Lecture Notes in Mathematics 1602

Jörg Winkelmann

The Classification of Three-dimensional Homogeneous Complex Manifolds

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Preface

In this monograph we give a classification of all three–dimensional homogeneous complex manifolds. A complex manifold X is called homogeneous if there exists a connected complex or real Lie group G acting transitively on X as a group of biholomorphic transformations. The goal is to classify these manifolds up to biholomorphic equivalence.

Since the class of homogeneous complex manifolds is much too big for any serious attempt of complete classification, it is necessary to impose further conditions. For example E. Cartan classified in [Ca] symmetric homogeneous domains in $\mathbb{C}^n$. Here we will require that X is of small dimension. For $dim_{\mathbb{C}}(X) = 1$ the classification follows from the Uniformization Theorem. In 1962 J. Tits classified the compact homogeneous complex manifolds in dimension two and three [Ti1]. In 1979 J. Snow classified all homogeneous manifolds $X = G/H$ with $dim_{\mathbb{C}}(X) \leq 3$, G being a solvable complex Lie group and H discrete [SJ1]. The classification of all complex-homogeneous (i.e. G is a complex Lie group) two-dimensional manifolds was completed in 1981 by A. Huckleberry and E. Livorni [HL]. Next, in 1984 K. Oeljeklaus and W. Richthofer classified all those homogeneous two-dimensional complex manifolds $X = G/H$ where G is only a real Lie group [OR]. The classification of three-dimensional complex-homogeneous manifolds was completed in 1985 [W1]. Finally in 1987 the general classification of the three-dimensional homogeneous complex manifolds was given by in [W2]. The purpose of this monograph is to give the complete proof of the classification of three-dimensional complex manifolds G/H.

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PART I

Survey

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Survey

1. Introduction

A complex manifold X is called homogeneous if there exists a connected complex or real Lie group G acting transitively on X as a group of biholomorphic transformations. Our goal is a general classification of homogeneous complex manifolds up to dimension three as complex manifolds, i.e. we identify two manifolds if they are biholomorphic. Thus we do not intend to classify all holomorphic transitive Lie group actions on complex manifolds. We just want to classify all complex manifolds for which there exists at least one holomorphic transitive Lie group action.

One should note that there exist complex manifolds which are not homogeneous in our sense, i.e. there does not exist any Lie group acting holomorphically on transitively, but nevertheless the whole group of all automorphisms does act transitively (see [Ka, p.70] and [W4]).

The classification is done in two steps. First we consider only homogeneous complex manifolds which are *complex-homogeneous*, i.e. for which there exists a *complex* Lie group acting transitively. Second we discuss those homogeneous complex manifolds on which only *real*, but no *complex* Lie groups act transitively.

On page 21 and 86 diagrams show how the classification is organized.

2. The complete List

The following list covers all homogeneous complex manifolds $X = G/H$ with $dim_{\mathbb{C}}(X) \leq 3$:

We distinguish the cases G solvable, G mixed and G semisimple. Here G is mixed means that G has a Levi–Malcev decomposition $G = S \ltimes R$ with $dim_{\mathbb{R}}(R) > 0$ and $dim_{\mathbb{R}}(S) > 0$, i.e. G is neither semisimple nor solvable.

G complex solvable

(1) Quotients G/Γ of solvable complex Lie groups G with $dim_{\mathbb{C}}(G) \leq 3$ by discrete subgroups.

This class contains in particular $\mathbb{C}^n$, $\mathbb{C}^*$ and Tori. These manifolds have been studied in detail in [SJ1,SJ2]. She gives a fine classification of the discrete subgroups of these solvable Lie groups.

G complex semisimple

(2) Quotients $SL_2(\mathbb{C})/\Gamma$ with Γ being a discrete subgroup of $SL_2(\mathbb{C})$.

This is a very large class. For example let M be an arbitrary Riemann surface. Then there is a holomorphic action of $\pi_1(M)$ on the universal covering $\tilde{M}$ of M. Since $\tilde{M} \simeq \mathbb{P}_1$, $\mathbb{C}$, or Δ_1, the universal covering $\tilde{M}$ is equivariantly embeddable in $\mathbb{P}_1$. Thus for any Riemann surface the fundamental group $\pi_1(M)$ can be embedded in $SL_2(\mathbb{C})/\mathbb{Z}_2 \simeq Aut_{\mathcal{O}}(\mathbb{P}_1)$ as a discrete subgroup. For more informations on discrete subgroups in semisimple Lie groups see [Kra,Mar,Ra,Z].

