Lecture Notes in Mathematics

1474

S. Jackowski B. Oliver K. Pawałowski (Eds.)

Algebraic Topology Poznań 1989

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Lecture Notes aim to report new developments - quickly, informally and at a high level. The following describes criteria and procedures for multi-author volumes. For convenience we refer throughout to "proceedings" irrespective of whether the papers were presented at a meeting.

The editors of a volume are strongly advised to inform contributors about these points at an early stage.

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Further remarks and relevant addresses at the back of this book.

Lecture Notes in Mathematics 1474

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Preface

In June, 1989, the International Conference on Algebraic Topology was held in Poznań, Poland. The conference was part of the scientific activity in connection with the 70-th anniversary of the Adam Mickiewicz University in Poznań. It was supported by the Adam Mickiewicz University, Warsaw University, and Polish government grant RP.I.10.

There were many of our colleagues and students from both Poznań and Warszawa who helped to contribute to the success of the conference. We would especially like to mention Agnieszka Bojanowska, Adam Neugebauer and Bogdan Szydło, who helped with the organizational work, and the two conference secretaries Danuta Marciniak and Katarzyna Kacperska-Panek.

The conference consisted of 10 plenary talks, as well as 49 talks in special sessions in various fields. These proceedings contain papers presented at the conference, as well as some other papers (mostly) submitted by conference participants. We tried—and with some success—to encourage the submission of survey papers.

All papers in the volume have been refereed. We would like to thank the referees for their work, and Andrzej Weber for proofreading of several manuscripts which had to be retyped during the editorial process.

Stefan Jackowski
Bob Oliver
Krzysztof Pawałowski

Warszawa/Århus/Poznań,
November 1990

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SOME APPLICATIONS OF SHIFTED SUBGROUPS IN TRANSFORMATION GROUPS

by C. Allday and V. Puppe

If a torus G of rank r acts on a compact space X, and if all isotropy subgroups have rank at most s, then there is a subtorus $K \subseteq G$ of rank $r - s$ such that the action of K on X is almost–free. When G is an elementary abelian p–group (i.e., $G \cong (\mathbb{Z}/(p))^r$, where p is a prime number), then there is no immediate analogue of the very useful fact above, since a finite number of proper subgroups can cover G. In order to overcome this difficulty, and others, shifted subgroups (to be defined in detail in Section 2 below), which have been used in modular representation theory for some time (see, e.g., [Benson, 1984]), have been introduced into the cohomological study of finite transformation groups. The use of shifted subgroups is quite natural; and indeed they seem to have appeared in transformation groups through the work of at least four different authors: A. Adem introduced them explicitly in his thesis ([Adem, 1986], and see [Adem, 1988]); they also appear explicitly in the work of A. Assadi ([Assadi, 1988], [Assadi, 1989a], [Assadi, 1989b] and [Assadi]); and shifted subgroups of rank one appeared implicitly in our paper [Allday, Puppe, 1985].

In this paper we intend to give a survey of some of these applications of shifted subgroups. We shall concentrate on the work of A. Adem and ourselves and closely related results. Since it would require a substantial amount of background material, we have not included Assadi's work concerning applications of the theory of varieties of G–modules in transformation groups: for this see Assadi's papers cited above. We have included one of Adem's theorems (Theorem (4.14) below), the proof of which makes substantial use of varieties of G–modules, but, for the same reason, we have not included Adem's proof. Otherwise, for the most part, we have included proofs, although we have referred some proofs, especially the proofs of some technical details, to our forthcoming book ([Allday, Puppe]).

In the first two sections we summarize some background material from algebra, including, in Section 2, the definition and some of the main properties of shifted subgroups. In the third section we give some of the basic topological notations and definitions which we shall use. We have chosen to work with paracompact finitistic spaces. There is only a small amount of technical difficulty in extending the results given here from finite–dimensional G–CW–complexes to paracompact finitistic G–spaces; and yet many more applications are included amongst the latter, for example, continuous actions on topological manifolds.

The last three sections give some of the applications of shifted subgroups. In Section 4 we treat equivariant Tate cohomology (as defined by R. Swan), in Section 5 we give an application in the manner of P. A. Smith's original method, and in Section 6 we give an application to equivariant cohomology (as defined by A. Borel).

1. k[G]–modules

Here we collect a few useful facts about k[G]–modules. Throughout this section G will be a finite group and k will be a field.

(1.1) **Theorem.** (1) A k[G]–module is projective if and only if it is injective.
(2) Any product of projective k[G]–modules is projective.

Proof. (2) follows at once from (1).

(1) follows from [Brown, 1982], Chap. VI, Corollaries (2.2) and (2.3). (1) also follows since k[G] is a symmetric algebra, and hence Frobenius: see, for example, [Fuller, 1989].

(1.2) **Corollary.** If M is a projective k[G]–module, then the dual module $M^* := \mathrm{Hom}_{k[G]}(M, k[G])$ is also projective.

(1.3) **Definition.** We shall say that a k[G]–module M is Tate acyclic if $\hat{H}^*(G; M) = 0$. (This is a slight simplification of the notion of a cohomologically trivial module: see [Brown, 1982], Chap. VI, Sec. 8.)

