Lecture Notes in Mathematics 1627

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Probabilistic Models for Nonlinear Partial Differential Equations

Montecatini Terme, 1995

Editors: D. Talay, L. Tubaro

Springer

Fondazione
C.I.M.E.

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Lectures given at the 1st Session of the
Centro Internazionale Matematico Estivo
(C.I.M.E.)
held in Montecatini Terme, Italy,
May 22-30, 1995

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Springer

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Cataloging-Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Centro Internazionale Matematico Estivo <Firenze>: Lectures given at the ... session of the Centro Internazionale Matematico Estivo (CIME) ... - Berlin ; Heidelberg ; New York ; London ; Paris ; Tokyo ; Hong Kong : Springer
Früher Schriftenreihe. - Früher angezeigt u.d.T.: Centro
Internazionale Matematico Estivo: Proceedings of the ... session of the Centro Internazionale Matematico Estivo (CIME)
NE: HST
1995,1. Probabilistic models for nonlinear partial differential equations. - 1996

Probabilistic models for nonlinear partial differential equations : held in Montecatini Terme, Italy, May 22 -30, 1995 / Carl Graham ... Ed.: D. Talay ; L. Tubaro. - Berlin ; Heidelberg ; New York ; Barcelona ; Budapest ; Hong Kong ; London ; Milan ; Paris ; Santa Clara ; Singapore ; Tokyo : Springer, 1996
(Lectures given at the ... session of the Centro Internazionale Matematico Estivo (CIME) ... ; 1995,1)
(Lecture notes in mathematics ; Vol. 1627 : Subseries: Fondazione CIME)
ISBN 3-540-61397-8
NE: Graham, Carl; Talay, Denis [Hrsg.]; 2. GT

Mathematics Subject Classification (1991): 60H10, 60H30, 60H15, 60K35, 60K30, 60G07, 65C20, 65M15, 65U05

ISSN 0075-8434
ISBN 3-540-61397-8 Springer-Verlag Berlin Heidelberg New York

Printed in Germany

Typesetting: Camera-ready TEX out put by the authors
SPIN: 10479756 46/3142-543210 - Printed on acid-free paper

Preface

These last years, there have been important developments in the probabilistic interpretation of nonlinear Partial Differential Equations, the theory of the convergence of law of stochastic processes and the numerical approximation of stochastic processes.

All these developments offer the appropriate theoretical background to analyse probabilistic algorithms used to solve equations as important in practice as the Navier-Stokes equation, the Boltzmann equation and certain Stochastic Partial Differential Equations. They also permit us to construct new methods. For example, all the works around the propagation of chaos, particularly those of A-S. Sznitman, permit a quite new and fruitful point of view on the random vortex methods in Fluid Mechanics and on Monte–Carlo methods for Boltzmann-like equations. Likewise, the numerical analysis of stochastic differential equations has recently progressed in several interesting directions (variance reduction techniques, simulation of reflected diffusion processes, convergence in law of the normalized trajectorial error, asymptotic expansions of the discretization error).

Weak limit theorems for stochastic integrals naturally are among the main ingredients in the study of interacting particle systems, approximation procedures for solutions of stochastic differential equations, etc. A selection of such theorems in view of the analysis of applied problems should be useful. Besides, quite new weak limit theorems have just appeared for stochastic integrals with respect to infinite dimensional semimartingales.

We therefore enthusiastically answered Prof. V. Capasso's suggestion to submit to the CIME a proposal for a Summer School on the probabilistic models for nonlinear PDE's and their numerical applications with a three-fold emphasis: first, on the weak convergence of stochastic integrals; second, on the probabilistic interpretation and the particle approximation of equations coming from Physics (conservation laws, Boltzmann-like and Navier-Stokes equations); third, on the modelling of networks by interacting particle systems.

We thank all the participants to this Summer School which was held in Montecatini from May 22^{th} to May 30^{th}. The exchanges between the lecturers and the audience were very useful for everybody.

We thank all the lecturers (Carl Graham, Tom Kurtz, Sylvie Méléard, Philip Protter, Mario Pulvirenti) for having given fascinating lectures and for having written pedagogic and deep contributions to the present volume.

