Algebraic Hyperstructures and Applications

ALGEBRAIC HYPERSTRUCTURES AND APPLICATIONS

PROCEEDINGS OF THE
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PREFACE

The Fourth International Congress on Algebraic Hyperstructures and Applications (AHA), held at Democritus University of Thrace in Xanthi from 27th to 30th of June 1990, was organized by the International Democritean Foundation. The previous congresses on the AHA series was held in Italy: Taormina 1978, Taormina 1983, Udine 1985.

The congress covered the recent developments on semi-hypergroups, hypergroups, hyperrings, hypermatrices, ordered hyperstructures and related topics such as join spaces, cogroups, polygroups. Other multivalued structures and applications of the above structures were included in the topics of the congress.

60 mathematicians, from 10 countries participated in the congress and there were 32 research announcements.

The Scientific Organizing Committee was: S. Comer (USA), P. Corsini (Italy), J. Jantosciak (USA), L. Konguetsof (Greece), Y. Sureau (France), T. Vougiouklis (Greece).

In these proceedings the first three papers were the invited introductory lectures to the topics and the rest were contributed papers selected after the refereeing process. Finally, a selected bibliography on the topics is presented at the end of this volume.

Sincere thanks are due to the referees for their assistance.

THOMAS VOUGIOUKLIS

INTERNATIONAL DEMOCRITEAN FOUNDATION

Democritus was one of the greatest philosophers of Greek antquity. His work is considered to be equivalent to Plato's. Many people believe that he was the learned man and writer who knew and wrote the most, before Aristotle. Democritus was Leucippus's student. His books cover all the branches of knowledge. As we know, he invented the theory of the atom. he was born in 470 B.C., in Abdera of Thrace, very near the town of Xanthi of today.

The "International Democritean Foundation" like the "Democritus University of Thrace" is dedicated to the great offspring of Thrace, the immortal philosopher and scientist Democritus. The IDF was founded in 1975, and has its headquarters in Xanthi. The IDF, in the relatively short period of its existence, has to exhibit an important scientific, cultural and published work. The periodic organising of international congresses, is amongst the means used for the achievement of the goals of the IDF. Let us remind you of the 1st International Congress on Democritus, 25 centuries since the philosopher's birth, which was held in Xanthi, from 6 to 9 October 1983. In this Congress, which took place under the high patronage of H. E. the President of the Hellenic Republic, and the auspices of UNESCCO and the Greek Ministry of Culture, a large number amongst the most distinguished researchers on the work of Democritus, participated. Also the AHA Congress, under the auspices of the IDF, does honour to this great thinker of ancient Abdera, as three of his books refer to Mathematics. We remind you of the most interesting scientific communications, made during the 1st International Congress on Democritus, referring to the work of Democritus on Mathematics.

Professor Leonidas Konguetsof
Vice-President of the International democritean Foundation

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THEMES AND PROBLEMS IN HYPERGROUP THEORY

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It gives me great satisfaction to participate at this Fourth Congress on Hyperstructures. When I began to work on this subject about twenty years ago, I was one of the very few in Italy and the world to understand just how significant and euristically important these themes were. Against the predictions of the many who thought the subject to be by then exhausted, we can state today that it is now alive and productive in four continents.

At this conference I will define the present situation of research and recall some of the most recent results and topics which at the moment seem to be the most interesting and promising.

Let me remind you of the fundamental definitions.

DEFINITION 1: We call a **Marty hypergroupoid** (or hypergroupoid) a non-empty set H endowed with a binary multivalued operation, that is, with a function $\circ : H\times H \longrightarrow \mathcal{P}^*(H)$.
We note $x\circ y$ the image of the pair (x,y) and $<H;\circ>$ the hypergroupoid.

DEFINITION 2: A (Marty) hypergroupoid $<H;\circ>$ is called a (Marty) **semi-hypergroup** if

i) $\forall (x,y,z) \in H^3 \quad (x\circ y)\circ z = x\circ (y\circ z)$

DEFINITION 3: A (Marty) hypergroupoid $<H;\circ>$ is called a (Marty) **hyper-quasigroup** if

ii) $\forall (x,y) \in H^2 \quad x\circ H = H\circ x = H$

DEFINITION 4: A (Marty) hypergroupoid $<H;\circ>$ satisfying both conditions i) and ii) is called a (Marty) **hypergroup**.

Now a short historical introduction.