(1.4) **Theorem.** Suppose that G is a finite p–group, where p is any prime number, and that k is a field of characteristic p. Then the following conditions on a k[G]–module M are equivalent.

(1) M is free.
(2) M is projective.
(3) M is Tate acyclic.

(4) $\hat{H}^i(G; M) = 0$ for at least one $i \in \mathbb{Z}$.

Proof. This is contained explicitly in [Brown, 1982], Chap. VI, Theorem (8.5).

(1.5) **Corollary**. Let G and k be as in Theorem (1.4). Then
(1) any direct limit of free k[G]–modules is free; and
(2) if k' is an extension field of k, if M is a k[G]–module, and if $M \otimes_k k'$ is a free $k'[G]$–module, then M is a free k[G]–module.

Proof. Since G has a complete resolution of finite type, P_*, say, for any $i \in \mathbb{Z}$, and k[G]–module M, $\mathrm{Hom}_{k[G]}(P_i, M) \cong P_i^* \otimes_{k[G]} M$. Hence, if $\{M_j | j \in J\}$ is a directed system of k[G]–modules, then $\hat{H}^*(G; \varinjlim_j M_j) \cong \varinjlim_j \hat{H}^*(G; M_j)$.

Similarly for (2), $\hat{H}^*(G; M \otimes_k k') \cong \hat{H}^*(G; M) \otimes_k k'$.

2. Shifted subgroups

In this section we recall the defnition of shifted subgroups and state some of their basic properties. Throughout this section G will be an elementary abelian p–group (also known as a p–torus), where p is a prime number; i.e. $G \cong (\mathbb{Z}/(p))^r$ for some $r \geq 0$: and k will be a field of characteristic p. The number r is called the rank of G, denoted rk G.

Suppose that G is generated by $g_1, \ldots, g_r$. For $1 \leq i \leq r$, let $\tau_i = 1 - g_i \in k[G]$. Let $\nu_i = (1 - g_i)^{p-1}$ for $1 \leq i \leq r$. Since $\tau_i^p = 0$, it follows that the homomorphism from the polynomial ring $k[X_1, \ldots, X_r] \longrightarrow k[G]$ given by $X_i \longrightarrow \tau_i$, for $1 \leq i \leq r$, induces an isomorphism $k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p) \xrightarrow{\sim} k[G]$.

Let $\mathbf{m}(G; k)$, or more simply just $\mathbf{m}$, denote the ideal in k[G] generated by $\tau_1, \ldots, \tau_r$. So $\mathbf{m}$ is the one and only prime ideal in k[G]. Note that $\mathbf{m}^{(p-1)r+1} = 0$.

(2.1) **Definitions.** (1) We shall say that any element $u \in k[G]$ is a non–trivial unit if there are $\alpha_1, \ldots, \alpha_r \in k$, not all zero, such that $u = 1 - \Sigma_{i=1}^r \alpha_i \tau_i$ modulo $\mathbf{m}^2$. Clearly $u^p = 1$; and so u generates a subgroup of order p in the group of units of k[G]. Denote this subgroup by $\Gamma(u)$. As follows from (3) below, $\Gamma(u)$ is a shifted subgroup of rank 1.

(2) If $u_1, \ldots, u_s$ are non–trivial units in k[G], then let $\Gamma(u_1, \ldots, u_s)$ denote the elementary abelian p–subgroup of the group of units of k[G] generated by $u_1, \ldots, u_s$.

The inclusion of $\Gamma = \Gamma(u_1, \ldots, u_s)$ in k[G] induces a homomorphism of group rings $i_\Gamma\colon k[\Gamma] \longrightarrow k[G]$.

(3) Suppose that $u_1, \ldots, u_s$ are non–trivial units in k[G]. For $1 \leq j \leq s$, let $u_j = 1 - \Sigma_{i=1}^r \alpha_{ji}\tau_i$ modulo $\mathbf{m}^2$. Then $\Gamma(u_1, \ldots, u_s)$ is said to be a shifted subgroup of rank s in k[G] if the vectors $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jr}) \in k^r$, for $1 \leq j \leq s$, are linearly independent. It is easy to see that a shifted subgroup of rank s is indeed an elementary abelian p–group of rank s.

(4) If a nontrivial unit $u = 1 - \Sigma_{i=1}^{r} \alpha_i \tau_i$, then we also denote $\Gamma(u)$ by $\Gamma(\alpha)$, where $\alpha = (\alpha_1, ..., \alpha_r) \in k^r$.

(5) Given $u, v \in k[G]$, we shall write $u \sim v$ if $u - v \in \mathbf{m}^2$.

Some important properties of shifted subgroups are listed in the following theorem. Proofs may be found in [Carlson, 1983]; but we shall include the proof of (1), which is most often used subsequently in this paper.

(2.2) **Theorem.** (1) If $\Gamma \subsetneq k[G]$ is a shifted subgroup of rank s, then $i_\Gamma : k[\Gamma] \longrightarrow k[G]$ is injective, and $k[G]$ is a free $k[\Gamma]$–module (via i_Γ). If $s = r$, then i_Γ is an isomorphism.

(2) If $u_1, ..., u_s$ are non–trivial units in $k[G]$ such that $\Gamma = \Gamma(u_1, ..., u_s)$ has order p^s, then $k[G]$ is a free $k[\Gamma]$–module if and only if $1 - u_1, ..., 1 - u_s$ are linearly independent modulo $\mathbf{m}^2$: i.e. if and only if $u_1, ..., u_s$ generate Γ as a shifted subgroup.

(3) Suppose that $u, v \in k[G]$ are non– trivial units such that $u \sim v$. Let M be a finitely generated $k[G]$–module. The M is a free $k[\Gamma(u)]$–module if and only if M is a free $k[\Gamma(v)]$–module.

(4) Dade's Lemma. If k is algebraically closed, and if M is a finitely generated $k[G]$–module, then M is a free $k[G]$–module if and only if M is a free $k[\Gamma(\alpha)]$–module for all non–zero $\alpha \in k^r$.

Proof. (1) Suppose $\Gamma = \Gamma(u_1, ..., u_s)$ where $u_j = 1 - \Sigma_{i=1}^{r} \alpha_{ji} \tau_i$ modulo $\mathbf{m}^2$ for $1 \leq j \leq s$. Let $\sigma_j = 1 - u_j \in k[\Gamma]$, and let $\sigma'_j = \Sigma_{i=1}^{r} \alpha_{ji} \tau_i$. So $\sigma_j \sim \sigma'_j$. And since Γ is a shifted subgroup of rank s we may assume that $\sigma'_1, ..., \sigma'_s$ are linearly independent. Now choose $\sigma'_{s+1}, ..., \sigma'_r$ so that $\{\sigma'_1, ..., \sigma'_r\}$ is a basis for the k–vector subspace of $k[G]$ spanned by $\tau_1, ..., \tau_r$. Let $u_i = 1 - \sigma'_i$ for $s + 1 \leq i \leq r$ and let $\Gamma' = \Gamma(u_{s+1}, ..., u_r)$.

Now $i_\Gamma : k[\Gamma] \longrightarrow k[G]$ and $i_{\Gamma'} : k[\Gamma'] \longrightarrow k[G]$ induce, via the direct sum of commutative k–algebras, a homomorphism $\varphi : k[\Gamma] \otimes k[\Gamma'] \longrightarrow k[G]$. Since $\sigma'_1, ..., \sigma'_r$ generate $k[G]$ as a k–algebra, since $\sigma'_j - \sigma_j \in \mathbf{m}^2$ for $1 \leq j \leq s$, and since $\mathbf{m}^{(p-1)r+1} = (0)$, it is clear that φ is surjective. Hence φ is an isomorphism, since its domain and codomain are finite–dimensional k–vector spaces of the same dimension.

For (2) see [Carlson, 1983], Theorem 6.2, and Corollary (1.5)(2) above. (Carlson assumes that k is algebraically closed.)

For (3) see [Carlson, 1983], Lemma 6.4, and Corollary (1.5)(2) above.

For (4) see [Carlson, 1983], Theorem 4.4, or [Dade, 1978].

(2.3) **Remarks**. (1) A subgroup Γ of the group of units of $k[G]$ is a shifted subgroup of rank s if and only if there exist $u_1, ..., u_s \in k[G]$, such that $\{1 - u_1, ..., 1 - u_s\} \subsetneq \mathbf{m}$, the image of $\{1 - u_1, ..., 1 - u_s\}$ under the quotient map $\mathbf{m} \longrightarrow \mathbf{m}/\mathbf{m}^2$ is linearly independent over k, and $\Gamma = \Gamma(u_1, ..., u_s)$.

And to put it another way, a subgroup Γ of the group of units of $k[G]$ is a shifted subgroup if and only if Γ is generated by a finite number of elements of the coset $1 + \mathbf{m}$, and the homomorphism $\mathbf{m}(\Gamma; k)/\mathbf{m}(\Gamma; k)^2 \longrightarrow \mathbf{m}/\mathbf{m}^2$ induced by i_Γ is injective.

(2) Let H be a subgroup of G of rank s. Then $G \cong H \times G/H$; and so there is a homomorphism $q_H : G \to H$ such that $q_H j_H = 1_H$, where $j_H : H \to G$ is the inclusion. It follows

that $H \subseteq k[G]$ is a shifted subgroup of rank s; and $i_H : k[H] \longrightarrow k[G]$ induces an injection $m(H; k)/m(H; k)^2 \longrightarrow m/m^2$.

In the following proposition we are concerned with a subgroup $H \subseteq G$ and a shifted subgroup $\Gamma \subseteq k[G]$; and we want to know when $k[G/H]$ is a free $k[\Gamma]$–module via the homomorphism $k[\Gamma] \longrightarrow k[G/H]$ obtained by composing i_Γ with the homomorphism $k[G] \longrightarrow k[G/H]$ induced by the quotient map. Let V be the k–vector space m/m^2, and let V_H be the image of $m(H; k)/m(H; k)^2$ in V. By Remarks (2.3)(2) above, $\dim_k V_H = \mathrm{rk}\, H$. For a shifted subgroup $\Gamma \subseteq k[G]$ let V_Γ be the image of $m(\Gamma; k)/m(\Gamma; k)^2$ in V under the homomorphism induced by i_Γ. By definition of a shifted subgroup $\dim_k V_\Gamma = \mathrm{rk}\Gamma$.

(2.4) **Proposition.** In the situation described above, for a subgroup $H \subseteq G$ and a shifted subgroup $\Gamma \subseteq k[G]$, $k[G/H]$ is a free $k[\Gamma]$–module if and only if $V_\Gamma \cap V_H = 0$.

Proof. Suppose $V_\Gamma \cap V_H = 0$. Let $\pi : k[G] \longrightarrow k[G/H]$ be induced by the quotient map, and let $\pi' : V \longrightarrow m(G/H; k)/m(G/H; k)^2$ be induced by π. Clearly π' is surjective and $V_H \subseteq \ker \pi'$. Hence $V_H = \ker \pi'$. And so $\pi' \,|\, V_\Gamma$ is injective. Hence π maps Γ isomorphically onto a shifted subgroup $\pi(\Gamma) \subseteq k[G/H]$. Since $k[G/H]$ is a free $k[\pi(\Gamma)]$–module by Theorem (2.2)(1), $k[G/H]$ is a free $k[\Gamma]$–module.

If $V_\Gamma \cap V_H \neq 0$, then there are two possibilities: (i) π does not map Γ isomorphically onto $\pi(\Gamma)$, or (ii) π maps Γ isomorphically onto $\pi(\Gamma)$, but $\pi(\Gamma)$ is not a shifted subgroup of $k[G/H]$. In case (i) the result is clear. In case (ii) the result follows from Theorem (2.2)(2).

The following corollary is very useful in the applications of shifted subgroups to transformation groups.

(2.5) **Corollary**. Let $H_1, \ldots, H_n$ be subgroups of G such that, for $1 \leq i \leq n$, $\mathrm{rk}\, H_i \leq t < r = \mathrm{rk}\, G$. Then there is an extension field E of k with finite degree over k, and a shifted subgroup $\Gamma \subseteq E[G]$ of rank $r - t$ such that $E[G/H_i]$ is a free $E[\Gamma]$–module for $1 \leq i \leq n$.

Proof. Let K be the algebraic closure of k. Using the notation of Proposition (2.4) for $K[G]$ instead of $k[G]$, since K is an infinite field, there is a subspace $W \subseteq V$ such that $W \cap V_{H_i} = 0$ for $1 \leq i \leq n$, and $\dim_k W = r - t$. Hence there is a shifted subgroup $\Gamma \subseteq K[G]$ of rank $r - t$ such that $K[G/H_i]$ is a free $K[\Gamma]$–module for $1 \leq i \leq n$.

But Γ is defined in terms of the elements of G using a finite number of coefficients in K. Let E be the extension field of k generated by these coefficients. So Γ may be viewed as a shifted subgroup in $E[G]$. Now it follows that $E[G/H_i]$ is a free $E[\Gamma]$–module, for $1 \leq i \leq n$, by Corollary (1.5)(2).

(2.6) **Remark**. For an important generalization of this result to $k[G]$–modules which are not necessarily permutation modules see [Kroll, 1984].

3. Topological notation and constructions

Let G be a finite group and let k be a commutative ring (with identity). Let $P_* \longrightarrow k \longrightarrow 0$ be a projective resolution of k viewed as a trivial $k[G]$–module; and let $\hat{P}_*$ be a complete resolution of k (see [Brown,1982], Chap. VI, Section 3). We shall assume, as we may, that P_* and $\hat{P}_*$ have finite type, and that $\hat{P}_i = P_i$ for $i \geq 0$. In particular there is an obvious map $\hat{P}_* \longrightarrow P_*$ which is the identity on $\hat{P}_i$ for $i \geq 0$ and zero on $\hat{P}_i$ for $i < 0$.

(3.1) **Definitions.** Let C^* be a cochain complex of k[G]–modules with $C^i = 0$ for $i < 0$.

(1) Let $\beta^*_G(C^*) = \mathrm{Hom}_{k[G]}(P_*,C^*)$. In particular $\beta^n_G(C^*) = \oplus_{i=0}^{n} \mathrm{Hom}_{k[G]}(P_i,C^{n-i})$. The differential d on $\beta^*_G(C^*)$ is given in terms of the differentials d_P and d_C on P_* and C^*, respectively, by the formula $df(x) = d_C(f(x)) - (-1)^n f(d_P(x))$ for $f \in \beta^n_G(C^*)$.

(2) Let $H^*_G(C^*) = H(\beta^*_G(C^*),d)$.

(3) Let $\hat{\beta}^*_G(C^*) = \mathrm{Hom}_{k[G]}(\hat{P}_*,C^*)$, where $\hat{\beta}^n_G(C^*) = \oplus_{i=-\infty}^{n} \mathrm{Hom}_{k[G]}(\hat{P}_i,C^{n-i})$. Note that we are taking the direct sum here not the direct product. The differential, d, on $\hat{\beta}^*_G(C^*)$ is defined in the same way as for $\beta^*_G(C^*)$.

(4) Let $\hat{H}^*_G(C^*) = H(\hat{\beta}^*_G(C^*),d)$.

(5) The first filtration on $\beta^*_G(C^*)$ is defined by $F^p\beta^n_G(C^*) = \oplus_{i=p}^{n}\mathrm{Hom}_{k[G]}(P_i,C^{n-i})$.

The second filtration on $\beta^*_G(C^*)$ is defined by $F^q\beta^n_G(C^*) = \oplus_{j=q}^{\infty}\mathrm{Hom}_{k[G]}(P_{n-j},C^j)$.

The first and second filtrations on $\hat{\beta}_G(C^*)$ are defined similarly.

The following spectral sequences are standard. (See [Brown, 1982], Chap. VII, and [Allday, Puppe], §4.6.)

(3.2) **Proposition**. The first filtrations gives rise to spectral sequences

(1) $E_2^{p,q} = H^p(G;H^q(C^*)) \Rightarrow H^*_G(C^*)$; and

(2) $E_2^{p,q} = \hat{H}^p(G;H^q(C^*)) \Rightarrow \hat{H}^*_G(C^*)$.

The second filtration gives rise to a spectral sequence

(3) $E_1^{p,q} = H^q(G;C^p) \Rightarrow H^*_G(C^*)$;

and, if C^* is bounded above, i.e. there is an integer n such that $C^i = 0$ for $i > n$, a spectral sequence

(4) $E_1^{p,q} = \hat{H}^q(G;C^p) \Rightarrow \hat{H}^*_G(C^*)$.

(3.3) **Corollary**. The map $\hat{P}_* \longrightarrow P_*$ induces a natural homomorphism $\theta^*: H^*_G(C^*) \longrightarrow \hat{H}^*_G(C^*)$. And, if $H^j(C^*) = 0$ for all $j > n$, then $\theta^*: H^j_G(C^*) \longrightarrow \hat{H}^j_G(C^*)$ is an isomophism for all $j > n$.

Proof. The existence of θ^* is immediate. So suppose $H^j(C^*) = 0$ for all $j > n$. Let C^*_t be the cochain complex with $C^i_t = C^i$ for $i < n$, $C^n_t = Z^n$, the cocycles of degree n, and $C^i_t = 0$ for

$i > n$. The inclusion $C^*_t \longrightarrow C^*$ is a weak equivalence, and hence the first spectral sequences show that $H^*_G(C^*_t) \cong H^*_G(C^*)$ and $\hat{H}^*_G(C^*_t) \cong \hat{H}^*_G(C^*)$.

Now $\beta^j_G(C^*_t) = \hat{\beta}^j_G(C^*_t)$ for $j \geq n$. Hence $H^j_G(C^*_t) \cong \hat{H}^j_G(C^*_t)$ for $j > n$.

In order to work with paracompact finitistic spaces using Alexander–Spanier or Čech cohomology we need to review some notation and terminology concerning coverings.

(3.4) **Definitions.** Let X be a paracompact G–space, and let $A \subseteq X$ be a closed invariant subspace. Let $\mathcal{U}$ be an open covering of X.

(1) Let $\mathcal{U}_A = \{U \in \mathcal{U} \mid U \cap A \neq \phi\}$.

(2) The Čech nerve of $\mathcal{U}$, denoted $\check{\mathcal{U}}$, is the abstract simplicial complex with vertices the non–empty members of $\mathcal{U}$, and $\{U_0, ..., U_n\}$ a simplex of $\check{\mathcal{U}}$, where $U_i \in \mathcal{U}$ for $0 \leq i \leq n$, if $\bigcap_{i=0}^n U_i \neq \phi$. The subcomplex $\check{\mathcal{U}}_A$ is defined by saying that a simplex $\{U_0, ..., U_n\}$ of $\check{\mathcal{U}}$ is a simplex of $\check{\mathcal{U}}_A$ if $\bigcap_{i=0}^n U_i \cap A \neq \phi$.

(3) The Vietoris nerve of $\mathcal{U}$, denoted $\overline{\mathcal{U}}$, is the abstract simplicial complex with vertices the points of X, and $\{x_0, ..., x_n\}$ a simplex of $\overline{\mathcal{U}}$ if $\{x_0, ..., x_n\} \subseteq U$ for some $U \in \mathcal{U}$. The subcomplex $\overline{\mathcal{U}}_A$ is defined by saying that a simplex $\{x_o, ..., x_n\}$ of $\overline{\mathcal{U}}$ is a simplex of $\overline{\mathcal{U}}_A$ if $\{x_0, ..., x_n\} \subseteq A$.

(4) $\mathcal{U}$ is said to be finite–dimensional if $\check{\mathcal{U}}$ is a finite–dimensional abstract simplicial complex, in which case the geometric realization $|\check{\mathcal{U}}|$ is a finite–dimensional CW–complex.

(5) X is said to be finitistic if every open covering of X has a finite–dimensional refinement: i.e. if finite–dimensional coverings are cofinal.

(6) $\mathcal{U}$ is said to be an invariant covering of X if for any $U \in \mathcal{U}$ and $g \in G$, $gU \in \mathcal{U}$.

(7) $\mathcal{U}$ is said to be a Čech–G–covering of X if $\mathcal{U}$ is invariant and if $gU \cap U \neq \phi$ implies $gU = U$ for any $U \in \mathcal{U}$ and $g \in G$. In this case, for any $U \in \mathcal{U}$, let $G_U = \{ g \in G \mid gU = U\}$. (Čech–G–coverings are just called G–coverings in [Bredon, 1972]).

(8) $\mathcal{U}$ is said to be faithful if $\mathcal{U}$ is a Čech–G–covering, and if, for any $U \in \mathcal{U}$, there is a $x \in X$ such that $G_U \subseteq G_x$.

(3.5) **Lemma.** If X is a paracompact G–space, then locally finite faithful Čech–G–coverings are cofinal. If X is also finitistic, then locally finite finite–dimensional faithful Čech–G–coverings are cofinal.

Proof. Most of this is contained in [Bredon, 1972], Chap. III, Theorem 6.1. Since G is finite, X has a covering by open slices, which is a faithful Čech–G–covering. And clearly any Čech–G–covering which refines a faithful Čech–G–covering is also faithful.

(3.6) **Definitions.** Let X be a paracompact G–space, let $A \subseteq X$ be a closed invariant subspace, and let Λ be a k–module.

(1) Let $\overline{C}^*(X, A; \Lambda) = \lim_{\mathcal{U}} C^*(\overline{\mathcal{U}}, \overline{\mathcal{U}}_A; \Lambda)$, where $\mathcal{U}$ ranges over the faithful Čech–G–coverings of X, and $C^*(\overline{\mathcal{U}}, \overline{\mathcal{U}}_A; \Lambda)$ is the ordered cochain complex of the pair $(\overline{\mathcal{U}}, \overline{\mathcal{U}}_A)$

with coefficients in Λ. Then $\bar{C}^*(X, A; \Lambda)$ is the Alexander–Spanier cochain complex of (X, A) with coefficients in Λ as defined, for example, in [Spanier, 1966], Chap. 6, sec. 4. Clearly $\bar{C}^*(X, A; \Lambda)$ is a cochain complex of $k[G]$–modules. If G is an elementary abelian p–group, k is a field of characteristic p, and if $\Gamma \subseteq k[G]$ is a shifted subgroup, then $\bar{C}^*(X, A; \Lambda)$ is also a cochain complex of $k[\Gamma]$–modules.

(2) Let $H^*_G(X, A; \Lambda) = H^*_G(\bar{C}^*(X, A; \Lambda))$, and let $\hat{H}^*_G(X, A; \Lambda) = \hat{H}^*_G(\bar{C}^*(X, A; \Lambda))$.

Define $H^*_\Gamma(X, A; \Lambda)$ and $\hat{H}^*_\Gamma(X, A; \Lambda)$ similarly if G is an elementary abelian p–group, k is a field of characteristic p, and $\Gamma \subseteq k[G]$ is a shifted subgroup.

(3) If X is a G–CW–complex, and A is G–CW–subcomplex, or more generally, if (X, A) is a relative G–CW–complex, then let $W_*(X, A; k)$, respectively $W^*(X, A; \Lambda)$, be the cellular chain complex of (X, A) with coefficients in k, respectively the cellular cochain complex of (X, A) with coefficients in Λ.

The following lemma is proven in detail in [Allday, Puppe].

(3.7) **Lemma.** If X is a paracompact G–space and $A \subseteq X$ is a closed invariant subspace, then

$$\hat{H}^*_G(X, A; \Lambda) \cong \varinjlim_{\mathcal{U}} \hat{H}^*_G(|\check{\mathcal{U}}|, |\check{\mathcal{U}}_A|; \Lambda)$$

$$\cong \varinjlim_{\mathcal{U}} \hat{H}^*_G(W^*(|\check{\mathcal{U}}|, |\check{\mathcal{U}}_A|; \Lambda)),$$

and similarly with H^*_G instead of $\hat{H}^*_G$.

If G is an elementary abelian p–group, k is a field of characteristic p, and if $\Gamma \subseteq k[G]$ is a shifted subgroup, then the corresponding results also hold for H^*_Γ and $\hat{H}^*_\Gamma$.

Furthermore $H^*_G(X, A; \Lambda) \cong H^*(X_G, A_G; \Lambda)$, the Alexander–Spanier cohomology of the pair (X_G, A_G) with coefficients in Λ, where X_G, for example, is the Borel construction on X; i.e. $X_G = (EG \times X)/G$.

(3.8) **Remarks.** (1) $H^*_G(X, A; \Lambda)$, respectively $\hat{H}^*_G(X, A; \Lambda)$, is called the equivariant, respectively the equivariant Tate, cohomology of (X, A) with coefficients in Λ.

(2) $\hat{H}^*_G$, like H^*_G, has natural long exact sequences for pairs, Mayer–Vietoris sequences, and tautness properties. For example, if A and B are closed invariant subspaces of X with $X = A \cup B$, then there is a long exact Mayer–Vietoris sequence

$$\ldots \to \hat{H}^j_G(X; \Lambda) \to \hat{H}^j_G(A; \Lambda) \oplus \hat{H}^j_G(B; \Lambda) \to \hat{H}^j_G(A \cap B; \Lambda) \to \hat{H}^{j+1}_G(X; \Lambda) \to \ldots$$ and

$\hat{H}^*_G(A; \Lambda) \cong \varinjlim_V \hat{H}^*_G(V; \Lambda)$, where V ranges over the closed invariant neighborhoods of A. The same holds for H^*_Γ and $\hat{H}^*_\Gamma$. (See [Allday, Puppe], § 4.6.)

We finish this section by recalling W.–Y. Hsiang's definition of the p–rank of a space.

(3.9) **Definitions.** Let $G \cong (\mathbb{Z}/(p))^r$, and let $\Phi : G \times X \to X$ be an action of G on a space X.

(1) The rank of Φ is $\mathrm{rk}\Phi := r - \max\{\mathrm{rk}\, G_x \mid x \in X\}$. Thus if $\mathrm{rk}\Phi = \rho$, then p^ρ is the order of the smallest orbit. $\mathrm{rk}\Phi = r$ if and only if G is acting freely; and $\mathrm{rk}\Phi = 0$ if and only if

$X^G \neq \phi$. Thus $\mathrm{rk}\Phi$ measures in a certain sense the extent to which the action is free.

(2) The p–rank of X is $\mathrm{rk}_p(X) := \sup \mathrm{rk}\Phi$, where Φ ranges over all elementary abelian p–group actions on X.

(3.10) **Remark.** We could also define the free p–rank of X to be $\mathrm{frk}_p(X) := \sup \{r \mid (\mathbb{Z}/(p))^r$ can act freely on $X\}$. There are well known examples where $\mathrm{rk}_p(X) > \mathrm{frk}_p(X)$. For example, $\mathrm{rk}_3(\mathbb{C}P^2) = 1$ but $\mathrm{frk}_3(\mathbb{C}P^2) = 0$: see Examples (5.5)(2) below.

4. Equivariant Tate cohomology.

Equivariant Tate cohomology and finitistic spaces were introduced by Swan in [Swan, 1960]. We shall recall Swan's main theorem immediately following the next definition.

(4.1) **Definition.** Let G be a compact Lie group and let X be a paracompact G–space. Then the singular set of X is defined to be $X_1 := \{x \in X \mid G_x \neq \{1\}\}$. X_1 is clearly invariant, and by the Slice Theorem it is closed.

(4.2) **Theorem.** Let G be a finite group, k a commutative ring with identity, Λ a k–module, X a paracompact finitistic G–space and $A \subseteq X$ is a closed invariant subspace. Then restriction induces an isomorphism

$$\hat{H}^*_G(X, A; \Lambda) \xrightarrow{\sim} \hat{H}^*_G(X_1, A_1; \Lambda).$$

For a proof see [Swan, 1960] or [Allday, Puppe], §4.6.

In applying Swan's Theorem the following easy lemma to be found in [Adem, 1988] and inspired by [Heller, 1959] is useful.

(4.3) **Lemma.** Let G be a finite group and let k be a field of characteristic p where p divides $|G|$. Let C^* be a cochain complex of k[G]–modules such that $C^i = 0$ for all $i < 0$ and $H^j(C^*) = 0$ for all $j > N$, where N is some integer. Then, for any integer m,

$$\dim_k \hat{H}^{m+1}(G; H^0(C^*)) \leq \dim_k \hat{H}^{m+1}_G(C^*) + \sum_{j=1}^{N} \dim_k \hat{H}^{m-j}(G; H^j(C^*)).$$

(4.4) **Corollary** ([Heller, 1959], [Adem, 1988]). Let X be a paracompact finitistic space such that $H^*(X; \mathbb{F}_p) \cong H^*(S^a x S^b; \mathbb{F}_p)$ as graded $\mathbb{F}_p$–vector spaces where a and b are integers such that $0 < a < b$. Then $\mathrm{frk}_p(X) \leq 2$. (See Remark (3.10))

Proof. Suppose that $G = (\mathbb{Z}/(p))^3$ is acting freely on X. By Swan's Theorem, $\hat{H}^*_G(X; \mathbb{F}_p) = 0$. Now Lemma (4.3) with $m = a + b$ and $C^* = \overline{C}^*(X; \mathbb{F}_p)$ yields a contradiction.

Now we would like to prove that, under the conditions of the Corollary (4.4), $\mathrm{rk}_p(X) \leq 2$. In [Adem, 1988], Adem did this by introducing shifted subgroups. Here then, following the next definition, is a shifted version of Swan's Theorem.

(4.5) **Definition.** Let $G \cong (\mathbb{Z}/(p))^r$ and let k be a field of characteristic p. Let X be a paracompact G–space. Then, using the notation introduced immediately above Proposition (2.4), for any shifted subgroup $\Gamma \subseteq k[G]$, let $X(\Gamma;k) = \{x \in X \mid V_\Gamma \cap V_{G_x} \neq 0\}$. Note that $X(\Gamma; k)$ is invariant, and, by the Slice Theorem, it is closed. Also $X(G; k) = \{x \in X \mid V_{G_x} \neq 0\} = X_1$.

(4.6) **Theorem**. Let $G \cong (\mathbb{Z}/(p))^r$, let k be field of characteristic p, let X be a paracompact finitistic G–space, let $A \subseteq X$ be a closed invariant subspace, and let $\Gamma \subseteq k[G]$ be a shifted subgroup. Then restriction induces an isomorphism

$$\hat{H}^*_\Gamma(X, A; k) \xrightarrow{\sim} \hat{H}^*_\Gamma(X(\Gamma; k), A(\Gamma; k); k).$$

Proof. Thanks to the long exact sequences (see Remarks (3.8)(2)) it is enough to prove the result when $A = \phi$. Suppose the result has been proven in case $X(\Gamma; k) = \phi$. If $X(\Gamma; k) \neq \phi$, let W_1 be a closed invariant neighbourhood of $X(\Gamma; k)$ and let W_2 be the complement of the interior of W_1. So $W_2(\Gamma; k) = \phi$ and $(W_1 \cap W_2)(\Gamma; k) = \phi$. By the Mayer–Victoris sequence, therefore, $\hat{H}^*_\Gamma(X; k) \cong \hat{H}^*_\Gamma(W_1; k)$. The result now follows by the tautness property (Remarks (3.8(2))).

So it remains to show that $\hat{H}^*_\Gamma(X; k) = 0$ if $X(\Gamma; k) = \phi$. Let $\mathcal{U}$ be a faithful finite–dimensional Čech–G–covering of X. For any $y \in |\check{\mathcal{U}}|$, there is a $x \in X$ such that $G_y \subseteq G_x$. (Since $\mathcal{U}$ is a Čech–G–covering, the maximal isotropy groups of $|\check{\mathcal{U}}|$ occur at the vertices; and these are all contained in isotropy groups of X since $\mathcal{U}$ is faithful.) Now, by Proposition (2.4), each $k[G/G_y]$, for $y \in |\check{\mathcal{U}}|$, is a free $k[\Gamma]$–module. Thus each $W_i(|\check{\mathcal{U}}|; k)$, and hence, by Theorem (1.1)(2), also each $W^i(|\check{\mathcal{U}}|; k)$, is a free $k[\Gamma]$–module.

By the second spectral sequence (Proposition (3.2)(4)), $\hat{H}^*_\Gamma(W^*(|\check{\mathcal{U}}|; k)) = 0$. So $\hat{H}^*_\Gamma(X; k) = 0$ by Lemmas (3.5) and (3.7).

(4.7) **Corollary**. Let $G \cong (\mathbb{Z}/(p))^r$, let X be a paracompact finitistic G–space, and let $A \subseteq X$ be a closed invariant subspace. Let ρ be the rank of the action on $X - A$: i.e. $\rho = r - \max \{\mathrm{rk}\, G_x | x \in X - A\}$. Then there is a finite field k of characteristic p and a shifted subgroup $\Gamma \subseteq k[G]$ of rank ρ such that

$$\hat{H}^*_\Gamma(X, A; k) = 0.$$

Proof. Let $H_1, \ldots, H_n$ be the isotropy groups of G on $X - A$. By Corollary (2.5) and its proof, there is a field k, which is of finite degree over $\mathbb{F}_p$, and a shifted subgroup $\Gamma \subseteq k[G]$ of rank ρ, such that $V_\Gamma \cap V_{H_i} = 0$ for $1 \leq i \leq n$. So $X(\Gamma; k) = A(\Gamma; k)$; and the result follows.

Combining Corollary (4.7) with Lemma (4.3) as in the proof of Corollary (4.4) we get immediately the first part of the following corollary. The second part of the following requires a little more work with the first spectral sequence.

(4.8) **Corollary**. Let X be a paracompact finistic G–space such that $H^*(X; \mathbb{F}_p) \cong H^*(S^a \times S^b; \mathbb{F}_p)$ as graded $\mathbb{F}_p$–vector spaces where a and b are integers such that $0 < a < b$. Then

(1) $\mathrm{rk}_p(X) \leq 2$. If $X = S^a \times S^b$, then $\mathrm{rk}_2(X) = 2$. If $X = S^a \times S^b$ and a, b and p are odd, then $\mathrm{rk}_p(S^a \times S^b) = 2$.

(2) If $a + b$ and p are odd, then $\mathrm{rk}_p(X) \leq 1$. (If a and b are even, and p is odd, then $\mathrm{rk}_p(X) = 0$. This follows from Theorem (5.1) below.)