An Introduction to Continuum Mechanics

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1981

ACADEMIC PRESS

A Subsidiary of Harcourt Brace Jovanovich, Publishers

New York London Toronto Sydney San Francisco

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data

Gurtin, Morton E.
An introduction to continuum mechanics.

(Mathematics in science and engineering)
Bibliography: p.
Includes index.
1. Continuum mechanics. I. Title. II. Series.
QA808.2.G86 531 80-2335
ISBN 0-12-309750-9 AACR2

AMS (MOS) 1970 Subject Classification: 73B99

PRINTED IN THE UNITED STATES OF AMERICA

81 82 83 84 9 8 7 6 5 4 3 2 1

Preface

This book presents an introduction to the classical theories of continuum mechanics; in particular, to the theories of ideal, compressible, and viscous fluids, and to the linear and nonlinear theories of elasticity. These theories are important, not only because they are applicable to a majority of the problems in continuum mechanics arising in practice, but because they form a solid base upon which one can readily construct more complex theories of material behavior. Further, although attention is limited to the classical theories, the treatment is modern with a major emphasis on foundations and structure.

I have used direct—as opposed to component—notation throughout. While engineers and physicists might at first find this a bit difficult, I believe that the additional effort required is more than compensated for by the resulting gain in clarity and insight. For those not familiar with direct notation, and to make the book reasonably self-contained, I have included two lengthy chapters on tensor algebra and analysis.

The book is designed to form a one- or two-semester course in continuum mechanics at the first-year graduate level and is based on courses I have taught over the past fifteen years to mathematicians, engineers, and physicists at Brown University and at Carnegie-Mellon University.

With the exception of a list of general references at the end of each chapter, I have omitted almost all reference to the literature. Those interested in questions of priority or history are referred to the encyclopedia articles of Truesdell and Toupin [1], Truesdell and Noll [1], Serrin, [1], and Gurtin [1].

Acknowledgments

I owe so much to so many people. To begin, this book would never have been written without the inspiration and influence of Eli Sternberg, Walter Noll, and Clifford Truesdell, who, through discussions, courses, and writings, both published and unpublished, have endowed me with the background necessary for a project of this type.

I want to thank D. Carlson, L. Martins, I. Murdoch, P. Podio-Guidugli, E. Sternberg, and W. Williams for their detailed criticism of the manuscript, and for their many valuable discussions concerning the organization and presentation of the material. I would also like to thank J. Anderson, J. Marsden, L. Murphy, D. Owen, D. Reynolds, R. Sampaio, K. Spear, S. Spector, and R. Temam for valuable comments.

The proof (p. 23) that the square-root function (on tensors) is smooth comes from unpublished lecture notes of Noll; the proof of the smoothness lemma (p. 65) is due to Martins; Section 33, concerning bending and torsion of linear elastic bodies, is based on unpublished lecture notes of Sternberg; the first paragraph of Section 21, concerning invariance under a change in observer, is a paraphrasing of a remark by Truesdell [2].

Because this book has been written over a period of fifteen years, and because of a "fading memory", I have probably neglected to mention many other sources from which I have profited; if so, I apologize.

Finally, I am extremely grateful to Nancy Colmer for her careful and accurate typing (and retyping!) of the manuscript.

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CHAPTER I

Tensor Algebra

1. POINTS. VECTORS. TENSORS

The space under consideration will always be a **three-dimensional euclidean point space** $\mathscr{E}$. The term **point** will be reserved for elements of $\mathscr{E}$, the term **vector** for elements of the associated vector space $\mathscr{V}$. The difference

$$\mathbf{v} = \mathbf{y} - \mathbf{x}$$

of two points is a vector (Fig. 1); the sum

$$\mathbf{y} = \mathbf{x} + \mathbf{v}$$

of a point $\mathbf{x}$ and a vector $\mathbf{v}$ is a point. The sum of two points is *not* a meaningful concept.

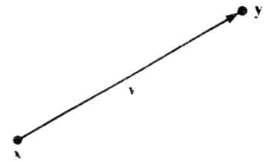

Figure 1

The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ will be designated by $\mathbf{u} \cdot \mathbf{v}$, and we define

$$|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2}, \qquad \mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u}.$$

We use the symbol $\mathbb{R}$ for the reals, $\mathbb{R}^+$ for the strictly positive reals.

Representation Theorem for Linear Forms.[1] *Let $\psi: \mathscr{V} \to \mathbb{R}$ be linear. Then there exists a unique vector* **a** *such that*

$$\psi(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}$$

for every vector **v**.

