Lecture Notes in Mathematics 1518

H. Stichtenoth M. A. Tsfasman (Eds.)

Coding Theory and Algebraic Geometry

Proceedings, Luminy 1991

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Lecture Notes aim to report new developments - quickly, informally and at a high level. The following describes criteria and procedures for multi-author volumes. For convenience we refer throughout to "proceedings" irrespective of whether the papers were presented at a meeting.

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Further remarks and relevant addresses at the back of this book.

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Foreword

The workshop "Algebraic Geometry and Coding Theory - 3" organized by the Institute of Information Transmission (Moscow), University of Essen, Équipe Arithmétique et Théorie de l'Information de C.N.R.S. (Marseille-Luminy), and Group d'Étude du Codage de Toulon took place in the Centre International de Rencontres Mathématiques, June 17-21, 1991.

The workshop was a continuation of AGCT-1 and AGCT-2 that took place in 1987 and 1989, respectively. It is to be followed by AGCT-4 in 1993, etc., each time held in C.I.R.M.

The list of participants follows.

It is our pleasure to thank the staff of C.I.R.M. for their hospitality, the participants for their interest, all supporting organizations for their financial support, and Springer-Verlag for the Proceedings.

Organizers,

H.Stichtenoth
M.Tsfasman
G.Lachaud
J.Wolfmann

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Algebraic Geometry and Coding Theory
An Introduction

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About ten years ago V.D.Goppa discovered an amazing connection between the theory of algebraic curves over a finite field $\mathbf{F}_q$ and the theory of error-correcting block q-ary codes. The idea is quite simple and generalizes the well known construction of Reed-Solomon codes. The latter use polynomials in one variable over $\mathbf{F}_q$ and Goppa generalized this idea using rational functions on an algebraic curve.

Here is the definition of an *algebraic geometric code* (or a *geometric Goppa code*). Let X be an absolutely irreducible smooth projective algebraic curve of genus g over $\mathbf{F}_q$. Consider an (ordered) set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of distinct $\mathbf{F}_q$-rational points on X and an $\mathbf{F}_q$-divisor D on X. For simplicity let us assume that the support of D is disjoint from $\mathcal{P}$. The linear space $L(D)$ of rational functions on X associated to D yields the linear evaluation map

$$Ev_{\mathcal{P}} : L(D) \to \mathbf{F}_q^n$$
$$f \mapsto (f(P_1), \ldots, f(P_n))$$

The image of this map is the linear code $C = (X, \mathcal{P}, D)_L$ we study.

The parameters of such a code can be easily estimated. Indeed, let $\mathbf{P} = P_1 + \ldots + P_n$, then the dimension k is given by

$$k = \ell(D) - \ell(D - \mathbf{P})$$

and in particular if $0 < \deg D < n$ then

$$k = \ell(D) \geq \deg D - g + 1.$$

The minimum distance

$$d \geq n - \deg D$$

since the number of zeroes of a function cannot be greater than the number of its poles. We get the lower bound

$$k + d \geq n + 1 - g$$

which is by g worse than the simplest upper bound valid for any code

$$k + d \leq n + 1.$$

An equivalent description of these codes can be given in terms of algebraic function fields in one variable over $\mathbf{F}_q$. The curve X corresponds to the function field $F = \mathbf{F}_q(X)$, and $\mathbf{F}_q$-points on X correspond to places of F of degree one.

Originally, Goppa used the dual construction using differentials on X rather than functions, and the residue map.

Unfortunately, there are at least two different traditions of notation. The second one uses D for our $\mathbf{P}$ and G for our D , and the code is denoted $C_L(D, G)$.

The construction can be generalized in several directions. In particular one can use sheaves (or some other tricks) to avoid the condition $\mathcal{P} \cap \mathrm{Supp} D = \emptyset$. The generalization to the case of higher dimensional algebraic varieties looks very promising but so far the results are few.

There are several main streams of the development of the theory. Let us briefly discuss some of them.

Asymptotic problems. One of the fundamental problems of coding theory is to construct long codes with good parameters (rate and relative minimum distance). One of the starting points of the theory was the construction of long codes which are asymptotically better than the Gilbert-Varshamov bound. The other asymptotic question is which codes can be constructed in polynomial time.

