Undergraduate Texts in Mathematics

Gérard Iooss
Daniel D. Joseph

Elementary Stability and Bifurcation Theory

Second Edition

稳定性和分歧理论〔英〕

Springer-Verlag
World Publishing Corp

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With 58 Illustrations

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AMS Subject Classification (1980): 34-01, 34, A34, 34D30, 34D99, 34C99

Library of Congress Cataloging-in-Publication Data
Iooss, Gérard.
Elementary stability and bifurcation theory / Gérard Iooss, Daniel D. Joseph. — 2nd ed.
p. cm. — (Undergraduate texts in mathematics)
Includes bibliographical references.
ISBN 0-387-97068-1
1. Differential equations—Numerical solutions. 2. Evolution equations—Numerical solutions. 3. Stability. 4. Bifurcation theory. I. Joseph, Daniel D. II. Title. III. Series.
QA372.168 1989
515'.35—dc20 89-21765

Reprinted by World Publishing Corporation, Beijing, 1991
for distribution and sale in The People's Republic of China only
ISBN 7-5062-1025-8

ISBN 0-387-97068-1 Springer-Verlag New York Berlin Heidelberg
ISBN 3-540-97068-1 Springer-Verlag Berlin Heidelberg New York

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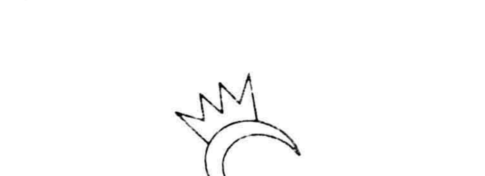
(continued after Index)

Everything should be made as simple as possible,
but not simpler.

ALBERT EINSTEIN



List of Frequently Used Symbols

All symbols are fully defined at the place where they are first introduced. As a convenience to the reader we have collected some of the more frequently used symbols in several places. The largest collection is the one given below. Shorter lists, for later use can be found in the introductions to Chapters X and XI.

$\stackrel{\text{def}}{=}$	equality by definition
$\in$	"$a \in A$" means "a belongs to the set A" or "a is an element of A"
$\mathbb{N}$	the set of nonnegative integers (0 included)
$\mathbb{N}^*$	the set of strictly positive integers (0 excluded)
$\mathbb{Z}$	the set of positive and negative integers including 0
$\mathbb{R}$	the set of real numbers (the real line)
$\mathbb{R}^n$	the set of ordered n-tuples of real numbers $\mathbf{a} \in \mathbb{R}^n$ may be represented as $\mathbf{a} = (a_1, \ldots, a_n)$. Moreover, $\mathbb{R}^n$ is a Euclidian space with the scalar product $$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{n} a_i b_i$$ where $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$. $\mathbb{R}^1 = \mathbb{R}$; $\mathbb{R}^2$ is the plane
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}^n$	the set of ordered n-tuples of complex numbers. The scalar product in $\mathbb{C}^n$ is denoted as in $\mathbb{R}^n$, but $$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{n} a_i \overline{b_i} = \overline{\langle \mathbf{b}, \mathbf{a} \rangle}.$$

$\mathscr{C}^n(\mathscr{V})$ — the set of n-times continuously differentiable functions on a domain $\mathscr{V}$. We may furthermore specify the domain E where these functions take their values by writing $\mathscr{C}^n(\mathscr{V}; E)$.

$\|\mathbf{u}\|$ — the norm of $\mathbf{u}$. For instance, if $\mathbf{u} \in \mathbb{C}^n$ we have $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2}$; if $\mathbf{u} \in \mathscr{C}(\mathscr{V})$, $\|\mathbf{u}\| = \operatorname{l.u.b}_{x \in \mathscr{V}} \|\mathbf{u}(x)\|$, where $\|\mathbf{u}(x)\|$ is the norm of $\mathbf{u}(x)$ in the domain of values for $\mathbf{u}$; $\|\mathbf{u}\| = 0$ implies that $\mathbf{u} = 0$.