(3) The following homogeneous–rational manifolds:
- a) $\mathbb{P}_n$ for $n \leq 3$,
- b) the projective quadric $\mathbf{Q}_3$ and
- c) the flag manifold $F_{1,2}(3)$ of full flags in $\mathbb{C}^3$.

(4) The affine quadric Q_2 and $\mathbb{P}_2 \setminus \mathbf{Q}_1$.

Both are quotients of $SL_2(\mathbb{C})$ by reductive subgroups and $\mathbb{P}_2 \setminus \mathbf{Q}_1 \simeq Q_2/\mathbb{Z}_2$. Furthermore Q_2 is biholomorphic to $\{(z,w) \in \mathbb{P}_1 \times \mathbb{P}_1 \mid z \neq w\}$ and may be realized as affine bundle over $\mathbb{P}_1$. In contrast $\mathbb{P}_2 \setminus \mathbf{Q}_1$ has no equivariant fibration at all.

(5) All $\mathbb{C}^*$– and Torus–principal bundles over homogeneous rational manifolds.

This class contains in particular $\mathbb{C}^2 \setminus \{(0,0)\}$, $\mathbb{C}^3 \setminus \{(0,0,0)\}$, homogeneous Hopf surfaces and $\mathbb{P}_3 \setminus (L_1 \dot{\cup} L_2)$ where L_1 and L_2 are two disjoint complex lines in $\mathbb{P}_3$.

G complex mixed

(6) The non–trivial $\mathbb{C}^*$– and torus–principal bundles over $\mathbb{P}_2 \setminus \mathbf{Q}_1$.

The non–trivial $\mathbb{C}^*$ – and torus–principal bundles over Q_2 are also homogeneous manifolds, but are already contained in the class $SL_2(\mathbb{C})/\Gamma$.

(7) Every line bundle over a homogeneous–rational manifold which is generated by a positive divisor.

This class contains in particular $\mathbb{P}_n \setminus \{x_0\}$.

(8) Holomorphic vector bundles of rank two over $\mathbb{P}_1$ which are direct sums of line bundles generated by positive divisors.

Any vector bundle of rank two over $\mathbb{P}_1$ is a direct sum of line bundles (see [GrR, p.237]), but of course not necessarily generated by positive divisors.

The total space of the vector bundle $E \simeq H^1 \oplus H^1$ is $\mathbb{P}_3 \setminus L$, where L denotes a complex line in $\mathbb{P}_3$. Furthermore E may be realized as a $\mathbb{C}$–principal bundle over H^2. Here H^2 denotes the 2^{nd} power of the hyperplane bundle over $\mathbb{P}_1$.

(9) Quotients of $\mathbb{P}_3 \setminus L$ realized as principal bundle over H^2 by discrete subgroups of the structure group.

It is easy to list all these quotients, since it suffices to determine the discrete subgroups of the one-dimensional structure group.

(10) Every line bundle over Q_2 and the unique non-trivial line bundle over $\mathbb{P}_2 \setminus \mathbb{Q}_1$.

(11) Quotients of $\mathbb{C} \times Q_2$ by discrete subgroups of $\mathbb{Z}_2 \ltimes (\mathbb{C}, +)$ with the $\mathbb{Z}_2 \ltimes \mathbb{C}$-action on $\mathbb{C} \times Q_2$ given by

$$([z], [w], y) \mapsto ([z], [w], y + x)$$

for $(e, x) \in \mathbb{Z}_2 \ltimes \mathbb{C}$ and

$$([z], [w], y) \mapsto ([w], [z], -y)$$

for $(\phi, 0)$, where ϕ denotes the non-trivial element of $\mathbb{Z}_2$.
(Here $([z], [w]) \in \mathbb{P}_1 \times \mathbb{P}_1 \setminus \Delta \simeq Q_2$).

(12) Quotients of $\mathbb{C} \times (\mathbb{C}^2 \setminus \{(0,0)\})$ by discrete subgroups of $\mathbb{C}^* \ltimes \mathbb{C}$ acting by

$$(\lambda, z) : (x, v) \mapsto (\lambda^k x + z, \lambda v)$$

for $k \in \mathbb{Z}$.

