

# COLLEGE ALGEBRA

H. T. DAVIS

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by

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NORTHWESTERN UNIVERSITY

WITH REVISIONS AND ADDITIONAL PROBLEMS

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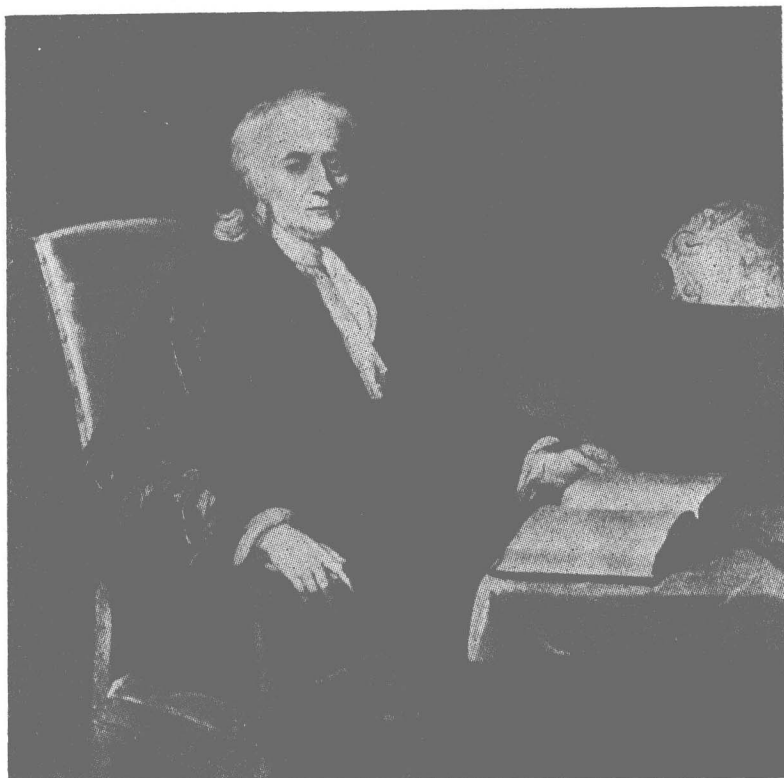
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# COLLEGE ALGEBRA



SIR ISAAC NEWTON (1642-1727)

## P R E F A C E

This text on college algebra presents in the first sixteen chapters the material usually treated in a first course in American colleges and universities. Since most students welcome a review of their earlier studies, the first three chapters develop the subject from the beginning. The concept of number is explained and deepened. Operations with polynomials, the use of simple identities in factoring, the combination of fractions, the application of simple equations, and other elementary topics are introduced and explained without the assumption of previous knowledge on the part of the student. In this way, preparation is made for an understanding of the more advanced topics which constitute the basis of a college course in algebra.

In recent years, the fact that mathematical teaching gains much from an intimate association with historical origins has been realized. Thus we find in the final report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics the following statement:

"The Commission believes that it should stress one subject, namely the history of mathematics. If the study of secondary mathematics is to reveal mathematics as one of the fundamental enterprises of man, which, though rooted in daily need, is an expression of deep, irrepressible, and idealistic impulses, then the teaching of it should constantly be associated with its history. One recalls the statement of Glaisher, 'I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history.' ""\*

With this conclusion the author of the present text is in full accord. Consequently unusual attention has been given

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\* *Fifteenth Yearbook of the National Council of Teachers of Mathematics: The Place of Mathematics in Secondary Education*, p. 197. Edited by W. D. Reeve Columbia University, New York City, 1940.

to the historical incidents which marked the development of algebra. Numerous references to the history of the subject have been interpolated in the text, and the final chapter is devoted to a discussion of the meaning of mathematics. A brief account is given of a few of the contributions of mathematics to the development of knowledge. Biographical sketches of some of the greatest mathematicians are included, and pictures of a few of them are given in the text.

Recognition of the growing interest of the social sciences in algebra is made by the inclusion of a chapter on statistics. This material is related to the graphing of data, to the binomial theorem, and to other topics treated earlier in the book.

A course in algebra may be illuminated by reference to some of the problems included under the general title of "Mathematical Recreations." A chapter on this subject provides ample material for use in connection with other topics in the book.

An elementary course must provide among other things a systematic drill in operational techniques. Numerous examples have been included to provide the necessary instruction in the processes described, and extensive lists of problems supply the student with material for testing his manipulative skill.