$\mathbf{A}(\cdot)$ — a linear operator:

$$\mathbf{A}(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha \mathbf{A}(\mathbf{u}) + \beta \mathbf{A}(\mathbf{v})$$

$\mathbf{B}(\cdot, \cdot)$ — a bilinear operator:

$$\mathbf{B}(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2, \beta_1 \mathbf{v}_1 + \beta_1 \mathbf{v}_2) = \alpha_1 \beta_1 \mathbf{B}(\mathbf{u}_1, \mathbf{v}_1) + \alpha_1 \beta_2 \mathbf{B}(\mathbf{u}_1, \mathbf{v}_2) + \alpha_2 \beta_1 \mathbf{B}(\mathbf{u}_2, \mathbf{v}_1) + \alpha_2 \beta_2 \mathbf{B}(\mathbf{u}_2, \mathbf{v}_2)$$

$\mathbf{C}(\cdot, \cdot, \cdot)$ — a trilinear operator

$\mathbf{N}(\cdot)$ — a general nonlinear operator with no constant term and no linear term in a neighborhood of 0:

$$\mathbf{N}(\mathbf{u}) \stackrel{\text{def}}{=} B(\mathbf{u}, \mathbf{u}) + C(\mathbf{u}, \mathbf{u}, \mathbf{u}) + O(\|\mathbf{u}\|^4)$$

Sometimes we assign a slightly different meaning to **A**, **B**, **C**:

$$(\mathbf{A} \cdot \mathbf{u})_i = A_{ij} u_j = A_{i1} u_1 + A_{i2} u_2 + \cdots + A_{in} u_n$$

$$(\mathbf{B} \cdot \mathbf{u} \cdot \mathbf{v})_i = B_{ijk} u_j v_k$$

$$(\mathbf{C} \cdot \mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w})_i = C_{ijkl} u_j v_k w_l$$

where we use the summation convention for repeated indices and where

(A_{ij}) is the matrix of a linear operator

(B_{ijk}) is the matrix of a bilinear operator

(C_{ijkl}) is the matrix of a trilinear operator

$\mathbf{F}(t, \mu, \mathbf{U})$ — a nonlinear operator—see the opening paragraph of Chapter I

$\mathbf{f}(t, \mu, \mathbf{u})$ — reduction of $\mathbf{F}$ to "local form," see §I.3

$\mathbf{F}_u$, $\mathbf{F}_{uu}$, etc. — derivatives of $\mathbf{F}$; see §I.6–7

$\mathbf{F}_u(t, \mu, \mathbf{U}_0 | \cdot)$ — the linear operator associated with the derivative of $\mathbf{F}$ at $\mathbf{U} = \mathbf{U}_0$

$\mathbf{F}_u(t, \mu, \mathbf{U}_0 | \mathbf{v})$ — first derivative of $\mathbf{F}(t, \mu, \mathbf{U})$, evaluated at $\mathbf{U} = \mathbf{U}_0$, acting on $\mathbf{v}$

$\sigma = \xi + i\eta$ — an eigenvalue of a linear operator arising in the study of stability of $\mathbf{u} = 0$

When $\mathbf{u} = 0$ corresponds to a time-periodic $\mathbf{U}(t) = \mathbf{U}(t + T)$, then σ is a *Floquet exponent*

$\lambda = e^{\sigma T}$	a *Floquet multiplier*; see preceding entry and §VII.6.2
$\gamma = \xi + i\eta$	an eigenvalue of a linear operator arising in the study of bifurcating solution. We use the same notation, ξ and η, for the real and imaginary part of σ and γ and depend on the context to define the difference.
ω; $T = 2\pi/\omega$	frequency ω and period T
ε	amplitude of a bifurcating solution defined in various ways: under (II.2), (V.2), (VI.72), $(\text{VII.6})_2$, (VIII.22), Figure X.1.
$\langle \mathbf{a}, \mathbf{b} \rangle$	notation for a scalar product $\overline{\langle \mathbf{a}, \mathbf{b} \rangle} = \langle \mathbf{b}, \mathbf{a} \rangle$ with the usual conventions. For vectors in $\mathbb{C}^n$, $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a} \cdot \bar{\mathbf{b}}$. For vector fields in $\mathscr{V}$, $\langle \mathbf{a}, \mathbf{b} \rangle = \int_{\mathscr{V}} \mathbf{a}(\mathbf{x})\bar{\mathbf{b}}(\mathbf{x})\, d\mathscr{V}$. See (IV.7), under (VI.4), §VI.6, under $(\text{VI.134})_2$, and under (VI.144)
$[\mathbf{a}, \mathbf{b}]$	another scalar product for 2π-periodic functions, defined above (VIII.15)
$[\mathbf{a}, \mathbf{b}]_{nT}$	see (IX.16)