A **cartesian coordinate frame** consists of an orthonormal basis $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ together with a point **o** called the **origin**. We assume once and for all that a single, fixed cartesian coordinate frame is given. The (cartesian) components of a vector **u** are given by

$$u_i = \mathbf{u} \cdot \mathbf{e}_i,$$

so that

$$\mathbf{u} \cdot \mathbf{v} = \sum_i u_i v_i.$$

Similarly, the coordinates of a point **x** are

$$x_i = (\mathbf{x} - \mathbf{o}) \cdot \mathbf{e}_i.$$

The **span** $\mathrm{sp}\{\mathbf{u}, \mathbf{v}, \dots, \mathbf{w}\}$ of a set $\{\mathbf{u}, \mathbf{v}, \dots, \mathbf{w}\}$ of vectors is the *subspace* of $\mathscr{V}$ consisting of all linear combinations of these vectors:

$$\mathrm{sp}\{\mathbf{u}, \mathbf{v}, \dots, \mathbf{w}\} = \{\alpha\mathbf{u} + \beta\mathbf{v} + \cdots + \gamma\mathbf{w} \,|\, \alpha, \beta, \dots, \gamma \in \mathbb{R}\}.$$

(We will also use this notation for vector spaces other than $\mathscr{V}$.)

Given a vector **v**, we write

$$\{\mathbf{v}\}^\perp = \{\mathbf{u} \,|\, \mathbf{u} \cdot \mathbf{v} = 0\}$$

for the subspace of $\mathscr{V}$ consisting of all vectors perpendicular to **v**.

We use the term **tensor** as a synonym for "linear transformation from $\mathscr{V}$ into $\mathscr{V}$." Thus a tensor **S** is a linear map that assigns to each vector **u** a vector

$$\mathbf{v} = \mathbf{S}\mathbf{u}.$$

The set of all tensors forms a vector space if **addition** and **scalar multiplication** are defined pointwise; that is, $\mathbf{S} + \mathbf{T}$ and $\alpha\mathbf{S}$ ($\alpha \in \mathbb{R}$) are the tensors defined by

$$(\mathbf{S} + \mathbf{T})\mathbf{v} = \mathbf{S}\mathbf{v} + \mathbf{T}\mathbf{v},$$

$$(\alpha\mathbf{S})\mathbf{v} = \alpha(\mathbf{S}\mathbf{v}).$$

The zero element in this space is the **zero tensor 0** which maps every vector **v** into the zero vector:

$$\mathbf{0}\mathbf{v} = \mathbf{0}.$$

[1] Cf., e.g., Halmos [1, §67].

Another important tensor is the **identity I** defined by

$$\mathbf{I}\mathbf{v} = \mathbf{v}$$

for every vector **v**.

The **product ST** of two tensors is the tensor

$$\mathbf{S}\mathbf{T} = \mathbf{S} \circ \mathbf{T};$$

that is,

$$(\mathbf{S}\mathbf{T})\mathbf{v} = \mathbf{S}(\mathbf{T}\mathbf{v})$$

for all **v**. We use the standard notation

$$\mathbf{S}^2 = \mathbf{S}\mathbf{S}, \quad \text{etc.}$$

Generally, $\mathbf{ST} \neq \mathbf{TS}$. If $\mathbf{ST} = \mathbf{TS}$, we say that **S** and **T** **commute.**

We write $\mathbf{S}^T$ for the **transpose** of **S**; $\mathbf{S}^T$ is the unique tensor with the property

$$\mathbf{S}\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{S}^T\mathbf{v}$$

for all vectors **u** and **v**. It then follows that

$$\begin{aligned} (\mathbf{S} + \mathbf{T})^T &= \mathbf{S}^T + \mathbf{T}^T, \\ (\mathbf{S}\mathbf{T})^T &= \mathbf{T}^T\mathbf{S}^T, \\ (\mathbf{S}^T)^T &= \mathbf{S}. \end{aligned} \tag{1}$$

A tensor **S** is **symmetric** if

$$\mathbf{S} = \mathbf{S}^T,$$

skew if

$$\mathbf{S} = -\mathbf{S}^T.$$

Every tensor **S** can be expressed uniquely as the sum of a symmetric tensor **E** and a skew tensor **W**:

$$\mathbf{S} = \mathbf{E} + \mathbf{W};$$

in fact,

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{S} + \mathbf{S}^T),$$

$$\mathbf{W} = \tfrac{1}{2}(\mathbf{S} - \mathbf{S}^T).$$

We call **E** the **symmetric part** of **S**, **W** the **skew part** of **S**.