(13) Certain $\mathbb{C}^2$-bundles over $\mathbb{P}_1$ which are given by the following transition functions

$$w_1 = -\left(\frac{z_0}{z_1}\right)^n w_0$$

$$v_1 = \left(\frac{z_0}{z_1}\right)^{np+n-2} v_0 - \left(\frac{z_0}{z_1}\right)^{np+n-1} w_0^{p+1}$$

for $p \geq 1$, $n \geq 1$.

Here v_i and w_i denote fibre coordinates over $U_i = \{[z_0 : z_1] \mid z_i \neq 0\}$.

These bundles arise as quotients of $SL_2(\mathbb{C}) \ltimes N$ by a three-codimensional subgroup where N is a complex nilpotent Lie group with $\dim_{\mathbb{C}}(N/N') = n+1$ and $N^{(p)} \neq \{e\} = N^{(p+1)}$. The commutator N' is abelian and induces a fibration which realizes these manifolds as affine bundles over H^n where H^n denotes the n-th power of the hyperplane bundle over $\mathbb{P}_1$. These affine bundles have no holomorphic section and the manifolds have only constant holomorphic functions.

For $n = 1$ and $p = 1$ the group N is the three-dimensional complex Heisenberg group, i.e.

$$N = \left\{ \left. \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \right| x, y, z \in \mathbb{C} \right\},$$

and the affine bundle over $H^n = H^1$ is actually a principal bundle. Moreover for $n = p = 1$ the manifold which arises is biholomorphic to $\mathbb{Q}_3 \setminus L$, where $\mathbb{Q}_3$ denotes the projective quadric and L an arbitrary complex line in $\mathbb{Q}_3$.

(14) Quotients E/Γ where E is the $\mathbb{C}$–principal bundle over H^1 which is contained in the above class for $n = p = 1$ and Γ is a discrete subgroup of the structure group $(\mathbb{C}, +)$ acting from the right on E.

(15) Simply–connected $\mathbb{C}^*$–principal bundles over H^1 which are given as a quotient $(SL_2(\mathbb{C}) \ltimes N)/H$ where N is the three–dimensional complex Heisenberg group, the representation of $SL_2(\mathbb{C})$ in $Aut(N/N')$ is irreducible and

$$H = \left\{ \left(\begin{pmatrix} e^z & w \\ & e^{-z} \end{pmatrix} ; \begin{pmatrix} 1 & x & \alpha z \\ & 1 & \\ & & 1 \end{pmatrix} \right) \middle| x, z, w \in \mathbb{C} \right\}$$

for $\alpha \in \mathbb{C}^*$.

(16) Quotients of the above principal bundles by discrete subgroups of the principal structure group acting from the right.

G real solvable

(17) An irreducible bounded homogeneous domain, i.e. a ball

$$\mathbb{B}_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} |z_i|^2 < 1\}$$
$$\simeq \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=2}^{n} |z_i|^2 < Re(z_1)\}$$

and

$$\Omega = \{(x, w, z) \in \mathbb{C}^3 \mid Im\, x > 0 \quad \text{and} \quad 4\, Im\, x\, Im\, z > (Im\, w)^2\}.$$

For $dim_{\mathbb{C}}(X) \leq 3$ every bounded homogeneous domain is also a hermitian symmetric space. In the notation of [Hel], $\mathbb{B}_n$ is a hermitian domain of type $AIII(p = 1, q = n)$ and Ω is of type $BD\,I(p = 3, q = 2) = C\,I(n = 2)$.

(18) A complement to a bounded domain in its equivariant embedding in $\mathbb{C}^n$, i.e.

$$X \simeq \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=2}^{n} |z_i|^2 > Re(z_1)\}$$

or

$$X \simeq \{(x, w, z) \in \mathbb{C}^3 \mid Im\, x > 0 \quad \text{and} \quad 4\, Im\, x\, Im\, z < (Im\, w)^2\}.$$

(19) $\mathbb{C}^2\backslash\mathbb{R}^2$ or a covering of this manifold.