Some operators whose domains are 2π-periodic functions of s:

$J(\cdot, \varepsilon)$, $J(\cdot, 0) = J_0$ (VII.38)

$\mathbb{J}_0$ (VIII.15); $\mathbb{J}_0^*$ (VIII.19); $\mathbb{J}(\varepsilon)$ above (VIII.37).

Similar operators for nT-periodic functions are defined under notation for Chapter IX, at the beginning of Chapter IX.

Order Symbols. we say that $f(\varepsilon) = O(\varepsilon^n)$ if

$$\frac{f(\varepsilon)}{\varepsilon^n} \text{ is bounded when } \varepsilon \to 0$$

we say that $f(\varepsilon) = o(\varepsilon^n)$

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\varepsilon^n} = 0.$$

Introduction

In its most general form bifurcation theory is a theory of asymptotic solutions of nonlinear equations. By asymptotic solutions we mean, for example, steady solutions, time-periodic solutions, and quasi-periodic solutions. The purpose of this book is to teach the theory of bifurcation of asymptotic solutions of evolution problems governed by nonlinear differential equations. We have written this book for the broadest audience of potentially interested learners: engineers, biologists, chemists, physicists, mathematicians, economists, and others whose work involves understanding asymptotic solutions of nonlinear differential equations.

To accomplish our aims, we have thought it necessary to make the analysis:

(1) general enough to apply to the huge variety of applications which arise in science and technology; and
(2) simple enough so that it can be understood by persons whose mathematical training does not extend beyond the classical methods of analysis which were popular in the nineteenth century.

Of course, it is not possible to achieve generality and simplicity in a perfect union but, in fact, the general theory is simpler than the detailed theory required for particular applications. The general theory abstracts from the detailed problems only the essential features and provides the student with the skeleton on which detailed structures of the applications must rest.

It is generally believed that the mathematical theory of bifurcation requires some functional analysis and some of the methods of topology and dynamics. This belief is certainly correct, but in a special sense which it is useful to specify as motivation for the point of view which we have adopted in this work.

The main application of functional analysis of problems of bifurcation is the justification of the reduction of problems posed in spaces of high or infinite dimension to one and two dimensions. These low-dimensional problems are associated with eigenfunction projections, and in some special cases, like those arising in degenerate problems involving symmetry-breaking steady bifurcations, analysis of problems of low dimension greater than two is required. But the one- and two-dimensional projections are the most important. They fall under the category of problems mathematicians call bifurcation at a simple eigenvalue.

The existence and nature of bifurcation and the stability of the bifurcating solutions are completely determined by analysis of the nonlinear ordinary differential and algebraic equations which arise from the methods of reduction by projections. The simplest way, then, to approach the teaching of the subject is to start with the analysis of low-dimensional problems and only later to demonstrate how the lower-dimensional problems may be projected out of high-dimensional problems. In the first part of the analysis we require only classical methods of analysis of differential equations and functions. In the second part of the analysis, which is treated in Chapters VI and VIII, we can proceed in a formal way without introducing the advanced mathematical tools which are required for the ultimate justification of the formal analysis. It goes almost without saying that we believe that all statements which we make are mathematically justified in published works which are cited and left for further study by courageous students.