#### PREFACE TO FIFTH PRINTING

In response to requests for more problems for classroom drill in certain parts of the text, this new edition has been provided. Some thousand new problems have been added and the supplementary lists for review have been incorporated in the body of the text and extended in their scope. In recognition of the need for problems illustrating the application of the algebraic processes to various domains of science, a number of new exercises of this character have been added.

The opportunity has also been used to enlarge several sections of the text and to include certain topics, which, although usually omitted in a course of three or four semester-hours, are desired in longer courses.



## SIGNS AND SYMBOLS USED IN ALGEBRA

- $+$ , sign of addition, to be read *plus*;  $-$ , sign of subtraction, to be read *minus*.  
 $\times, \cdot$ , signs of multiplication, to be read *times*;  $\div$ , sign of division, to be read *divided by*. The solidus,  $/$ , is also used to indicate division.  
 $=$ , denotes equality, and is read *is equal to*;  $\equiv$  is the identity symbol, and is read, *is identical with*.  
 $\neq$ , the sign of inequality; to be read, *is not equal to*.  
 $>$ , to be read, *is greater than*;  $<$ , to be read, *is less than*.  
 $\geq$ , to be read, *is greater than or equal to*;  $\leq$ , to be read, *is less than or equal to*.  
 $( )$ , parentheses;  $[ ]$ , brackets;  $\{ \}$ , braces.  
 $|a|$ , denotes the absolute value of  $a$ .  
 $\sqrt{a}$ , means the square root of  $a$ ;  $\sqrt[n]{a}$ , denotes the  $n$ th root of  $a$ .  
 $n!$ , means *factorial n*. This symbol is sometimes written  $|n$ .  
 ${}_nC_r$ , denotes both the *r*th *binomial coefficient* and the *number of combinations of n things taken r at a time*.  
 ${}_nP_r$ , denotes the *number of permutations of n things taken r at a time*.  
 $s_n$ , denotes the *amount of unit principal at compound interest*, and  $v^n$  the *discounted amount of unit principal at compound interest*.  
 $s_{\overline{n}|i}$ , denotes the *amount of an annuity of 1*, and  $a_{\overline{n}|i}$  the *present value of an annuity of 1*.  
 $\rightarrow$ , to be read *approaches*. For example,  $x \rightarrow a$ , means  $x$  approaches  $a$ .  
 $\lim_{x=a}$ , means the limit as  $x$  approaches  $a$ .  
 $f(x)$ , denotes a *function of x*; to be read,  $f$  is a *function of x*.  
 $(x, y)$  designates a *point whose coordinates are x and y*.

## GREEK ALPHABET

LETTERS	NAMES	LETTERS	NAMES	LETTERS	NAMES
A $\alpha$	Alpha	I $\iota$	Iota	P $\rho$	Rho
B $\beta$	Beta	K $\kappa$	Kappa	$\Sigma$ $\sigma$ s	Sigma
$\Gamma$ $\gamma$	Gamma	$\Lambda$ $\lambda$	Lambda	T $\tau$	Tau
$\Delta$ $\delta$	Delta	M $\mu$	Mu	$\Upsilon$ $\upsilon$	Upsilon
E $\epsilon$	Epsilon	N $\nu$	Nu	$\Phi$ $\phi$	Phi
Z $\zeta$	Zeta	$\Xi$ $\xi$	Xi	X $\chi$	Chi
H $\eta$	Eta	O $\omicron$	Omicron	$\Psi$ $\psi$	Psi
$\Theta$ $\theta$	Theta	$\Pi$ $\pi$	Pi	$\Omega$ $\omega$	Omega



# CONTENTS

CHAPTER	PAGE
I. PRELIMINARY TOPICS . . . . .	1
1. Algebra—its purposes and historical develop- ment . . . . .	1
2. Numbers—the integers . . . . .	3
3. Rational and irrational numbers . . . . .	6
4. Negative numbers . . . . .	8
5. Elementary operations . . . . .	10
6. The use of parentheses . . . . .	11
7. Positive integral exponents . . . . .	13
8. The multiplication of polynomials . . . . .	13
9. The division of polynomials . . . . .	15
II. THE PROCESSES OF ALGEBRA . . . . .	19
1. Some fundamental axioms . . . . .	19
2. Use of the axioms in simplifying equations . . . . .	20
3. Problems involving equations . . . . .	25
4. Identities and equations of condition . . . . .	30
5. Some fundamental identities . . . . .	32
6. Factoring . . . . .	34
7. Completing the square—factors of $px^2 + qx + r$ . . . . .	36
8. Fractions . . . . .	38
9. The addition of fractions . . . . .	43
10. Historical note on fractions . . . . .	47
III. THE LAWS OF EXPONENTS AND LOGARITHMS . . . . .	51
1. Introduction . . . . .	51
2. The laws of exponents . . . . .	51
3. Fractional exponents . . . . .	55
4. Rationalization of denominators . . . . .	60
5. Logarithms . . . . .	65
6. The laws of logarithms . . . . .	67
7. Tables of logarithms . . . . .	70
8. Determination of logarithms . . . . .	74
9. Computation by logarithms . . . . .	77
10. Conversion of logarithms from one base to another . . . . .	82

CHAPTER		PAGE
IV.	ARITHMETICAL AND GEOMETRICAL PROGRESSIONS.....	84
	1. Series.....	84
	2. Arithmetical progressions.....	84
	3. Geometrical progressions.....	88
	4. Infinite geometrical progressions.....	92
V.	FUNCTIONAL RELATIONSHIPS—THE EQUATION.....	99
	1. The function concept.....	99
	2. Coordinate axes.....	103
	3. The graphical representation of functions.....	105
	4. The graphical representation of data.....	108
	5. Zeros and infinities of functions.....	113
VI.	THE LINEAR FUNCTION—LINEAR EQUATIONS.....	117
	1. The linear function.....	117
	2. Fitting a straight line to data.....	121
	3. The use of tables in fitting a straight line to data.....	123
	4. Systems of linear equations.....	125
	5. Solution of systems of equations by determinants.....	129
	6. Linear equations in three unknowns.....	131
VII.	QUADRATIC EQUATIONS—THE QUADRATIC FUNCTION.....	136
	1. Roots of the quadratic equation.....	136
	2. The sum and product of the roots.....	140
	3. A note on complex numbers.....	142
	4. Classification of the roots of a quadratic equation.....	147
	5. The quadratic function.....	149
	6. The method of least squares.....	153
	7. The graphical solution of quadratic equations... ..	155
	8. Irrational equations reducible to quadratic equations.....	156
	9. Simultaneous equations involving quadratics... ..	158
	10. Simultaneous systems in which both equations are quadratics.....	162
VIII.	THE BINOMIAL THEOREM—MATHEMATICAL INDUCTION.....	166
	1. The binomial formula.....	166
	2. A note on algebraic proofs—mathematical induction.....	166

VIII. THE BINOMIAL THEOREM (*Cont.*)

3. Examples illustrating mathematical induction . . .	168
4. Statement of the binomial formula . . . . .	171
5. The general term of the binomial formula . . . . .	172
6. The binomial series . . . . .	175
7. The number $e$ —the exponential series . . . . .	178
8. The binomial theorem . . . . .	182
9. Proof of the binomial theorem for positive integral values of the exponent . . . . .	182

IX. THE THEORY OF INVESTMENT . . . . . 185

1. The functions of investment . . . . .	185
2. Compound interest . . . . .	186
3. Present value . . . . .	188
4. Nominal and effective rates of interest . . . . .	189
5. Annuities . . . . .	191
6. Bonds . . . . .	194
7. Perpetuities . . . . .	196

X. PERMUTATIONS AND COMBINATIONS—PROB-  
ABILITY . . . . . 200

1. Permutations . . . . .	200
2. Combinations . . . . .	204
3. Probability . . . . .	206
4. The multiplication and addition of probabilities . . . . .	209
5. Empirical probability . . . . .	213

XI. THE THEORY OF PROPORTION—THE RATIOS  
OF TRIGONOMETRY . . . . . 218

1. Some elementary theorems of proportion . . . . .	218
2. Variation . . . . .	222
3. The trigonometric ratios . . . . .	226
4. The measurement of angles . . . . .	229
5. The functions of any acute angle . . . . .	230
6. The functions of obtuse angles . . . . .	234
7. Inverse trigonometric functions . . . . .	237
8. Trigonometric formulas . . . . .	239

XII. COMPLEX NUMBERS . . . . . 241

1. The complex number system . . . . .	241
2. The graphical representation of complex numbers . . . . .	241
3. The polar representation of complex numbers . . . . .	242
4. The addition and subtraction of complex numbers . . . . .	245

XII. COMPLEX NUMBERS (*Cont.*)

- |  |     |
|--|-----|
| 5. The multiplication and division of complex numbers..... | 245 |
| 6. Demoivre's theorem.....                                 | 248 |
| 7. Roots of complex numbers.....                           | 250 |

## XIII. THE THEORY OF EQUATIONS..... 255

- |  |     |
|--|-----|
| 1. The problem.....  | 255 |
| 2. The remainder and factor theorems.....                                      | 258 |
| 3. Synthetic division.....   | 260 |
| 4. The fundamental theorem of algebra.....                                     | 261 |
| 5. The relationship between the roots and the coefficients of an equation..... | 264 |
| 6. Transformation of the roots of an equation....                              | 266 |
| 7. Descartes' rule of signs.....   | 269 |
| 8. The location of the real roots of an equation....                           | 271 |
| 9. Horner's method for finding the real roots of an equation.....              | 273 |
| 10. Tartaglia's solution of the cubic equation.....                            | 277 |
| 11. Ferrari's solution of the quartic.....                                     | 281 |

## XIV. DETERMINANTS..... 284

- |  |     |
|--|-----|
| 1. Definitions.....  | 284 |
| 2. Properties of determinants.....   | 285 |
| 3. Minors.....   | 288 |
| 4. Expansion of a determinant by minors.....                                 | 288 |
| 5. A note on the evaluation of determinants.....                             | 290 |
| 6. Systems of linear equations containing $n$ equations in $n$ unknowns..... | 292 |
| 7. Homogeneous equations.....  | 296 |
| 8. Systems of equations containing fewer unknowns than equations.....        | 298 |
| 9. Systems of equations containing fewer equations than unknowns.....        | 299 |

## XV. INEQUALITIES..... 301

- |   |     |
|---|-----|
| 1. Definitions.....                       | 301 |
| 2. Properties of inequalities.....        | 301 |
| 3. Linear and quadratic inequalities..... | 302 |
| 4. Special inequalities.....              | 304 |

# CONTENTS

xi

CHAPTER		PAGE
XVI.	INFINITE SERIES . . . . .	306
	1. Examples and definition . . . . .	306
	2. Limits . . . . .	308
	3. The convergence and divergence of infinite series . . . . .	310
	4. The comparison test . . . . .	314
	5. The ratio test . . . . .	318
	6. Extension of the ratio test . . . . .	321
	7. Power series . . . . .	324
XVII.	INTRODUCTION TO STATISTICS . . . . .	328
	1. Historical note . . . . .	328
	2. Frequency distributions . . . . .	329
	3. The arithmetic average, or mean . . . . .	330
	4. The standard deviation . . . . .	331
	5. Computation of $A$ and $\sigma$ . . . . .	332
	6. Binomial frequency distribution . . . . .	334
	7. The normal frequency curve . . . . .	337
	8. The purpose of averages . . . . .	339
	9. The correlation coefficient . . . . .	341
	10. Lines of regression . . . . .	344
XVIII.	SPECIAL TOPICS . . . . .	347
	1. Euclid's algorithm . . . . .	347
	2. Continued fractions . . . . .	350
	3. Partial fractions . . . . .	354
	4. The extraction of roots of positive numbers . . . . .	358
	5. Interpolation . . . . .	362
XIX.	MATHEMATICAL RECREATIONS . . . . .	367
	1. Mathematical recreations . . . . .	367
	2. Scales of notation . . . . .	367
	3. The binary scale . . . . .	370
	4. The game of Nim . . . . .	372
	5. Criteria for divisibility . . . . .	374
	6. Prime numbers . . . . .	376
	7. Magic squares . . . . .	384
	8. Mathematical concepts of infinity . . . . .	391
	9. The four-color problem . . . . .	395
	10. Trisection of an angle . . . . .	395
	11. Squaring the circle . . . . .	398

CHAPTER		PAGE
XIX.	MATHEMATICAL RECREATIONS ( <i>Cont.</i> )	
12.	Duplication of the cube.....	398
13.	Paradromic rings.....	400
14.	The sailors and the coconuts.....	400
15.	Pythagorean triangles.....	403
XX.	THE MEANING OF MATHEMATICS.....	405
1.	Why study mathematics?.....	405
2.	Is mathematics a tool?.....	407
3.	The history of mathematics.....	409
4.	Mathematical heroes.....	411
5.	The achievement of mathematics.....	422
ANSWERS.	.....	439
INDEX.	.....	463

# TABLES

TABLE		PAGE
I.	Four-Place Table of Logarithms . . . . .	72, 428
II.	Table of Sines, Cosines, and Tangents . . . . .	232, 430
III.	Table of the Logarithms of the Sine, Cosine, and Tangent . . . . .	233, 431
IV.	Powers and Roots . . . . .	432
V.	American Experience Table of Mortality . . . . .	433
VI.	$s_n = (1 + i)^n$ (Amount of 1 at Compound Interest) . . .	434
VII.	$v^n = (1 + i)^{-n}$ (Present Value of 1 at Compound Interest) . . . . .	435
VIII.	$s_{\overline{n} } = \frac{(1 + i)^n - 1}{i}$ (Amount of an Annuity of 1) . . . . .	436
IX.	$a_{\overline{n} } = \frac{1 - (1 + i)^{-n}}{i}$ (Present Value of an Annuity of 1) . . .	437
X.	Coefficients for Fitting Straight Line . . . . .	438



## CHAPTER I

### PRELIMINARY TOPICS

1. **Algebra—its purposes and historical development.** Algebra is the branch of mathematics that reasons about quantities by means of letters and other symbols. It is essentially a generalized form of common arithmetic in which symbols are used to represent numbers, and specific processes of arithmetic are stated more comprehensively. The word **algebra** is derived from the Arabic word *al-jebr*, which means *restoration*—that is to say, the restoration of an equation by transposing negative terms from one side to the other. The first book in which the word *al-jebr* was known to have been used was written by Mohammed ibn Mûsâ al-Khowârizmî, who lived during the reign of the Caliph Ma'mûn (813–833). The name of this author was in later years contracted to Algoritmi, from which we derive the word **algorithm**, which means the art of computing by some special method or process.

The subject of algebra originated in the very distant past. We find that simple problems appeared in an Egyptian papyrus written by Ahmes some time before 1700 B.C., and this work is believed to have been founded on an earlier work that may date as far back as 3400 B.C. Both arithmetical and geometrical progressions, which we shall study later in this book, are found in the Ahmes' papyrus. For example, the author requires that 100 loaves be so divided among five people that the number received will decrease from person to person by a constant amount and that the last two will receive one seventh of what the first three get. The student may verify the answer:  $38\frac{1}{3}$ ,  $29\frac{1}{6}$ , 20,  $10\frac{5}{6}$ ,  $1\frac{2}{3}$ . In this ancient manuscript also appears a problem which historians have interpreted as equivalent to the following: 7 people each have 7 cats; each cat eats 7 mice; each mouse eats 7 ears of barley; from each ear, 7 measures of barley

may grow. What is the total number of people, cats, mice, ears of barley, and measures of barley? The student may verify Ahmes' answer of 19,607.

Among the Greeks the only algebraist of significance was Diophantus. His death occurred about 300 A.D. We know that he lived to be 84 years old from the following epitaph: Diophantus passed one sixth of his life in childhood, one twelfth in youth, and one seventh more as a bachelor; five years after his marriage, a son was born; the son died four years before his father at half his father's age. Diophantus wrote a treatise, entitled *Arithmetica*, in 13 books, of which seven are extant. This work is largely confined to problems in algebra and the theory of integers. He solved a special case of quadratic equations, but rejected both negative and irrational numbers.

Although the roots of some of the problems in algebra are to be found in Arabian and Hindu mathematics, and although some sporadic attempts to develop algebra were made in the Middle Ages (notably by Leonardo of Pisa, who flourished around 1200), the beginnings of algebra in any modern sense are found in the sixteenth century. The *Ars Magna* of G. Cardano (1501–1576), which was published in 1545, laid the foundations for the general theory of algebraic equations; the germs of the theory of exponents and of logarithms are to be found in the *Arithmetica Integra* of Michael Stifel (1486?–1567), which appeared in Germany in 1544.

It is a significant fact that the development of algebra and the development of science were simultaneous. Thus François Vieta (1540–1603), in his numerous writings on algebra, was introducing the use of letters for algebraical quantities, while Tycho Brahe (1546–1601), his contemporary, was collecting the first modern set of observations of the motions of the planets. John Napier (1550–1617), Baron of Merchiston in Scotland, published his theory of logarithms in 1614, and Henry Briggs (1561–1630) produced the first table of common logarithms in 1624—works contemporary with the discoveries of Johann Kepler (1571–1630), the astronomer. The *Géométrie* of René Descartes (1596–1650), which connected the appar-