One can consider 1934 as the year of birth of the hyperstructures when Marty, at the Fourth Congress of Scandinavian Mathematicians, presented the definition of a hypergroup and illustrated some of its properties. Subsequently, during the forties, the same Marty, Krasner, Kuntzmann and Croisot in France, Dresher, Ore, Eaton, Wall, Campaigne, Griffith and Prenowitz in the United States, Utumi in Japan, Dietzman and Vikrov in USSR, and Zappa in Italy investigated the subject of both general theory and applications in other sectors of mathematics: Geometry (Prenowitz), Group Theory and Field Theory (Krasner).

In the fifties Drbohlav in Czechoslovakia worked on the hypergroups of classes, Boccioni in Italy on conditions of associativity in hypergroupoids, Benado in Romania on multilattices; in the sixties Orsati in Italy re-examined and investigated problems of structure, Pickett and Graetzer in the United States extended various notions in the context of multi-algebras (which have been taken up again more recently by Hoft, Howard and Schweigert), Nakano in Japan studied multimodules and d-groups, Sade in France worked on hypergroupoids. At the end of the decade the most productive period began, which is still continuing.

In Greece, Mittas founded a theory of canonical hypergroups and studied hyperstructures with many operations. Stratigopoulos described artinian hyper-rings and hypermodules. In the United States, Roth investigated the theory of canonical hypergroups, Prenowitz and Jantosciak analysed join spaces and utilized them in geometry; in France Koskas discovered the semi-hypergroupoid of associativity, the same Koskas, Krasner and Sureau obtained results in general theory. In Australia, McMullen and Price developed an algebraic theory of the hypergroups, utilized in harmonic analysis, dealing with certain Wall hypergroups which in the case of multiplicity one are canonical hypergroups. Finally in Italy Corsini and Rossi studied questions of general theory and applications respectively to ordered abelian groups and geometry.

Schools are founded: At Thessaloniki Mittas, Konstantinidou, Serafimidis and Ioulidis studied ordered hyperstructures, hyper-rings and hyperlattices. In Xanthi Vougiouklis, Konguetsof and Spartalis analysed cyclic hypergroups, P-hypergroups and hyper-rings; At Patras Stratigopoulos and, currently in Athens, Massouros work on hyper-rings, hyperfields and hypermodules; At Messina Corsini, De Salvo, De Maria, Bonansinga, Romeo, Gionfriddo, Marino, and subsequently Freni and Migliorato; obtain results on complete hypergroups and semi-hypergroups of associativity, homomorphism theory and quasicanonical hypergroups. In the United States: Comer worked on polygroups (quasi-canonical hypergroups) and studied their relation with graph theory and algebras of various types, McAlister on multilattices. In Czechoslovakia in Prague, Kepka especially and more recently Drbohlav, Nemec and Jezek investigated the subject of semi-hypergroups of associativity, M. Tomkova, O. Klančová, J. Livková, K. Repasky, M.

Jasem worked on multilattices, J. Jakubik on multistructures, J. Močkor utilized hypergroups in the theory of ordered groups and in that of rings. On median algebras (which can be turned into join spaces), Isbell worked in U.S.A., Nieminen in Finland, H.J. Bandelt in Germany; J. Hedlikova and M. Kolibiar in Czechoslovakia.

The investigation into hyperstructures knew no respite. In Italy it continued at Udine where Corsini, Freni, Azzali and Marchi obtained results in canonical hypergroups, join spaces, general theory, finite hypergroups, correlated hypergroupoids with geometrical structures. It expanded to Rome where Rosaria Rota, the Tallinis and R. Procesi worked on hyper-rings, vectorial hyperspaces and connections of hyperstructures with geometry, in Florence where A. Barlotti (with K. Strambach introduced a metric in join spaces, and at Lecce where A. Letizia worked on Λ-hyperspaces.

At present the theory and its applications are also cultivated by the Gutans (simplifiable semi-hypergroups) in Rumania, by V. Dacic (multihomomorphisms) in Yugoslavia, by E. Degreef (join spaces) in Belgium, by I. Rosenberg in Canada and M.M. Deza in France (hyper-algebras and hypergroupoids), by A. Nakassis (hyper-rings) and D. Hansen (multilattices) in U.S.A., by M. Niemenmaa (semi-hypergroups of associativity) in Finland.

We come to themes and problems.

A subject already presented, historically one of the most important and still at present full of problems and a source of significant research, is that of canonical hypergroups (i.c.) and their generalizations.

Strongly canonical hypergroups (f.c.) have been defined and analysed by Mittas who has identified their notable properties and has also shown their usefulness in the theory of metric spaces.

DEFINITION: A canonical hypergroup H ia said to be **i.p.s.** if it satisfies the condition

1) $\forall\ (a,x) \in H^2\ , \quad a+x \ni x \Longrightarrow a+x = x$

DEFINITION: An i.p.s. hypergroup H is said to be **strongly canonical (f.c)** if

2) $\forall\ (a,b,c,d) \in H^4\ ,$

$$(a+b) \cap (c+d) \neq \emptyset \Longrightarrow [(a+b) \subset (c+d)] \vee [(a+b) \supset (c+d)]$$

If we denote C_k(i.p.s.) and C_k(f.c.) the classes, respectively, of (i.p.s.) hypergroups and (f.c.) hypergroups of cardinality k, it can be shown that if $k \leq 8$ $C_k(f.c.) = C_k(i.p.s.)$ and if $k > 8$

$C_k(f.c.) \neq C_k(i.p.s.)$.

All the i.p.s. of cardinality < 9 have been determined. The class of i.p.s. of cardinality greater than 9 deserves, in my opinion, a further investigation.

The same can be said of K-hypergroups:

DEFINITION: Let H be a canonical hypergroup, $K \in \mathcal{P}^*(H)$. H is said to be a **K-hypergroup** if

$$\forall\, x \in H-K \;,\quad I_p(x) = I_p(x) \cap S(H) = K$$

(where $I_p(x)$ is the set of partial identities of x in H, S(H) is the set of scalars of H).

It is known that in each K-hypergroup H, K is a group; moreover

$K = \omega_H = U_H$ and $\forall\, s \in \mathbb{N}^*$, $\forall\, (x_1 \ldots x_s) \in H^s$, if $\sum_{i=1}^{s} x_i \ni 0$

and $j \le s$ exists such that $x_j \in H-K$, then $\sum_{i=1}^{s} x_i = K$.

Another class of canonical hypergroups which has aroused interest is that of the A-hypergroups, introduced by Corsini in 1987 with the following definition.

DEFINITION: An **A-hypergroup** is a canonical hypergroup H which satisfies the following condition:

(O) $\forall\, x \in H$, $x-x$ is a sub-hypergroup of H.

The class of A-hypergroups which actually contain i.p.s. has recently been studied by Freni who has shown among other things the

THEOREM: If H is a A-hypergroup, R the equivalence on H defined $x \,R\, y \iff x-x = y-y$, and $H/R = \{x_1, x_2, \ldots, x_k\}$, then the heart ω_H is the set $\sum_{i=1}^{k} (x_i - x_i)$.

Not only have classes of hypergroups contained in the canonicals been studied and are still being studied, but also classes which actually contain them. For example, the feebly canonicals.

DEFINITION: A hypergroup H is called **feebly quasicanonical** (D.Q.C.) if

1) H is regular reversible

2) $\forall\, x \in H\,,\quad \forall\, \{x', x''\} \subset i_H(x)\,,\quad \forall\, a \in H$

$$a \circ x' = a \circ x'' \quad , \quad x' \circ a = x'' \circ a$$

If moreover $\forall\, (x,y) \in H^2\,,\quad x \circ y = y \circ x\,,$ H is called **feebly canonical** (D.C.).

A recently obtained result by De Salvo is the following.

Recall that if $\langle M; \circ \rangle$ is a hypergroup, $\langle C; \otimes \rangle$ is a K_M**-hypergroup** if

$\forall\, m \in M\,,\ A_m$ is a non-empty set such that

$$m' \neq m'' \Longrightarrow A_{m'} \cap A_{m''} = \varnothing \quad , \quad C = \bigcup_{m \in M} A_m \,,$$

and the hyperproduct in C is defined for all $(x,y) \in C^2$ as

$$x \otimes y = \bigcup_{z \in g(x) \circ g(z)} A_z$$

where $g : C \longrightarrow M$ is defined as $g(x) = m \Longleftrightarrow x \in A_m$.

THEOREM: (De Salvo) Let H be a hypergroup. H is D.C. if and only if H is representable as a K_M-hypergroup with $M = H/E$ where M is a canonical hypergroup, $A_m = g^{-1}(m)$ for all $m \in M$, and E is the equivalence relation defined for $(x,y) \in H^2$ by

$$x\, E\, y \Longleftrightarrow \exists\, z \in H \quad \{x, y\} \subset i_H(z) \,.$$

The theorem reduces the study of D.C.'s to that of canonical hypergroups.

Other classes of hypergroups different from the canonicals have seemed particularly significant and are presently being investigated. Complete hypergroups (and semi-hypergroups) have been studied not only in the multivalued context, but also in that of the theory of groupoids, quasigroups and semigroups. Recently the combinatorial aspects have also been pointed out by De Salvo, Freni and Migliorato and the notion in those of the m-completes has been generalized by the latter. Prenowitz and Jantosciak have particularly stressed the importance of join spaces in geometry and others (Corsini and Nieminen) in graph theory. D-hypergroups, that is double-sided hypergroups, have recently been re-examined and investigated by Sureau and

Haddad. Cyclic hypergroups already defined and analysed by Vougiouklis and De Salvo have been resumed by the former utilizing P-hypergroups successfully.

Finally, 1-hypergroups, that is those whose heart is reduced to one element, introduced and analysed by Corsini, have been reconsidered by Migliorato and characterized in the case $|H/\beta| < \aleph_0$; in the general case the structure problem is still open.

Many problems are still open, even in general theory.

Let's look at some:

1) Given a hypergroup H, various classes of sub-hypergroups have been identified and studied (Krasner, Koskas, Corsini, Sureau): The class of closed sub-hypergroups (which I shall denote by $S_1(H)$), the the class of the invertible sub-hypergroups ($S_2(H)$), the class of ultraclosed sub-hypergroups ($S_3(H)$), and the class of the complete part sub-hypergroups ($S_4(H)$). It is known for all (i,j) with i<j, that $S_i(H) \supset S_j(H)$.

If we denote by $S_0(H)$ the class of all the sub-hypergroups of H, the problem is posed to determine $\forall$ (i,j), $0 \le i < j$ the class C(i,j) of those hypergroups H such that $S_i(H) = S_j(H)$. Only some partial answers are known today. For example, join spaces, quasi-canonical and 1-hypergoups are in C(3,4), reversible regular hypergroups are in C(1,2).

2) In a semi-hypergroup and in a hypergroup, the analysis of the properties of the relation $\beta = \bigcup_{n \ge 1} \beta_n$ (with the corresponding problem of the determination of the heart structure), begun by Orsatti, carried on by Koskas, Corsini and more recently by Migliorato and Freni, is far from being concluded. There is information on the heart structure of regular hypergroups and the n-complete hypergroups, i.e. those in which $\beta = \beta_n$. Recently it has been proved by Freni that in any hypergroup β is transitive.

Open problems remain on the heart structure of regular hypergroups which are neither reversible nor commutative and of hypergroups wtich are neither regular nor n-complete and finally on β^* in semi-hypergroups.

3) Another subject, fundamental in the theory, on which significant results have already been found, is that of homomorphisms between hyperstructures.

DEFINITION: Let H, H' be hypergroupoids. A function $f: H \longrightarrow H'$ is called (0)-homomorphism if it satisfies the condition

(0) $\forall\ (x,y) \in H^2, \quad f(x \circ y) \subset f(x) \circ f(y)$

The notion (0) of homomorphism seems the most appropriate. It is equivalent to that of homomorphism among the associated models (that is provided with the ternary relation R defined:

$$(x,y,z) \in R \iff z \in x \circ y \text{).}$$

However it is not that which allows to extend more easily to multivalued structures results valid in classical ones. For example, the image of a sub-hypergroup in general, is not even a sub-semihypergroup.

Several stronger kinds of homomorphisms between hypergroups have been identified, for instance the j-homomorphisms, for $0 \le j \le 7$ (see the 2nd edition of the "Prolegomena").

A wide theme of research is to study the classes $H(i,j)$ of hypergroups such that if $f: H \longrightarrow H'$, for $\{H, H'\} \subset H(i,j)$, then f is an i-homomorphism if and only if it is a j-homomorphism. Not much is known about this class.

To conclude, we have seen together an inevitably incomplete but, I think, sufficiently wide outline to illustrate the variety of research and problems of hyperstructure theory. I hope it will spur a further deepening of the subject.