It is perhaps useful to emphasize that we confine our attention to problems which can be reduced to one or two dimensions. In this setting we can discuss the following types of bifurcation: bifurcation of steady solutions in one dimension (Chapter II) and for general problems which can be projected into one dimension (Chapter VI); isolated solutions which perturb bifurcation in one dimension (Chapter III) and for general problems which can be projected into one dimension (Chapter VI); bifurcation of steady solutions from steady solutions in two dimensions (Chapters IV and V) and for general problems which can be projected into two dimensions (Chapter VI); bifurcation of time-periodic solutions from steady ones in two dimensions (Chapter VII) and for general problems which can be projected into two dimensions (Chapter VIII); the bifurcation of subharmonic solutions from T-periodic ones in the case of T-periodic forcing (Chapter IX), the bifurcating torus of "asymptotically quasi-periodic" solutions which bifurcate from T-periodic ones in the case of T-periodic forcing (Chapter X), the bifurcation of subharmonic solutions and tori from self-excited periodic solutions (the autonomous case, Chapter XI). It is not possible to do much better in an elementary book because even apparently benign systems of three nonlinear ordinary differential equations give rise to very complicated dynamics with turbulentlike attracting sets which defy description in simple terms. In one dimension all solutions lie on the real line, in two dimensions all solutions of the initial value problem lie in the plane and their trajectories cannot intersect transversally because the solutions are unique. This severe restriction on solutions of two-dimensional problems has already much less force in three dimensions where nonintersecting trajectories can ultimately generate attracting sets of considerable complexity (for example, see E. N. Lorenz, Deterministic nonperiodic flow, *J. Atmos. Sci.* 20, 130 (1963)).

We regard this book as a text for the teaching of the principles of bifurcation. Our aim was to give a complete theory for all problems which in a sense, through projections, could be said to be set in two dimensions. To do this we had to derive a large number of new research results. In fact new results appear throughout this book but most especially in the problem of bifurcation of periodic solutions which is studied in Chapters X and XI. Students who wish to continue their studies after mastering the elementary theory may wish to consult some of the references listed at the end of Chapter I.

There are many very good and important papers among the thousands published since 1963. We have suppressed our impulse to make systematic reference to these papers because we wish to emphasize only the elementary parts of the subject. It may be helpful, however, to note that some papers use the "method of Liapunov–Schmidt" to decompose the space of solutions and equations into a finite-dimensional and an infinite-dimensional part. The infinite part can be solved and the resulting finite dimensional problem has all the information about bifurcation. Other papers use the "center manifold" to reduce the problems to finite dimensions. This method uses the fact that in problems like those in this book, solutions are attracted to the center manifold, which is finite dimensional. Both methods are good for proving existence theorems. Though they can also be used to construct solutions, they in fact involve redundant computations. These methods are systematically avoided in this book. Instead, we apply the *implicit function theorem* to justify the direct, sequential computation of *power series solutions* in an amplitude ε, using the *Fredholm alternative*, as the most economic way to determine qualitative properties of the bifurcating solutions and to compute them.

Acknowledgments

This book was begun in 1978 during a visit of G. Iooss to the University of Minnesota, made possible by a grant from the Army Research Office in Durham. Continued support of the research of D. D. Joseph by the Fluid Mechanics program of the N.S.F. is most gratefully acknowledged.

Preface to the Second Edition

This second edition of *Elementary Stability and Bifurcation Theory* is an expanded and simplified revision of our earlier work. We have removed and corrected the small number of errors which readers brought to our attention and we have tried to clarify and simplify wherever possible, most especially in the less elementary parts of the book relating to the bifurcation of periodic solutions, which is developed in Chapters X and XI. It has to be confessed that many readers did not find the final chapters of our previous book elementary. We have made simplifications and hope at least that this treatment is as near to being elementary as is possible. We set down a unified theory in the previous version and focused only on the well-established parts of the subject. In this, and the old version of the book, we restrict our attention to what has become known as local bifurcation theory, analysis of stability, and branching in the neighborhood of points of bifurcation. Global methods require tools of geometry and topology and some of these are very well explained in a recent book by Guckenheimer and Holmes*. Our book leans more heavily on analysis than on topology, and it is basically restricted to analysis near points of bifurcation.

Of all the interesting developments which have taken place in bifurcation theory in the past decade, at least two have attained the status of established parts of bifurcation theory. The first of these concerns problems which are invariant under symmetries. A fairly complete exposition of this theory, which was started by David Sattinger, can be found in the two-volume work by Golubitsky and Schaeffer†. The second development, the use of normal forms,

* Guckenheimer, J. and Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*. Applied Mathematical Sciences, Vol. 42. New York: Springer-Verlag, 1983.