The **tensor product a ⊗ b** of two vectors **a** and **b** is the *tensor* that assigns to each vector **v** the vector $(\mathbf{b}\cdot\mathbf{v})\mathbf{a}$:

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{v} = (\mathbf{b}\cdot\mathbf{v})\mathbf{a}.$$

Then

$$(\mathbf{a}\otimes\mathbf{b})^{\mathrm{T}} = (\mathbf{b}\otimes\mathbf{a}),$$

$$(\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d}) = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}\otimes\mathbf{d},$$

$$(\mathbf{e}_i\otimes\mathbf{e}_i)(\mathbf{e}_j\otimes\mathbf{e}_j) = \begin{cases}\mathbf{0}, & i\neq j\\ \mathbf{e}_i\otimes\mathbf{e}_i, & i=j,\end{cases} \tag{2}$$

$$\sum_i \mathbf{e}_i\otimes\mathbf{e}_i = \mathbf{I}.$$

Let **e** be a unit vector. Then $\mathbf{e}\otimes\mathbf{e}$ applied to a vector **v** gives

$$(\mathbf{v}\cdot\mathbf{e})\mathbf{e},$$

which is the projection of **v** in the direction of **e**, while $\mathbf{I}-\mathbf{e}\otimes\mathbf{e}$ applied to **v** gives

$$\mathbf{v} - (\mathbf{v}\cdot\mathbf{e})\mathbf{e},$$

which is the projection of **v** onto the plane perpendicular to **e** (Fig. 2).

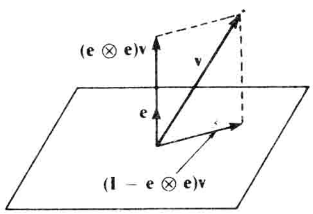

Figure 2

The components S_{ij} of a tensor **S** are defined by

$$S_{ij} = \mathbf{e}_i\cdot\mathbf{S}\mathbf{e}_j.$$

With this definition $\mathbf{v} = \mathbf{S}\mathbf{u}$ is equivalent to

$$v_i = \sum S_{ij}u_j.$$

Further,

$$\mathbf{S} = \sum S_{ij}\mathbf{e}_i\otimes\mathbf{e}_j \tag{3}$$

and

$$(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j.$$

We write [S] for the matrix

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}.$$

It then follows that

$$[\mathbf{S}^{\mathrm{T}}] = [\mathbf{S}]^{\mathrm{T}},$$
$$[\mathbf{ST}] = [\mathbf{S}][\mathbf{T}],$$

and

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The **trace** is the linear operation that assigns to each tensor $\mathbf{S}$ a scalar $\operatorname{tr} \mathbf{S}$ and satisfies

$$\operatorname{tr}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$. By (3) and the linearity of tr,

$$\begin{aligned} \operatorname{tr} \mathbf{S} = \operatorname{tr}\left(\sum_{i,j} S_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \right) &= \sum_{i,j} S_{ij} \operatorname{tr}(\mathbf{e}_i \otimes \mathbf{e}_j) \\ &= \sum_{i,j} S_{ij} \mathbf{e}_i \cdot \mathbf{e}_j = \sum_i S_{ii}. \end{aligned}$$

Thus the trace is well defined:

$$\operatorname{tr} \mathbf{S} = \sum_i S_{ii}.$$

This operation has the following properties:

$$\begin{aligned} \operatorname{tr} \mathbf{S}^{\mathrm{T}} &= \operatorname{tr} \mathbf{S}, \\ \operatorname{tr}(\mathbf{ST}) &= \operatorname{tr}(\mathbf{TS}). \end{aligned} \tag{4}$$

The space of all tensors has a natural **inner product**

$$\mathbf{S} \cdot \mathbf{T} = \operatorname{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}),$$

which in components has the form

$$\mathbf{S} \cdot \mathbf{T} = \sum_{i,j} S_{ij} T_{ij}.$$

Then

$$\begin{aligned} \mathbf{I}\cdot\mathbf{S} &= \operatorname{tr}\mathbf{S}, \\ \mathbf{R}\cdot(\mathbf{ST}) &= (\mathbf{S}^{\mathrm{T}}\mathbf{R})\cdot\mathbf{T} = (\mathbf{RT}^{\mathrm{T}})\cdot\mathbf{S}, \\ \mathbf{u}\cdot\mathbf{Sv} &= \mathbf{S}\cdot(\mathbf{u}\otimes\mathbf{v}), \\ (\mathbf{a}\otimes\mathbf{b})\cdot(\mathbf{u}\otimes\mathbf{v}) &= (\mathbf{a}\cdot\mathbf{u})(\mathbf{b}\cdot\mathbf{v}). \end{aligned} \tag{5}$$

More important is the following

Proposition

(a) *If* $\mathbf{S}$ *is symmetric,*

$$\mathbf{S}\cdot\mathbf{T} = \mathbf{S}\cdot\mathbf{T}^{\mathrm{T}} = \mathbf{S}\cdot\{\tfrac{1}{2}(\mathbf{T}+\mathbf{T}^{\mathrm{T}})\}.$$

(b) *If* $\mathbf{W}$ *is skew,*

$$\mathbf{W}\cdot\mathbf{T} = -\mathbf{W}\cdot\mathbf{T}^{\mathrm{T}} = \mathbf{W}\cdot\{\tfrac{1}{2}(\mathbf{T}-\mathbf{T}^{\mathrm{T}})\}.$$

(c) *If* $\mathbf{S}$ *is symmetric and* $\mathbf{W}$ *skew,*

$$\mathbf{S}\cdot\mathbf{W} = 0.$$

(d) *If* $\mathbf{T}\cdot\mathbf{S} = 0$ *for every tensor* $\mathbf{S}$, *then* $\mathbf{T} = \mathbf{0}$.
(e) *If* $\mathbf{T}\cdot\mathbf{S} = 0$ *for every symmetric* $\mathbf{S}$, *then* $\mathbf{T}$ *is skew.*
(f) *If* $\mathbf{T}\cdot\mathbf{W} = 0$ *for every skew* $\mathbf{W}$, *then* $\mathbf{T}$ *is symmetric.*

We define the **determinant** of a tensor $\mathbf{S}$ to be the determinant of the matrix $[\mathbf{S}]$:

$$\det\mathbf{S} = \det[\mathbf{S}].$$

This definition is independent of our choice of basis $\{\mathbf{e}_i\}$.

A tensor $\mathbf{S}$ is **invertible** if there exists a tensor $\mathbf{S}^{-1}$, called the inverse of $\mathbf{S}$, such that

$$\mathbf{SS}^{-1} = \mathbf{S}^{-1}\mathbf{S} = \mathbf{I}.$$

It follows that $\mathbf{S}$ is invertible if and only if $\det\mathbf{S} \neq 0$.

The identities

$$\begin{aligned} \det(\mathbf{ST}) &= (\det\mathbf{S})(\det\mathbf{T}), \\ \det\mathbf{S}^{\mathrm{T}} &= \det\mathbf{S}, \\ \det(\mathbf{S}^{-1}) &= (\det\mathbf{S})^{-1}, \\ (\mathbf{ST})^{-1} &= \mathbf{T}^{-1}\mathbf{S}^{-1}, \\ (\mathbf{S}^{-1})^{\mathrm{T}} &= (\mathbf{S}^{\mathrm{T}})^{-1} \end{aligned} \tag{6}$$

will be useful. For convenience, we use the abbreviation

$$\mathbf{S}^{-\mathrm{T}} = (\mathbf{S}^{-1})^{\mathrm{T}}.$$

A tensor $\mathbf{Q}$ is **orthogonal** if it preserves inner products:

$$\mathbf{Qu} \cdot \mathbf{Qv} = \mathbf{u} \cdot \mathbf{v}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$. A necessary and sufficient condition that $\mathbf{Q}$ be orthogonal is that

$$\mathbf{QQ}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I},$$

or equivalently,

$$\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}.$$

An orthogonal tensor with positive determinant is called a **rotation**. (Rotations are sometimes called proper orthogonal tensors.) Every orthogonal tensor is either a rotation or the product of a rotation with $-\mathbf{I}$. If $\mathbf{R} \neq \mathbf{I}$ is a rotation, then the set of all vectors $\mathbf{v}$ such that

$$\mathbf{Rv} = \mathbf{v}$$

forms a one-dimensional subspace of $\mathscr{V}$ called the **axis** of $\mathbf{R}$.

A tensor $\mathbf{S}$ is **positive definite** provided

$$\mathbf{v} \cdot \mathbf{Sv} > 0$$

for all vectors $\mathbf{v} \neq \mathbf{0}$.

Throughout this book we will use the following notation:

Lin = the set of all tensors;
Lin^+ = the set of all tensors $\mathbf{S}$ with $\det \mathbf{S} > 0$;
Sym = the set of all symmetric tensors;
Skw = the set of all skew tensors;
Psym = the set of all symmetric, positive definite tensors;
Orth = the set of all orthogonal tensors;
Orth^+ = the set of all rotations.

The sets Lin^+, Orth, and Orth^+ are *groups* under multiplication; in fact, Orth^+ is a subgroup of both Orth and Lin^+. Orth is the *orthogonal group*; Orth^+ is the *rotation group* (proper orthogonal group).

On any three-dimensional vector space there are exactly two cross products, and one is the negative of the other. We assume that one such cross product, written

$$\mathbf{u} \times \mathbf{v}$$

for all **u** and **v**, has been singled out. Intuitively, $\mathbf{u} \times \mathbf{v}$ will represent the right-handed cross product of **u** and **v**; thus if

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2,$$

then the basis $\{\mathbf{e}_i\}$ is right handed and the components of $\mathbf{u} \times \mathbf{v}$ relative to $\{\mathbf{e}_i\}$ are

$$u_2 v_3 - u_3 v_2, \qquad u_3 v_1 - u_1 v_3, \qquad u_1 v_2 - u_2 v_1.$$

Further,

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u},$$

$$\mathbf{u} \times \mathbf{u} = \mathbf{0},$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}).$$

When **u**, **v**, and **w** are linearly independent, the magnitude of the scalar

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

represents the volume of the parallelepiped $\mathscr{P}$ determined by **u**, **v**, **w**. Further,[1]

$$\det \mathbf{S} = \frac{\mathbf{S}\mathbf{u} \cdot (\mathbf{S}\mathbf{v} \times \mathbf{S}\mathbf{w})}{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}$$

and hence

$$|\det \mathbf{S}| = \frac{\operatorname{vol}(\mathbf{S}(\mathscr{P}))}{\operatorname{vol}(\mathscr{P})},$$

which gives a geometrical interpretation of the determinant (Fig. 3). Here $\mathbf{S}(\mathscr{P})$ is the image of $\mathscr{P}$ under **S**, and vol designates the volume.

There is a one-to-one correspondence between vectors and skew tensors: given any skew tensor **W** there exists a unique vector **w** such that

$$\mathbf{W}\mathbf{v} = \mathbf{w} \times \mathbf{v} \tag{7}$$

for every **v**, and conversely; indeed,

$$[\mathbf{W}] = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}$$

corresponds to

$$w_1 = \alpha, \qquad w_2 = \beta, \qquad w_3 = \gamma.$$

[1] Cf., e.g., Nickerson, Spencer, and Steenrod [1, §5.2].

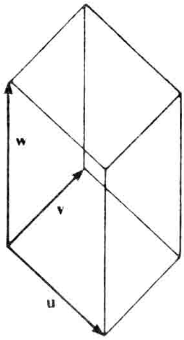

Figure 3

We call $\mathbf{w}$ the **axial vector** corresponding to $\mathbf{W}$. It follows from (7) that (for $\mathbf{W} \neq \mathbf{0}$) the null space of $\mathbf{W}$, that is the set of all $\mathbf{v}$ such that

$$\mathbf{W}\mathbf{v} = \mathbf{0},$$

is equal to the *one-dimensional* subspace spanned by $\mathbf{w}$. This subspace is called **axis** of $\mathbf{W}$.

We will frequently use the facts that $\mathscr{V}$ and Lin are normed vector spaces and that the standard operations of tensor analysis are continuous. In particular, on $\mathscr{V}$ and Lin, the sum, inner product, and scalar product are continuous, as are the tensor product on $\mathscr{V}$ and the product, trace, transpose, and determinant on suitable subsets of Lin.

Exercises

1. Choose $\mathbf{a} \in \mathscr{V}$ and let $\psi: \mathscr{V} \to \mathbb{R}$ be defined by $\psi(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}$. Show that $\mathbf{a} = \sum_i \psi(\mathbf{e}_i)\mathbf{e}_i$.
2. Prove the representation theorem for linear forms (page 1).
3. Show that the sum $\mathbf{S} + \mathbf{T}$ and product $\mathbf{ST}$ are tensors.
4. Establish the existence and uniqueness of the transpose $\mathbf{S}^T$ of $\mathbf{S}$.
5. Show that the tensor product $\mathbf{a} \otimes \mathbf{b}$ is a tensor.
6. Prove that

   (a) $\mathbf{S}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{Sa}) \otimes \mathbf{b}$,
   (b) $(\mathbf{a} \otimes \mathbf{b})\mathbf{S} = \mathbf{a} \otimes (\mathbf{S}^T\mathbf{b})$.
   (c) $\sum_i (\mathbf{S}\mathbf{e}_i) \otimes \mathbf{e}_i = \mathbf{S}$.
7. Establish (1), (2), (4), and (5).