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SUR LES STRUCTURES HYPERCOMPOSITIONNELLES

par

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ABSTRACT. In this article, which is the opening speech of the Fourth International Congress on Algebraic Hyperstructures and Applications, the basic hyperstructures are presented in brief, together with a more detailed presentation of their applications in the order theory, linear algebra, valuation theory and languages and automata, based on previous research done by me, my collaborators, as well as others.

1. Généralités

Comme il est bien connu, plusieurs mathématiciens ayant comme point de départ deux structures algèbriques basées sur la notion d' opération multivoque ou **hyperopération** ou, encore, **hypercomposition** [23] - qui pour cela ont été appelées **structures hypercompositionnelles** ou **hyperstructures** - ont présenté un très grand nombre de travaux avec des résultats remarquables, qui ont considérablement enrichi la théorie plus générale des structures algèbriques, bien que d' autres mathématiciens ont contesté plutôt par égoisme ou par frivolité leur importance et leur utilité. Ces deux hyperstructures sont, comme il est clair, l' **hypergroupe** qui a été introduit par Fr. Marty en 1934 comme une généralisation de la notion du groupe et après une remarque faite par lui et concernant la composition des fonctions multivoques [25] et l' **hypercorps** (même l' hypercorps **valué**), introduit par M. Krasner en 1956, qui généralise la notion de corps et auquel il avait abouti désirant obtenir une approximation bien determinée d' un corps valué complet de caractéristique $p\neq0$ par une suite de tels corps, mais de caractéristique $p=0$ [22].

L' hypergroupe est, comme tout le monde sait, un **hypergroupoide**, c'est-a-dire un ensemble $H \neq \emptyset$ muni d' une hyperopération

$H \times H \longrightarrow P(H)$ associative et reproductive [23] [32], pour lequel, donc, on a [46] [47]

H_1 $(xy)z = x(yz)$ $\forall x, y, z \in H$

H_2 $xH = Hx = H$ $\forall x \in H$

(avec, comme il est prouvé, $xy \neq \emptyset$, $x, y \in H$) d' où il résulte que tout groupe est un hypergroupe (non propre).

Quant à l' hypercorps, c'est une structure additivomultiplicative $(K,+,.)$, possédant l' addition comme hyperopération et la multiplication comme opération, dont, tout d' abord, la partie additive, considerée indépendemment comme une hyperstructure $(H,+)$, vérifie les axiomes:

C_1 $x+y = y+x$ $\forall x, y \in H$

C_2 $(x+y)+z = x+(y+z)$ $\forall x, y, z \in H$

C_3 $(\exists 0 \in H)(\forall x \in H)[0+x=x]$[1] (L' élément 0, évidemment unique est le zéro de H)

C_4 $(\forall x \in H)(\exists x' \in H)[0 \in x+x']$ [x' est (par definition) unique, appelé opposé ou symétrique de x et noté $-x$. On pose $x+(-y)=x-y$]

C_5 $(\forall (x,y,z) \in H^3)[z \in x+y \Longrightarrow y \in z-x]$

Le dernier axiome equivaut à

C_5' $(\forall (x,y) \in H^2)[-(x+y) = -x-y]$

et, encore, comme il est clair, à

C_5'' $(\forall (x,y,z,w) \in H^4)[(x+y) \cap (z+w) \neq \emptyset \Longrightarrow (z-x) \cap (y-w) \neq \emptyset]$

Il résulte facilement que l'hypergroupoide $(H,+)$ - demi-hypergroupe en raison de C_2 - vérifie la proprieté

$$x+H=H, \quad \forall x \in H$$

Il s' agit, donc, d' hypergroupe d' un type spécial, qui a été nommé canonique [31] et qui fournit un exemple montrant que l' on obtient différent types d' hypergroupes par l' adjonction a leur définition générale d' autres axiomes convenables. D' ailleurs, c' est Marty lui-même qui, pendant la durée très courte de sa vie (il est péri pendant la guerre 1939-1945) a étudie deux types d' hypergroupes et leurs applications [25]. Il s' agit des hypergroupes $(H,.)$ réguliers, quand il existe un élément unité $e \in H$ (non forcement unique), c' est-a-dire tel que

$$x \in ex \cap xe \quad \forall x \in H$$

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1 On identifie, quand il n' y a pas le risque de confusion les éléments x avec les singletons correspondents [23] [31].