We hope that this book will be useful for our colleagues working on stochastic particle methods and on the approximation of SPDE's and in particular, for Ph.D. students and for young researchers.

We thank CIME for its generous financial support and for arranging the location in Montecatini offering us the combined delights of beautiful Tuscan surroundings and gastronomical excellence.

Sophia–Antipolis and Trento, December 1995

Denis Talay and Luciano Tubaro

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Weak Convergence of Stochastic Integrals and Differential Equations

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1. Semimartingales

Let W denote a standard Wiener process with $W_0 = 0$. For a variety of reasons, it is desirable to have a notion of an integral $\int_0^1 H_s dW_s$, where H is a stochastic process; or more generally an indefinite integral $\int_0^t H_s dW_s$, $0 \leq t < \infty$. If H is a process with continuous paths, an obvious way to define a stochastic integral is by a limit of sums: let $\pi^n[0,t]$ be a sequence of partitions of $[0,t]$, with mesh $(\pi^n) = \sup_i (t_{i+1} - t_i)$, where $0 = t_0 < t_1 < \ldots < t_n = t$ are the successive points of the partition. Then one could define

$$\int_0^t H_s dW_s = \lim_{n \to \infty} \sum_{t_i \in \pi^n[0,t]} H_{t_i}(W_{t_{i+1}} - W_{t_i}) \tag{1.1}$$

when $\lim_{n\to\infty}$ mesh $(\pi^n) = 0$. If one wants the natural condition that (1.1) holds for all continuous processes H, then it is an elementary consequence of the Banach-Steinhaus theorem that W must have a.s. paths of finite variation on compacts. Of course this is precisely not the case for the Wiener process. The key insight of K. Itô in the 1940's was to ask for condition (1.1) to hold only for *adapted* continuous stochastic processes. We will both explain this idea and extend it to a large class of stochastic processes: exactly those for which both the integral exists as a limit of sums, and for which we also have a dominated convergence theorem.

We suppose given a filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$, where $\mathcal{F}$ is a P-complete σ-algebra and where $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t < \infty}$ is a filtration of σ-algebras: i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. We also assume that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}_0$ and that $\mathbf{F}$ is right continuous: that is, $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{u>t}\mathcal{F}_u$. (Note that if W is a standard Wiener process with its natural filtration $\mathbf{F}^0 = (\mathcal{F}_t^0)_{0 \leq t < \infty}$, where $\mathcal{F}_t = \sigma(W_s; s \leq t)$, then if one adds the P-null sets of $\mathcal{F}_t^0$ to $\mathcal{F}_t^0$, all t, the resulting filtration $\mathbf{F}$ satisfies the preceding hypotheses, which are known

* Research supported in part by NSF grants DMS 92-04866 and DMS 95-04323
** Research supported in part by NSF grant INT 94-01109 and NSA grant MDR 904-94-H-2049

as the *usual hypotheses*. The same holds for Lévy processes and for most strong Markov processes.)

Let X be an *adapted* process with càdlàg paths: that is, X_t is $\mathcal{F}_t$-measurable, each $t > 0$, and a.s. has paths which are right continuous with left limits.[1]

Definition 1.1. *A process H is* simple predictable *if H has a representation*

$$H_t = H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H_i 1_{(T_i, T_{i+1}]}(t) \tag{1.2}$$

where $0 = T_1 \leq \ldots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in \mathcal{F}_{T_i}$, $|H_i| < \infty$ a.s., $0 \leq i \leq n$. The collection of simple predictable processes is denoted $\mathbf{S}$.

Let $\mathbf{L}^0$ *denote all a.s. finite random variables. We topologize* $\mathbf{L}^0$ *with convergence in probability, and we topologize* $\mathbf{S}$ *with uniform convergence (in (t, ω)) and denote it* $\mathbf{S}_u$. *For a given X we define an operator I_X mapping* $\mathbf{S}$ *to* $\mathbf{L}^0$ *by (with H as in (1.2)):*

$$I_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i (X_{T_{i+1}} - X_{T_i}). \tag{1.3}$$

Definition 1.2. *A process X is a* semimartingale *if $I_X : \mathbf{S}_u \to \mathbf{L}^0$ is continuous on compact time sets.*

Definition (1.2) is not customary. We give the customary definition here, and to distinguish it from ours we call it a "classical" semimartingale.