Specific curves. There are many interesting examples of curves with many $\mathbf{F}_q$-points which lead to codes with good parameters. Sometimes such curves and codes have nice additional properties, such as large automorphism groups.

Spectra and duality. The study of weight distribution and of duality leads to interesting questions of algebraic geometry, such as the study of Weierstrass points and special divisors on a curve.

Decoding. Surprisingly enough the decoding problem can be set in purely algebraic geometric terms and again one needs information about special divisors.

Exponential sums. Another component of the picture is the theory of exponential sums closely related both to algebraic geometry and to coding theory.

Related areas. The theory of algebraic geometric codes has either analogues or applications in several other topics. Such are sphere packings and spherical codes, multiplication complexity in finite fields, graph theory, and so on. These applications also require subtle information about the geometry and arithmetic both of function fields and of number fields.

To conclude, the first ten years of development show that the connection between algebraic geometry and coding theory proves fruitful for both, giving new results and posing many exciting questions.

Several books and many papers on the subject are either published or in preparation. The papers are too numerous to list them here and we refer to the extensive bibliography in [Ts/Vl] and to references given in the papers of this volume. Here is the list of books.

[Go] **V.D.Goppa,** *Geometry and Codes.* Kluwer Acad. Publ., 1988

[Mo] **C.J. Moreno,** *Curves over Finite Fields.* Cambridge Univ. Press, 1991

[Sti] **H.Stichtenoth,** *Algebraic Function Fields and Codes.* Springer-Verlag (in preparation)

[Ts/Vl] **M.A.Tsfasman, S.G.Vladut,** *Algebraic Geometric Codes.* Kluwer Acad. Publ., 1991

[vG/vL] **J.H.van Lint, G.van der Geer,** *Linear Codes and Algebraic Curves.* Birkhäuser, 1988

Reed-Muller Codes Associated to Projective Algebraic Varieties

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Abstract

The classical generalized Reed-Muller codes introduced by Kasami, Lin and Peterson [5], and studied also by Delsarte, Goethals and Mac Williams [2], are defined over the affine space $\mathbf{A}^n(\mathbf{F}_q)$ over the finite field $\mathbf{F}_q$ with q elements. Moreover Lachaud [6], following Manin and Vladut [7], has considered projective Reed-Muller codes, i.e. defined over the projective space $\mathbf{P}^n(\mathbf{F}_q)$.
In this paper, the evaluation of the forms with coefficients in the finite field $\mathbf{F}_q$ is made on the points of a projective algebraic variety **V** over the projective space $\mathbf{P}^n(\mathbf{F}_q)$. Firstly, we consider the case where **V** is a quadric hypersurface, singular or not, Parabolic, Hyperbolic or Elliptic. Some results about the number of points in a (possibly degenerate) quadric and in the hyperplane sections are given, and also is given an upper bound of the number of points in the intersection of two quadrics.

In application of these results, we obtain Reed-Muller codes of order 1 associated to quadrics with three weights and we give their parameters, as well as Reed-Muller codes of order 2 with their parameters.

Secondly, we take **V** as a hypersurface, which is the union of hyperplanes containing a linear variety of codimension 2 (these hypersurfaces reach the Serre bound). If **V** is of degree h, we give parameters of Reed-Muller codes of order $d < h$, associated to **V**.

1. Construction of the Projective Reed-Muller codes

We denote by $\mathbf{P}^n(\mathbf{F}_q)$ the projective space of dimension n over the finite field $\mathbf{F}_q$ with q elements, q a power of a prime p. The number of (rational) points (over $\mathbf{F}_q$) of $\mathbf{P}^n(\mathbf{F}_q)$ is :

$$\pi_n = | \mathbf{P}^n(\mathbf{F}_q) | = q^n + q^{n-1} + \ldots + q + 1 = \frac{q^{n+1} - 1}{q - 1} .$$

Let W_i be the set of points with homogeneous coordinates $(x_0 : x_1 : ... : x_n) \in \mathbf{P}^n(\mathbf{F}_q)$ such that

$$x_0 = x_1 = ... = x_{i-1} = 0 \text{ and } x_i \neq 0.$$

The family $\{ W_i \}_{0 \leq i \leq n}$ is clearly a partition of $\mathbf{P}^n(\mathbf{F}_q)$.

Let $\mathbf{F}_q[X_0, X_1, ..., X_n]_d^0$ be the vector space of homogeneous polynomials of degree d with (n+1) variables and with coefficients in $\mathbf{F}_q$. Let **V** be a projective algebraic variety of $\mathbf{P}^n(\mathbf{F}_q)$ and let $|\mathbf{V}|$ denotes the number of theirs rational points over $\mathbf{F}_q$. Following G. Lachaud ([6]), we define the *projective Reed-Muller code* $\mathcal{R}(d,\mathbf{V})$ of order d associated to the variety **V** as the image of the linear map

$$\mathbf{c} : \mathbf{F}_q[X_0, X_1, ..., X_n]_d^0 \rightarrow \mathbf{F}_q^{|\mathbf{V}|}$$

defined by $\mathbf{c}(P) = (c_x(P))_{x \in \mathbf{V}}$, where

$$c_x(P) = \frac{P(x_0, ..., x_n)}{x_i^d} \text{ if } x = (x_0 : ... : x_n) \in W_i \ .$$

G. Lachaud has considered in [6] the case where $\mathbf{V} = \mathbf{P}^n(\mathbf{F}_q)$, with $d \leq q$.Moreover, A.B. Sorensen has considered in [12] the case where **V** is equal to $\mathbf{P}^n(\mathbf{F}_q)$ too, but with a weaker hypothesis on d.

Now we are going, firstly, to study the case where **V** is a quadric, degenerate or not, but before we have to establish results on quadrics and this is the subject of the following paragraph.

2. Results on quadrics

In what follows the characteristic of the field $\mathbf{F}_q$ is supposed to be arbitrary (the results hold in characteristic 2 as well as in characteristic different of 2).

2.1. The quadrics in $\mathbf{P}^n(\mathbf{F}_q)$.

In this paragraph, we recall some properties of quadrics in the projective space $\mathbf{P}^n(\mathbf{F}_q)$. J.F. Primrose has given in [8] the number of points in a nondegenerate quadric (see below the definition of the rank of a quadric), and D.K. Ray-Chaudhuri [9] gave more general results (which with, in a particular case, we recover those of Primrose's). We are going here to follow the notations of J.W.P. Hirschfeld in [4].

A quadric Q of $\mathbf{P}^n(\mathbf{F}_q)$ is the set of zeros in $\mathbf{P}^n(\mathbf{F}_q)$ of a quadratic form

$$F \in \mathbf{F}_q[X_0, X_1, ..., X_n]_2^0,$$

that is of an homogeneous polynomial of degree 2. We set $Q = Z_{\mathbf{P}^n}(F)$ or simply $Z(F)$ if no confusion is possible. The quadric Q is said to be *degenerate* if there exists a linear change of coordinates with which we can write the form F with a fewer number of variables. More precisely, if T is an invertible linear transformation defined over $\mathbf{P}^n(\mathbf{F}_q)$, denote by $F_T(X)$ the form $F(TX)$. Let $i(F)$ be the number of indeterminates appearing explicitly in F. The rank $r(F)$ of F (and by abuse of language, of the quadric Q), is defined by :

$$r(F) = \min_T i(F_T)$$

where T ranges over all the invertible transformations defined over $\mathbf{F}_q$. A form F (and by abuse the quadric Q) is said to be *degenerate* if

$$r(F) < n + 1.$$

Otherwise, the form and the quadric are *nondegenerate*.
Let us remark that a quadric is degenerate if and only if it is singular (see [4]).

We recall after J.W.P. Hirschfeld (see [4]) that in $\mathbf{P}^n(\mathbf{F}_q)$, the number of different types of nondegenerate quadrics Q is 1 or 2 as n is even or odd, and they are respectively called *Parabolic* ($\mathcal{P}$), and *Hyperbolic* ($\mathcal{H}$) or *Elliptic* ($\mathcal{E}$).
The maximum dimension $g(Q)$ of linear subspaces lying on the nondegenerate quadric Q is called the *projective index* of Q. The projective index has the following values (see [4]) :

$$g(\mathcal{P}) = \frac{n-2}{2} \quad , \quad g(\mathcal{H}) = \frac{n-1}{2} \quad , \quad g(\mathcal{E}) = \frac{n-3}{2} \quad .$$

The character $\omega(Q)$ of a nondegenerate quadric Q of $\mathbf{P}^n(\mathbf{F}_q)$ is defined by :

$$\omega(Q) = 2g(Q) - n + 3.$$

Consequently, we have :

$$\omega(\mathcal{P}) = 1 \quad , \quad \omega(\mathcal{H}) = 2 \quad , \quad \omega(\mathcal{E}) = 0 \quad .$$

Then, we have the following proposition (for a proof see [4]) :

Proposition 1 : The number of points of a nondegenerate quadric Q of $\mathbf{P}^n(\mathbf{F}_q)$ is :

$$|Q| = \pi_{n-1} + (\omega(Q) - 1)\, q^{(n-1)/2} .$$

We want now to evaluate the number of points of a degenerate quadric $Q = Z(F)$ of $\mathbf{P}^n(\mathbf{F}_q)$ of rank r (called a "cone" of rank r).
We have the following decomposition in disjoint union (an analogous decomposition is given by R.A. Games in [3]) :

$$Q = V_{n-r} \cup Q^*_{r-1}.$$

We have set

$$V_{n-r} = \{(0 : 0 : \ldots : 0 : y_r : \ldots : y_n) \in \mathbf{P}^n(\mathbf{F}_q)\} \cong \mathbf{P}^{n-r}(\mathbf{F}_q),$$

if we suppose that the r variables appearing in the quadratic form F are $X_0, X_1, \ldots, X_{r-1}$. The set V_{n-r} is called the vertex of Q, and is the set of singular points of Q. We note also

$$Q^*_{r-1} = \{(x_0 : \ldots : x_{r-1} : y_r : \ldots : y_n) \in \mathbf{P}^n(\mathbf{F}_q) \mid F(x_0, \ldots, y_n) = 0 \text{ and the } x_i \text{ are not all zero}\}.$$

Let Q_{r-1} be the nondegenerate quadric of $\mathbf{P}^{r-1}(\mathbf{F}_q)$ associated to Q, i.e. defined by

$$Q_{r-1} = Z_{\mathbf{P}^{r-1}}(F_{r-1})$$

or more precisely,

$$Q_{r-1} = \{ (x_0 : \dots : x_{r-1}) \in \mathbf{P}^{r-1}(\mathbf{F}_q) \mid F_{r-1}(x_0, \dots, x_{r-1}) = 0 \},$$

where $F_{r-1}(X_0, \dots, X_{r-1}) = F(X_0, \dots, X_n)$. The (degenerate) quadric Q will abusively be said to be parabolic, hyperbolic or elliptic according to the type of its associated nondegenerate quadric Q_{r-1}. Its character $\omega(Q)$ is by definition the character $\omega(Q_{r-1})$ of Q_{r-1}.
Then, we have the following result which can be found in R.A. Games [3] :

Theorem 1 : The number of points of a quadric Q of $\mathbf{P}^n(\mathbf{F}_q)$ of rank r is :

$$|Q| = \pi_{n-1} + (\omega(Q) - 1)\, q^{(2n-r)/2}$$

and we have $\omega(Q) = 1$ if r is odd, and $\omega(Q) = 0$ or $\omega(Q) = 2$ if r is even.

In particular, a quadric of odd rank is necessarily parabolic, and a quadric of even rank is hyperbolic or elliptic.

Corollary : Let Q be a quadric of $\mathbf{P}^n(\mathbf{F}_q)$, with $n \geq 2$. We have :

$$\pi_{n-2} \leq |Q| \leq \pi_{n-1} + q^{n-1},$$

and the bounds are reached.

Observe that the lower bound is the Warning bound and that the upper bound reaches the following Serre bound, conjectured by Tsfasman, which says that (see [11]) if $F \in \mathbf{F}_q[X_0,\dots,X_n]_d^0$ is a nonzero form of degree $d \leq q$, with $n \geq 2$, then the number N of zeros of F in $\mathbf{F}_q^n$ is such that :

$$N \leq d\, q^{n-1} - (d-1)\, q^{n-2}.$$

2.2. Hyperplane sections of quadrics.

This paragraph deals with the number of points in the intersection of a quadric and a hyperplane. When the quadric is nondegenerate, the result is known (see for example [13]). R.A. Games has given the result when the quadric has the size of a hyperplane, provided the quadric itself is not a hyperplane (see [3]). Furthermore, I.M. Chakravarti in [1] has solved the case when the quadric is 1-degenerate, that is a quadric of rank n in $\mathbf{P}^n(\mathbf{F}_q)$.
We are going, here, to consider the general case, i.e. quadrics in $\mathbf{P}^n(\mathbf{F}_q)$ of any rank.

We begin by the known nondegenerate case. If Q is a nondegenerate quadric of $\mathbf{P}^n(\mathbf{F}_q)$ (i.e. of rank $r = n + 1$) and if H is a hyperplane of $\mathbf{P}^n(\mathbf{F}_q)$, with $n > 1$, then $Q \cap H$ can be seen as a quadric in a space of dimension $n - 1$. We know (see for example [8]) that the rank of $Q \cap H$ is $r - 1$ or $r - 2$. Then, either $Q \cap H$ is nondegenerate (in $\mathbf{P}^{n-1}(\mathbf{F}_q)$), or $Q \cap H$ is of rank $r - 2 = n - 1$ (whence degenerate in $\mathbf{P}^{n-1}(\mathbf{F}_q)$) ; one says in this last case that H is *tangent* to Q.

Now we have to know what is the value of $\omega(Q \cap H)$, i.e. what happens to the type of the quadric. If the hyperplane H is not tangent to Q, it is obvious that $Q \cap H$ becomes parabolic if Q is hyperbolic or elliptic (indeed r(Q) is necessarily even, and if H is not tangent we have $r(Q \cap H) = r(Q) - 1$ hence odd, then $Q \cap H$ is parabolic) ; and $Q \cap H$ becomes hyperbolic or elliptic if Q is parabolic (same reason rest on the parity of the ranks).
Now if the hyperplane H is tangent to Q, we have the following proposition (see [13]) :

Proposition 2 : The quadric $Q \cap H$ is of the same type as the nondegenerate quadric Q if the hyperplane H is tangent to Q.

Then, we can give the result about the hyperplane sections of a quadric of any rank :

Theorem 2 : Let Q be a quadric of $\mathbf{P}^n(\mathbf{F}_q)$ of rank r whose decomposition is

$$Q = V_{n-r} \cup Q^*_{r-1}$$

and let H be a hyperplane of $\mathbf{P}^n(\mathbf{F}_q)$. Then :

a) If $H \supset V_{n-r}$ then

$$| Q \cap H | = \pi_{n-2} + (\omega(Q_{r-1} \cap H_*) - 1)\, q^{(2n-r-1)/2}$$

if H_* is not tangent to Q_{r-1}, and

$$| Q \cap H | = \pi_{n-2} + (\omega(Q) - 1)\, q^{(2n-r)/2}$$

if H_* is tangent to Q_{r-1}, where H_* is the hyperplane of $\mathbf{P}^{r-1}(\mathbf{F}_q)$ defined by

$$H_* = Z_{\mathbf{P}^{r-1}}(h)$$

where h is the linear form in $\mathbf{F}_q[X_0,\ldots,X_{r-1}]_1^0$ defining H ; moreover $\omega(Q_{r-1} \cap H_*)$ is equal to 1 if Q is hyperbolic or elliptic, and equal to 0 or 2 if Q is parabolic.

b) If $H \not\supset V_{n-r}$ then

$$| Q \cap H | = \pi_{n-2} + (\omega(Q) - 1)\, q^{(2n-r-2)/2} .$$

Proof : We suppose that the r variables appearing in the quadratic form F defining Q are $X_0, X_1, \ldots, X_{r-1}$.
If we set H_i the hyperplane whose equation is $X_i = 0$, we have

$$V_{n-r} = H_0 \cap H_1 \cap \ldots \cap H_{r-1} .$$

But

$$Q \cap H = (V_{n-r} \cup Q^*_{r-1}) \cap H = (V_{n-r} \cap H) \cup (Q^*_{r-1} \cap H),$$

Thus

$$| Q \cap H | = | V_{n-r} \cap H | + | Q^*_{r-1} \cap H | - | V_{n-r} \cap Q^*_{r-1} \cap H | ;$$

but $V_{n-r} \cap Q^*_{r-1} = \varnothing$, thus :

$$| Q \cap H | = | V_{n-r} \cap H | + | Q^*_{r-1} \cap H |.$$