† Golubitsky, M. and Schaeffer, D. *Singularities and Groups in Bifurcation Theory*. Applied Mathematical Sciences, Vol. 51. New York: Springer-Verlag, 1985.

can be related to the pioneering works of Landau and Stuart on amplitude equations but also follows, perhaps, more completely from the more recent techniques associated with the center manifold theorem. We referred to some of this in the last version; amplitude equations are better understood now and we have tried to pass on our understanding.

Symmetries often occur in physical systems and it is important to have efficient mathematical methods to extract their consequences. Symmetry often leads to multiple eigenvalues and to bifurcating solutions which break symmetry. Group theory allows an efficient classification of genuinely different types of bifurcating solutions. For example, in the Taylor–Couette flow of fluid between rotating concentric cylinders, symmetry arguments imply that bifurcating flows are either axisymmetric with horizontal cells (Taylor vortex flow), or spiral waves or ribbons breaking rotation symmetry and translation invariance along the axis of the cylinder‡. Group theory, combined with amplitude equations, has been used to predict, without explicit computations, certain types of motion§, which were only observed later in experiments‡.

We have continued to organize our studies of bifurcation theory with power series in the amplitude, using the Fredholm alternative to invert the perturbation equations which arise at different orders in the perturbation. In the present edition, however, we have greatly emphasized amplitude equations. The amplitude equations, unlike the amplitude expansions, do not commit us, at the outset, to name the invariant set describing the bifurcation. For example, it is not necessary to say at the start that you seek a steady solution or a time-periodic solution. Instead, the qualitative properties of the solutions which bifurcate can be found in the solution set of the amplitude equations.

Amplitude equations were used in applications to problems of hydrodynamic stability in the 1960s. More recent approaches emphasize the center manifold, whose tangent space is spanned by excited modes, and the normal forms which are the simplest forms for the amplitude equations. The equations are controlled by excited modes. The linearly damped modes, sometimes called "slaves," do not enter strongly into the dynamics. There are different ways to generate amplitude equations; the one we like uses expansions of all quantities relative to criticality and the Fredholm alternative, preserving the unity of method and presentation achieved in the first edition.

In the present edition we have added problems and discussions of simple symmetries, and emphasized methods which can be used to simplify amplitude equations in the presence of symmetry. In §VI.14 we show how a symmetry may lead to pitchfork bifurcation. In several examples in Chapter VI we show how to compute bifurcations and stability of bifurcating solutions in a symmetric

‡ Chossat, P. and Iooss, G. Primary and secondary bifurcations in the Couette–Taylor problem. *Japan J. Appl. Math.*, **2**, 37–68 (1985).

§ Iooss, G. Secondary bifurcations of Taylor vortices into wavy inflow or outflow boundaries, *J. Fluid Mech.*, 173, 273–288 (1986).

problem with a double eigenvalue in the presence of discrete symmetries. We compare the method of amplitude expansions with the method of amplitude equations, plus symmetry. Rotation symmetry combined with Hopf bifurcation may lead to "rotating waves" breaking both rotational and time-shift invariance, as we show in §§VIII.5 and XI.22.

We have added an additional Chapter XII to this edition, in an attempt to introduce some of the methods, which are used in the study of stability and bifurcation of conservative systems, to our study of dissipative systems. The omission of such considerations in the last edition may have been a source of confusion to a huge number of readers who confront bifurcation problems in their work on conservative systems. Students of elasticity, for example, fall into this category. In this final chapter we have confined our attention to equilibrium solutions of conservative problems, leaving aside many of the important but not elementary questions arising in the dynamics of Hamiltonian systems. Hamiltonian systems perturbed by small dissipation are important in the theory of chaos but are not treated here.

Finally, we take note of the huge impact made by modern computers on the various parts of the theory of stability and bifurcation. One can scarcely overestimate the importance of numerical methods which have made possible the actual realization of theoretical approaches in a form suitable for comparison with observation and experiment. Numerical analysis of bifurcations is a large specialized subject which deserves to be treated in its own right, in a dedicated manner not possible in this book.

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