Definition 1.3. *A process X is a* classical semimartingale *if it is adapted, càdlàg, and has a decomposition $X = M + A$, where M is a local martingale, and A (is adapted, càdlàg, and) has paths of finite variation on compacts.*

One of the deepest results in the theory of semimartingales is the following, proved around 1978, primarily by C. Dellacherie and K. Bichteler.

Theorem 1.4 (Bichteler-Dellacherie). *An adapted, càdlàg process X is a semimartingale if and only if it is a classical semimartingale.*

We remark that the deeper implication is the "only if".

Also note that the Bichteler-Dellacherie theorem gives us many *examples of semimartingales:*

(i) Any local martingale, such as the Wiener process, is a semimartingale.
(ii) Any finite variation process, such as the Poisson process, is a semimartingale.

[1] "càdlàg" is the French acronym for right continuous with left limits

(iii) The Doob-Meyer decomposition theorem states that any submartingale Y can be written $Y = M + A$, where M is a local martingale and A is an adapted, càdlàg process with nondecreasing paths. Thus, any submartingale (and hence any supermartingale) is a semimartingale.

(iv) If Z is a Lévy process (i.e., a càdlàg process with stationary and independent increments), then if $E\{|Z_t|\} < \infty$, each t, one has $E\{|Z_t|\} = \alpha t$ (assuming $Z_0 = 0$) and thus $Z_t = (Z_t - \alpha t) + \alpha t$ is a decomposition for Z,a nd Z is a semimartingale. More generally it can be shown that any Lévy process is a semimartingale.

(v) Most "reasonable" real valued strong Markov processes are semimartingales.

(vi) An illustrative example of a Lévy process that is a martingale is as follows: let N^i be a sequence of i.i.d. Poisson processes with arrival intensities $\alpha_i (\alpha_i > 0)$. Let $|\beta_i| \le c$ and assume $\sum_{i=1}^{\infty} \beta_i^2 \alpha_i < \infty$. Then

$$M_t = \sum \beta_i (N_t^i - \alpha_i t)$$

is a Lévy process. Note that if, for example, $\alpha_i = 1$ (all i) and $\beta_i = \frac{1}{i}$, then if $\Delta M_s = M_s - M_{s-}$ (the jump at time s), we have $\sum_{0<s\le t} |\Delta M_s| = \sum_{0<s\le t} \Delta M_s = \sum_{i=1}^{\infty} \frac{1}{i} N_t^i = \infty$ a.s. This is an example of a martingale that cannot be used, path by path, as a classical differential because of behavior arising purely from the jumps; that is, M has paths of infinite variation on compacts and one cannot define a Lebesgue-Stieltjes pathwise integral for M.

Finally let us note some simple but important properties of semimartingales.

Theorem 1.5. *The set of semimartingales is a vector space.*

Theorem 1.6. *If Q is another probability absolutely continuous with respect to P, then every P-semimartingale is a Q-semimartingale.*

Theorem 1.7 (Stricker). *If X is a semimartingale for a filtration* **F**, *and if* **G** *is a subfiltration such that X is adapted to* **G**, *then X is a* **G**-*semimartingale as well.*

Proof. Theorem 1.5 is immediate from the definition. For Theorem 1.6 it is enough to remark that if $Q \ll P$, then convergence in P-probability implies convergence in Q probability. For Theorem 1.7, let $\mathbf{S}(\mathbf{F})$ denote $\mathbf{S}$ for the filtration $\mathbf{F}$. Since $\mathbf{S}(\mathbf{G}) \subset \mathbf{S}(\mathbf{F})$, if I_X is continuous for $I_X : \mathbf{S}_u(\mathbf{F}) \to \mathbf{L}^0$, then it is *a fortiori* continuous for $\mathbf{S}_u(\mathbf{G})$. □

