *cofunctor* F^* from the category $\mathscr{C}$ to the category $\mathscr{D}$ is a function which

a) to each object $X \in \mathscr{C}$ assigns an object $F^*(X) \in \mathscr{D}$,

b) to each $f \in \hom_{\mathscr{C}}(X, Y)$ assigns a morphism

$$F^*(f) \in \hom_{\mathscr{D}}(F^*(Y), F^*(X))$$

satisfying the two axioms:

CF1) For each object $X \in \mathscr{C}$ we have $F^*(1_X) = 1_{F^*(X)}$.

CF2) For each $f \in \hom_{\mathscr{C}}(X, Y)$ and $g \in \hom_{\mathscr{C}}(Y, Z)$ we have

$$F^*(g \circ f) = F^*(f) \circ F^*(g) \in \hom_{\mathscr{D}}(F^*(Z), F^*(X)).$$

Remark. In the literature functors are often referred to as covariant functors and cofunctors as contravariant functors.

1.6. Examples. i) Define $F: \mathscr{T} \to \mathscr{S}$ as follows: if $X \in \mathscr{T}$, let $F(X)$ be the underlying set (forget the topology), and if $f: X \to Y$ is a continuous function, let $F(f)$ be the underlying function (forget continuity). For obvious reasons F is called a "forgetful functor". One can think of many examples of forgetful functors.

ii) Given a fixed ring R and fixed (left) R-module K, we can define a functor $F_K: \mathscr{M}_R \to \mathscr{A}$; we take $F_K(M) = \mathrm{Hom}_R(K, M)$, $M \in \mathscr{M}_R$, and for any homomorphism $\phi: M \to M'$, we take

$$F_K(\phi) = \mathrm{Hom}_R(1_K, \phi): \mathrm{Hom}_R(K, M) \to \mathrm{Hom}_R(K, M').$$

Similarly we get a cofunctor $F_K^*: \mathscr{M}_R \to \mathscr{A}$ by taking

$$F_K^*(M) = \mathrm{Hom}_R(M, K), \quad F_K^*(\phi) = \mathrm{Hom}_R(\phi, 1_K).$$

iii) In a similar vein, given a fixed pointed space $(K, k_0) \in \mathscr{P}\mathscr{T}$, we define a functor

$$F_K: \mathscr{P}\mathscr{T} \to \mathscr{P}\mathscr{S}$$

as follows: for each $(X, x_0) \in \mathscr{P}\mathscr{T}$ we take $F_K(X, x_0) = [K, k_0;\ X, x_0]$. Given $f: (X, x_0) \to (Y, y_0)$ in $\hom((X, x_0), (Y, y_0))$ we define $F_K(f)$ by

$$F_K(f)[g] = [f \circ g] \in [K, k_0;\ Y, y_0]$$

for every $[g] \in [K, k_0;\ X, x_0]$.

Similarly we obtain a cofunctor F_K^* by taking $F_K^*(X, x_0) = [X, x_0; K, k_0]$ and for $f: (X, x_0) \to (Y, y_0)$ in $\hom((X, x_0), (Y, y_0))$

$$F_K(f)[g] = [g \circ f] \in [X, x_0; K, k_0]$$

for every $[g] \in [Y, y_0;\ K, k_0]$.

Observe that $f \simeq f'$ rel x_0 implies $F_K(f) = F_K(f')$ and likewise $F_K^*(f) = F_K^*(f')$. Therefore F_K and F_K^* can equally well be regarded as defining a functor $\mathscr{P}\mathscr{T}' \to \mathscr{P}\mathscr{S}$, respectively a cofunctor.

iv) Define a functor $C: \mathscr{T} \to \mathscr{T}$ by taking $C(X) = \hat{X} =$ one point compactification of X for every $X \in \mathscr{T}$. If $f: X \to Y$ is a continuous func-