The manifold $\mathbb{C}^2\backslash\mathbb{R}^2$ *is not simply-connected,* $\mathbb{C}^2\backslash\mathbb{R}^2 \simeq S^1 \times \mathbb{R}^3$. *The universal covering* $\widetilde{\mathbb{C}^2\backslash\mathbb{R}^2}$ *which is diffeomorphic to* $\mathbb{R}^4$ *has some interesting complex-analytic properties. In particular* $\widetilde{\mathbb{C}^2\backslash\mathbb{R}^2}$ *is hypersurface-separable (i.e. for all* $x, y \in \widetilde{\mathbb{C}^2\backslash\mathbb{R}^2}$ *there exists a hypersurface* $H \subset \widetilde{\mathbb{C}^2\backslash\mathbb{R}^2}$ *such that* $x \in H \not\ni y$, *[O]) but it is not meromorphically separable. Actually any meromorphic function on* $\widetilde{\mathbb{C}^2\backslash\mathbb{R}^2}$ *is* $\pi_1(\mathbb{C}^2\backslash\mathbb{R}^2)$*-invariant. Hence two points in the same fibre over* $\mathbb{C}^2\backslash\mathbb{R}^2$ *can not be separated.*

(20) A quotient of $\mathbb{C} \times \widetilde{\mathbb{C}^2\backslash\mathbb{R}^2}$ by a discrete subgroup of $\mathbb{C} \times \mathbb{Z}$ acting naturally ($\mathbb{C}$ on $\mathbb{C}$ by translations and $\mathbb{Z} \simeq \pi_1(\mathbb{C}^2\backslash\mathbb{R}^2)$ on $\widetilde{\mathbb{C}^2\backslash\mathbb{R}^2}$ as a group of covering transformations).

(21) The following domains in $\mathbb{C}^3$

$$\begin{aligned}
\Omega_0 &= \{(x,w,z) \mid Im\, x > 0 \quad \text{and} \quad f_1(x,w,z) > 0\} \\
\Omega_1 &= \{(x,w,z) \mid f_1(x,w,z) > 0\} \\
\Omega_2 &= \{(x,w,z) \mid f_2(x,w,z) > 0\} \\
\Omega_3 &= \{(x,w,z) \mid f_2(x,w,z) < 0\}
\end{aligned}$$

with $f_1 = Im\, z - Re\, w\, Im\, x$ and $f_2 = Im\, z - Re\, w\, Im\, x + (Re\, x)^4$.

The manifold Ω_0 *is particular interesting for its Kobayashi-reduction.*

The Kobayashi-reduction identifies two points in a manifold if their Kobayashi-pseudometric is zero (see [Ko1, Ko3, L] for details about the Kobayashi-pseudometric in general and [W5] for a survey of the Kobayashi-pseudodistance on homogeneous manifolds). Now the Kobayashi-reduction of Ω_0 *is a fibration*

$$\pi : G/H \xrightarrow{\Delta_1 \times \mathbb{C}} G/I \simeq \Delta_1$$

compatible with the complex structure. In particular the fibre has a non-trivial Kobayashi-pseudometric. Nevertheless if one takes any open subset U *of* G/I *then the Kobayashi-pseudometric of* $\pi^{-1}(U)$ *degenerates along the fibres.*

One can define a "complex-line-reduction" for Ω_0 *which identifies two points* $x, y \in \Omega_0$ *if and only if there is a finite chain of holomorphic maps* $\phi_1, \ldots \phi_n : \mathbb{C} \to \Omega_0$ *with* $\phi_0(0) = x$, $\phi_i(1) = \phi_{i+1}(0)$ *and* $\phi_n(1) = y$. *Then* $\Omega_0 \to \Omega_0/\sim$ *is a* G*-equivariant real analytic fibre bundle and all the fibres are closed complex-analytic subsets of* Ω_0 *but there is no compatible complex structure on* $\Omega_0/\sim$.

That Ω_2 *and* Ω_3 *are not biholomorphic is proved in Lemma 6.6.1. in the following way: Assume to the contrary that* $\phi : \Omega_2 \to \Omega_3$ *is a biholomorphic map. Obviously* ϕ *is extendable to the envelopes of holomorphy i.e. to the whole* $\mathbb{C}^3$. *Then* $-f_2 \circ \phi$ *and* f_2 *must define the same boundary. Hence* $-f_2 \circ \phi = \lambda f_2$ *for some positive real-analytic function* λ. *One obtains a contradiction by writing down this equation in coordinates and comparing the coefficients of the power series up to degree 4.*