Stricker's theorem shows one can easily shrink the filtration since one is only shrinking the domain of a continuous operator. Expanding the filtration, on the other hand, is more delicate, since one is then asking a continuous operator to remain continuous for a larger domain. An elementary result in this direction is the following:

Theorem 1.8 (P. A. Meyer). *Let $\mathcal{A}$ be a countable collection of disjoint sets in $\mathcal{F}$. Let* **H** *be the filtration given by $\mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{A})$. Then every* **F** *semimartingale is an* **H** *semimartingale.*

Proof. Without loss of generality assume $\mathcal{A}$ is a partition of Ω, and $P(A_n) > 0$, each $A_n \in \mathcal{A}$. Define $Q_n \ll P$ by $Q_n(\Lambda) = P(\Lambda|A_n)$. Then X is a Q_n-semimartingale by Theorem 1.6. Let $\mathbf{I}^n$ be the filtration generated by **F** and all Q_n null sets. Let X be a $(\mathbf{I}^n, Q_n)$-semimartingale, each n. Moreover $\mathbf{F} \subset \mathbf{H} \subset \mathbf{I}^n$. By Stricker's theorem, X is an **H** semimartingale under Q_n. Note that $dP = \sum_{n=1}^{\infty} P(A_n) dQ_n$. Suppose $H^n \in \mathbf{S}(\mathbf{H})$ converges to $H \in \mathbf{S}(\mathbf{H})$ uniformly. Then $I_X(H^n)$ converges to $I_X(H)$ in Q_n-probability for each n, and it follows that it converges in P-probability as well. Thus X is an $(\mathbf{H}, P)$-semimartingale. □

2. Stochastic Integration

We wish to define a stochastic integral of the form $\int_0^t H_{s-} dX_s$, where H is càdlàg, adapted, and H_{s-} represents its left continuous version; and X is a semimartingale. We recall **S** is the space of simple predictable processes and $\mathbf{L}^0$ is the space of finite valued random variables.

We also define:

D = the space of adapted processes with càdlàg paths
L = the space of adapted processes with càdlàg paths (left continuous with right limits)

Note that if $H \in \mathbf{D}$, then H_- (its left continuous version) is in **L**; and if $H \in \mathbf{L}$, then H_+ is in **D**. We next define a new topology, ucp, which will replace uniform convergence.

Definition 2.1. *A sequence of processes Y^n converges to a process Y uniformly on compacts in probability (denoted ucp) if for each $t > 0$, $\sup_{s \le t} |Y_s^n - Y_s| = (Y^n - Y)_t^*$ tends to 0 in probability as n tends to ∞.*

We note that this topology is metrizable.

Theorem 2.2. **S** *is dense in* **L** *under ucp .*

Proof. By stopping, $b\mathbf{L}$ is dense in **L**, where $b\mathbf{L}$ denotes the bounded processes in **L**. For $Y \in b\mathbf{L}$, let $Z = Y_+$, and for $\varepsilon > 0$, define $T_0^\varepsilon = 0$ and

$$T_{n+1}^\varepsilon = \inf\{t : t > T_n^\varepsilon \text{ and } |Z_t - Z_{T_n^\varepsilon}| > \varepsilon\}.$$

Then T_n^ε are stopping times and they are increasing since Z is càdlàg. Pose $Z_1^\varepsilon = Y_0 1_{\{0\}} + \sum_{i=1}^n Z_{T_i^\varepsilon} 1_{(T_i^\varepsilon \wedge n, T_{i+1}^\varepsilon \wedge n]}$. This can be made arbitrarily close to $Y \in b\mathbf{L}$ by taking ε small enough and n large enough. □

The operator I_X defined in (1.3) was, effectively, an operator giving a definite integral for processes $H \in \mathbf{S}$ and semimartingales X. We now wish to define an operator which will be an indefinite integral operator. Thus its range should be processes rather than random variables. Therefore for a given process X and a process $H \in \mathbf{S}$ as given in (1.2) we define the operator $J_X : \mathbf{S} \to \mathbf{D}$ by:

$$J_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i (X^{T_{i+1}} - X^{T_i}), \tag{2.1}$$

where the notation X^T, for a stopping time T, denotes the process $X_t^T = X_{t \wedge T} (t \geq 0)$.

Definition 2.3. *For an adapted, càdlàg process X and $H \in \mathbf{S}$, the process $J_X(H)$ is called the* stochastic integral *of H with respect to X.*

We will also use the notations $\int_0^t H_s dX_s$ and $H \cdot X$ or $H \cdot X_t$ to denote the stochastic integral. That is

$$\begin{aligned} J_X(H) &= \int H dX &= H \cdot X \\ J_X(H)_t &= \int_0^t H_s dX_s &= H \cdot X_t. \end{aligned}$$

Theorem 2.4. *Let X be a semimartingale. Then $J_X : \mathbf{S}_{\mathrm{ucp}} \to \mathbf{D}_{\mathrm{ucp}}$ is continuous.*

Proof. Suppose $H^k \in \mathbf{S}$ tends to H uniformly. By linearity, we can suppose without loss H^k tends to 0. Let $T^k = \inf\{t : |(H^k \cdot X)_t| \geq \delta\}$. Then $H^k 1_{[0,T^k]} \in \mathbf{S}$ tends to 0 uniformly as k tends to ∞. Thus for every t

$$\begin{aligned} P\{(H^k \cdot X)_t^* > \delta\} &\leq P\{|H^k \cdot X_{T^k \wedge t}| \geq \delta\} \\ &= P\{|(H^k 1_{[0,T^k]} \cdot X)_t| \geq \delta\} \\ &= P\{|I_X(H^k 1_{[0,T^k \wedge t]})| \geq \delta\} \end{aligned}$$

which tends to 0 by definition because X is a semimartingale. Therefore $J_X : \mathbf{S}_u \to \mathbf{D}_{\mathrm{ucp}}$ is continuous. We next show $J_X : \mathbf{S}_{\mathrm{ucp}} \to \mathbf{D}_{\mathrm{ucp}}$ is continuous. Let $\delta > 0$, $\varepsilon > 0$, $t > 0$. We now know there exists η such that $\|H\|_u \leq \eta$ implies $P(J_X(H)_t^* > \delta) < \varepsilon/2$. Let $R^k = \inf\{s : |H_s^k| > \eta\}$, and set $\widetilde{H}^k = H^k 1_{[0,R_k]} 1_{\{R_k > 0\}}$. Then $\widetilde{H}^k \in \mathbf{S}$ and $\|\widetilde{H}^k\|_u \leq \eta$ by left continuity. When $R^k \geq t$ we have $(\widetilde{H}^k \cdot X)_t^* = (H^k \cdot X)_t^*$, whence

$$\begin{aligned} P\left((H^k \cdot X)_t^* > \delta\right) &\leq P\left((\widetilde{H}^k \cdot X)_t^* \delta\right) + P(R^k < t) \\ &\leq \varepsilon/2 + P\left((H^k)_t^* > \eta\right) \\ &< \varepsilon, \end{aligned}$$

if k is large enough, since $\lim_{k \to \infty} P((H^k)_t^* > \eta) = 0$. □

Definition 2.5. *Let X be a semimartingale. The continuous linear mapping $J_X : \mathbf{L}_{\text{ucp}} \to \mathbf{D}_{\text{ucp}}$ obtained as the extension of $J_X : \mathbf{S} \to \mathbf{D}$ is called the* stochastic integral.

Suppose H is a process in $\mathbf{D}$. We can write the stochastic integral $H_{s-} \cdot X = (\int_0^t H_{s-} dX_s)_{t \geq 0}$ as defined above, as a limit of sums. Let σ denote a finite sequence of stopping times:

$$0 = T_0 \leq T_1 \leq \ldots \leq T_k < \infty \text{ a.s.} \tag{2.2}$$

Such a sequence is called a *random partition*. A sequence of random partitions σ_n

$$\sigma_n : T_0^n \leq T_1^n \leq \ldots \leq T_{k_n}^n$$

is said to *tend to the identity* if

(i) $\lim_n \sup_i T_i^n = \infty$ a.s.
(ii) $\|\sigma_n\| = \sup_i |T_{i+1}^n - T_i^n|$ converges to 0 a.s.

For a process H and a random partition σ as in (2.2) we define

$$H^\sigma = H_0 1_{\{0\}} + \sum_{i=1}^{k} H_{T_i} 1_{(T_i, T_{i+1}]}. \tag{2.3}$$

Thus if H is in $\mathbf{L}$ or $\mathbf{D}$, we have

$$\int_0^t H_s^\sigma dX_s = H_0 X_0 + \sum_{i=1}^{k} H_{T_i}(X^{T_{i+1}} - X^{T_i}). \tag{2.4}$$

Theorem 2.6. *Let X be a semimartingale and let $H \in \mathbf{D}$. Let $(\sigma_n)_{n \geq 1}$ be a sequence of random partitions tending to the identity. Then*

$$H_- \cdot X = \lim_{n \to \infty} \sum_i H_{T_i^n}(X^{T_{i+1}^n} - X^{T_i^n})$$

with convergence in ucp .

Proof. Let $H^k \in \mathbf{S}$ converge to H in ucp. Then

$$(H_- - H^{\sigma_n}) \cdot X = (H_- - H^k) \cdot X + (H^k - (H_+^k)^{\sigma_n}) \cdot X + ((H_+^k)^{\sigma_n} - H^{\sigma_n}) \cdot X. \tag{2.5}$$

The first term on the right side of (2.5) equals $J_X(H_- - H^k)$, which goes to 0 because J_X is continuous on $\mathbf{L}_{\text{ucp}}$. The same applies to the third term for fixed k as n tends to ∞. Indeed, $(H_+^k)^{\sigma_n} - H^{\sigma_n}$ tends to 0 as $k \to \infty$ uniformly in n. As for the middle term on the right side of (2.5), for fixed k it tends to 0 as n tends to ∞. Thus one need only choose k so large that the first and third terms are small, and then choose n so large that the middle term is small. $\square$

Theorem 2.6 gives an appealing intuitive description of the stochastic integral as a limit of Riemann-type sums. Of course one can only do this because of the path regularity of the integrands.

Let us next note some simple and quite nice properties of the stochastic integral. H will be assumed to be in $\mathbf{D}$, and X a semimartingale in Theorems 2.7 through 2.11.

Theorem 2.7. *If X has paths of finite variation a.s., then $H_- \cdot X$ agrees with the Lebesgue-Stieltjes integral, denoted $\int_{LS} H_{s-}dX_s$.*

Proof. The result is evident for $H \in \mathbf{S}$. For $H \in \mathbf{D}$, let $H^n \in \mathbf{S}$ converge to H_- in ucp. Then there exists a subsequence n_k such that H^{n_k} converges uniformly on compacts a.s. to $H_- \cdot X$. Since the convergence is uniform, $\int_{LS} H_s^{n_k} dX_s$ converges as well to $\int_{LS} H_{s-}dX_s$, whence the result. □

Recall that for a process $Y \in \mathbf{D}$, $\Delta Y_t = Y_t - Y_{t-}$, and ΔY denotes the process $(\Delta Y_t)_{0 \leq t < \infty}$. An important feature of the stochastic integral is that the jumps behave "correctly" — that is, in the same manner as they do for the Lebesgue-Stieltjes integral. This is part of the reason we use $\mathbf{L}$, rather than, for example, $\mathbf{D}$, as our space of integrands. (See Pratelli [14] or Ahn-Protter [1] for more on this subject.)

Theorem 2.8. *The jump process $\Delta(H_- \cdot X)_s$ is indistinguishable*[2] *from the process $H_{s-}\Delta X_s$.*

Theorem 2.9. *Let $Q \ll P$. Then $H_{-Q} \cdot X$ is Q-indistinguishable from $H_{-P} \cdot X$.*

Theorem 2.10. *Let P and Q be any two probabilities and X a semimartingale for each. Then there exists $H_- \cdot X$ which is a version of both $H_{-P} \cdot X$ and $H_{-Q} \cdot X$.*

Theorem 2.11. *Let $\mathbf{G}$ be another filtration and suppose $H \in \mathbf{D}(\mathbf{G}) \cap \mathbf{D}(\mathbf{F})$, and that X is semimartingale for both $\mathbf{F}$ and $\mathbf{G}$. Then $H_{-\mathbf{G}} \cdot X = H_{-\mathbf{F}} \cdot X$.*

Proof. For Theorem 2.8 and 2.9, the result is clear for $H \in \mathbf{S}$ and follows for H_- with $H \in \mathbf{D}$ by taking limits in ucp (convergence in P-probability implies convergence in Q-probability). For Theorem 2.10, let $R = \frac{1}{2}(P+Q)$, and apply Theorem 2.9. For Theorem 2.11, we can use the construction in the proof of Theorem 2.2 to approximate $H \in \mathbf{D}$ constructively from H; thus the approximations $H^n \in \mathbf{S}$ are in $\mathbf{S}(\mathbf{F}) \cap \mathbf{S}(\mathbf{G})$; the result is clearly true for H in $\mathbf{S}$ and thus it follows by again taking limits. □

Theorem 2.9 can be used to show that many global results also hold locally.

We give an example.

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[2] Two processes Y and Z are *indistinguishable* if $P\{\omega : t \to Y_t(\omega) \neq t \to Z_t(\omega)\} = 0$.

Theorem 2.12. *Let X, Y be two semimartingales and H, J be two processes in* $\mathbf{D}$. *Let*

$$A = \{\omega : H.(\omega) = J.(\omega) \text{ and } X.(\omega) = Y.(\omega)\}$$

where $H.(\omega)$ denotes the path of $H : t \to H_t(\omega)$. Let

$$B = \{\omega : X.(\omega) \text{ is finite variation on compacts}\}.$$

Then $H_- \cdot X = J_- \cdot X$ on A a.s., and $H_- \cdot X = \int_{LS} H_{s-} dX$ on B a.s.

Proof. Without loss of generality assume $P(A) > 0$. Define a new Q by $Q(\Lambda) = P(\Lambda|A)$. Then $H_- = J_-$ and $X = Y$ under Q. Note that X and Y are also semimartingales under Q. Thus $H_{-Q} \cdot X = H_{-P} \cdot X$, and one need only apply Theorem 2.9. The second assertion is a combination of the above idea with Theorems 2.7 and 2.9. □

The next result is quite important.

Theorem 2.13. *Let $H \in \mathbf{D}$ and X be a semimartingale. Then $Y = H_- \cdot X$ is again a semimartingale. Moreover if $G \in \mathbf{D}$ as well, then*

$$G_- \cdot Y = G_- \cdot (H_- \cdot X) = (GH)_- \cdot X.$$

Proof. If $G, H \in \mathbf{S}$, then clearly $Y = H_- \cdot X$ is a semimartingale, and $J_Y(G) = J_X(GH)$. The associativity property extends to H_-, G_- with $G, H \in \mathbf{D}$ by continuity. Therefore it remains only to show $Y = H_- \cdot X$ is a semimartingale. By taking subsequences if necessary, assume $H^n \in \mathbf{S}$ converges to H_- in ucp and also $H^n \cdot X$ converges a.s. to $H_- \cdot X$. For $G \in \mathbf{S}$, $J_Y(G)$ is defined for any process Y and hence makes sense *a priori*. Thus

$$\begin{aligned} J_Y(G) &= \lim_{n\to\infty} G \cdot Y^n = \lim_{n\to\infty} G \cdot (H^n \cdot X) \\ &= \lim_{n\to\infty} (GH^n) \cdot X = J_X(GH_-), \end{aligned}$$

since X is a semimartingale. Next let G^n converge to G in $\mathbf{S}_u$. We wish to show $I_Y(G^n)$ converges to $I_Y(G)$. But

$$\lim_{n\to\infty} J_Y(G^n) = \lim_{n\to\infty} J_X(G^n H_-) = J_X(GH_-)$$

since $G^n H_-$ converges to GH_- in ucp. Then since $J_X(GH_-) = J_Y(G)$ we have the result. □

3. Quadratic Variation

A process which plays a key role in the theory of stochastic integration is the quadratic variation process. We define it using stochastic integration: