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THEORY and PROBLEMS

of

MATRICES

by FRANK AYRES, JR.

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SCHAUM'S OUTLINE SERIES
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THEORY AND PROBLEMS

OF

MATRICES

BY

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Preface

Elementary matrix algebra has now become an integral part of the mathematical background necessary for such diverse fields as electrical engineering and education, chemistry and sociology, as well as for statistics and pure mathematics. This book, in presenting the more essential material, is designed primarily to serve as a useful supplement to current texts and as a handy reference book for those working in the several fields which require some knowledge of matrix theory. Moreover, the statements of theory and principle are sufficiently complete that the book could be used as a text by itself.

The material has been divided into twenty-six chapters, since the logical arrangement is thereby not disturbed while the usefulness as a reference book is increased. This also permits a separation of the treatment of real matrices, with which the majority of readers will be concerned, from that of matrices with complex elements. Each chapter contains a statement of pertinent definitions, principles, and theorems, fully illustrated by examples. These, in turn, are followed by a carefully selected set of solved problems and a considerable number of supplementary exercises.

The beginning student in matrix algebra soon finds that the solutions of numerical exercises are disarmingly simple. Difficulties are likely to arise from the constant round of definition, theorem, proof. The trouble here is essentially a matter of lack of mathematical maturity, and normally to be expected, since usually the student's previous work in mathematics has been concerned with the solution of numerical problems while precise statements of principles and proofs of theorems have in large part been deferred for later courses. The aim of the present book is to enable the reader, if he persists through the introductory paragraphs and solved problems in any chapter, to develop a reasonable degree of self-assurance about the material.

The solved problems, in addition to giving more variety to the examples illustrating the theorems, contain most of the proofs of any considerable length together with representative shorter proofs. The supplementary problems call both for the solution of numerical exercises and for proofs. Some of the latter require only proper modifications of proofs given earlier; more important, however, are the many theorems whose proofs require but a few lines. Some are of the type frequently misnamed "obvious" while others will be found to call for considerable ingenuity. None should be treated lightly, however, for it is due precisely to the abundance of such theorems that elementary matrix algebra becomes a natural first course for those seeking to attain a degree of mathematical maturity. While the large number of these problems in any chapter makes it impractical to solve all of them before moving to the next, special attention is directed to the supplementary problems of the first two chapters. A mastery of these will do much to give the reader confidence to stand on his own feet thereafter.

The author wishes to take this opportunity to express his gratitude to the staff of the Schaum Publishing Company for their splendid cooperation.

FRANK AYRES, JR.

Carlisle, Pa.
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Chapter 1

Matrices

A RECTANGULAR ARRAY OF NUMBERS enclosed by a pair of brackets, such as

$$(a)\ \begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \quad \text{and} \quad (b)\ \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix},$$

and subject to certain rules of operations given below is called a **matrix**. The matrix (a) could be considered as the **coefficient matrix** of the system of homogeneous linear equations $\begin{cases} 2x + 3y + 7z = 0 \\ x - y + 5z = 0 \end{cases}$ or as the **augmented** matrix of the system of non-homogeneous linear equations $\begin{cases} 2x + 3y = 7 \\ x - y = 5 \end{cases}$ Later, we shall see how the matrix may be used to obtain solutions of these systems. The matrix (b) could be given a similar interpretation or we might consider its rows as simply the coordinates of the points $(1,3,1)$, $(2,1,4)$, and $(4,7,6)$ in ordinary space. The matrix will be used later to settle such questions as whether or not the three points lie in the same plane with the origin or on the same line through the origin.

In the matrix

$$\textbf{(1.1)} \qquad \begin{bmatrix} a_{11}\ a_{12}\ a_{13} \cdots\cdots\ a_{1n} \\ a_{21}\ a_{22}\ a_{23} \cdots\cdots\ a_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}\ a_{m2}\ a_{m3} \cdots\cdots\ a_{mn} \end{bmatrix}$$

the numbers or functions a_{ij} are called its **elements**. In the double subscript notation, the first subscript indicates the row and the second subscript indicates the column in which the element stands. Thus, all elements in the second row have 2 as first subscript and all the elements in the fifth column have 5 as second subscript. A matrix of m rows and n columns is said to be of order "m by n" or $m \times n$.

(In indicating a matrix pairs of parentheses, ( ), and double bars, $\|\ \|$, are sometimes used. We shall use the double bracket notation throughout.)

At times the matrix **(1.1)** will be called "the $m \times n$ matrix $[a_{ij}]$" or "the $m \times n$ matrix $A = [a_{ij}]$". When the order has been established, we shall write simply "the matrix A".

SQUARE MATRICES. When $m = n$, **(1.1)** is square and will be called a **square matrix** of order n or an n-square matrix.

In a square matrix, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called its **diagonal elements**.

The sum of the diagonal elements of a square matrix A is called the **trace** of A.

EQUAL MATRICES. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** ($A = B$) if and only if they have the same order and each element of one is equal to the corresponding element of the other, that is, if and only if

$$a_{ij} = b_{ij}, \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$$

Thus, two matrices are equal if and only if one is a duplicate of the other.

ZERO MATRIX. A matrix, every element of which is zero, is called a **zero matrix**. When A is a zero matrix and there can be no confusion as to its order, we shall write $A = 0$ instead of the $m \times n$ array of zero elements.

SUMS OF MATRICES. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices, their sum (difference), $A \pm B$, is defined as the $m \times n$ matrix $C = [c_{ij}]$, where each element of C is the sum (difference) of the corresponding elements of A and B. Thus, $A \pm B = [a_{ij} \pm b_{ij}]$.

Example 1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

and

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0-(-1) & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Two matrices of the same order are said to be **conformable** for addition or subtraction. *Two matrices of different orders cannot be added or subtracted.* For example, the matrices (a) and (b) above are non-conformable for addition and subtraction.

The sum of k matrices A is a matrix of the same order as A and each of its elements is k times the corresponding element of A. We define: If k is any **scalar** (we call k a scalar to distinguish it from $[k]$ which is a 1×1 matrix) then by $kA = Ak$ is meant the matrix obtained from A by multiplying each of its elements by k.

Example 2. If $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$, then

$$A + A + A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 6 & 9 \end{bmatrix} = 3A = A \cdot 3$$

and

$$-5A = \begin{bmatrix} -5(1) & -5(-2) \\ -5(2) & -5(3) \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ -10 & -15 \end{bmatrix}$$

In particular, by $-A$, called the **negative** of A, is meant the matrix obtained from A by multiplying each of its elements by -1 or by simply changing the sign of **all** of its elements. For every A, we have $A + (-A) = 0$, where 0 indicates the zero matrix of the same order as A.

Assuming that the matrices A, B, C are conformable for addition, we state:

(a) $A + B = B + A$ (commutative law)

(b) $A + (B + C) = (A + B) + C$ (associative law)

(c) $k(A + B) = kA + kB = (A + B)k$, k a scalar

(d) There exists a matrix D such that $A + D = B$.

These laws are a result of the laws of elementary algebra governing the addition of numbers and polynomials. They show, moreover,

1. Conformable matrices obey the same laws of addition as the elements of these matrices.

MULTIPLICATION. By the product AB **in that order** of the $1 \times m$ matrix $A = [a_{11}\ a_{12}\ a_{13}\ \dots\ a_{1m}]$ and the $m \times 1$ matrix $B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \cdot \\ \cdot \\ \cdot \\ b_{m1} \end{bmatrix}$ is meant the 1×1 matrix $C = [a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1}]$.

That is,

$$[a_{11}\ a_{12}\ \dots\ a_{1m}] \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_{m1} \end{bmatrix} = [a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1}] = \left[\sum_{k=1}^{m} a_{1k}b_{k1}\right].$$

Note that the operation is **row by column**; each element of the row is multiplied into the corresponding element of the column and then the products are summed.

Example 3. (*a*) $[2\ \ 3\ \ 4] \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = [2(1)+3(-1)+4(2)] = [7]$

(*b*) $[3\ \ -1\ \ 4] \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix} = [-6-6+12] = 0$

By the product AB **in that order** of the $m \times p$ matrix $A = [a_{ij}]$ and the $p \times n$ matrix $B = [b_{ij}]$ is meant the $m \times n$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}, \qquad (i = 1, 2, \dots, m;\ j = 1, 2, \dots, n).$$

Think of A as consisting of m rows and B as consisting of n columns. In forming $C = AB$ each row of A is multiplied once and only once into each column of B. The element c_{ij} of C is then the product of the ith row of A and the jth column of B.

Example 4.

$$A\ B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

The product AB is defined or A is **conformable** to B for multiplication only when the number of columns of A is equal to the number of rows of B. If A is conformable to B for multiplication (AB is defined), B is not necessarily conformable to A for multiplication (BA may or may not be defined).

See Problems 3-4.

Assuming that A, B, C are conformable for the indicated sums and products, we have

(*e*) $A(B+C) = AB + AC$ (first distributive law)

(*f*) $(A+B)C = AC + BC$ (second distributive law)

(*g*) $A(BC) = (AB)C$ (associative law)

However,

(*h*) $AB \neq BA$, generally,

(*i*) $AB = 0$ does not necessarily imply $A = 0$ or $B = 0$,

(*j*) $AB = AC$ does not necessarily imply $B = C$.

See Problems 3-8.

PRODUCTS BY PARTITIONING. Let $A = [a_{ij}]$ be of order $m \times p$ and $B = [b_{ij}]$ be of order $p \times n$. In forming the product AB, the matrix A is in effect partitioned into m matrices of order $1 \times p$ and B into n matrices of order $p \times 1$. Other partitions may be used. For example, let A and B be partitioned into matrices of indicated orders by drawing in the dotted lines as

$$A = \left[\begin{array}{c|c|c} (m_1 \times p_1) & (m_1 \times p_2) & (m_1 \times p_3) \\ \hline (m_2 \times p_1) & (m_2 \times p_2) & (m_2 \times p_3) \end{array}\right], \qquad B = \left[\begin{array}{c|c} (p_1 \times n_1) & (p_1 \times n_2) \\ \hline (p_2 \times n_1) & (p_2 \times n_2) \\ \hline (p_3 \times n_1) & (p_3 \times n_2) \end{array}\right]$$

or

$$A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right], \qquad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \end{array}\right]$$

In any such partitioning, it is necessary that the columns of A and the rows of B be partitioned in exactly the same way; however m_1, m_2, n_1, n_2 may be any non-negative (including 0) integers such that $m_1 + m_2 = m$ and $n_1 + n_2 = n$. Then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = C$$

Example 5. Compute AB, given $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 2 \end{bmatrix}$

Partitioning so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 3 & 2 & 0 \\ \hline 1 & 0 & 1 \end{array}\right] \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ \hline 2 & 3 & 1 & 2 \end{array}\right],$$

we have

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[2 \;\; 3 \;\; 1] & \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[2] \\ [1 \;\; 0]\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + [1][2 \;\; 3 \;\; 1] & [1 \;\; 0]\begin{bmatrix} 0 \\ 0 \end{bmatrix} + [1][2] \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 4 & 3 & 3 \\ 7 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [1 \;\; 1 \;\; 1] + [2 \;\; 3 \;\; 1] & [0] + [2] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 4 & 3 & 3 \\ 7 & 5 & 5 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [3 \;\; 4 \;\; 2] & [2] \end{bmatrix} = \begin{bmatrix} 4 & 3 & 3 & 0 \\ 7 & 5 & 5 & 0 \\ 3 & 4 & 2 & 2 \end{bmatrix}$$

See also Problem 9.

Let $A, B, C, \ldots$ be n-square matrices. Let A be partitioned into matrices of the indicated orders

$$\left[\begin{array}{c|c|c|c} (p_1 \times p_1) & (p_1 \times p_2) & \ldots & (p_1 \times p_s) \\ \hline (p_2 \times p_1) & (p_2 \times p_2) & \ldots & (p_2 \times p_s) \\ \hline \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ \hline (p_s \times p_1) & (p_s \times p_2) & \ldots & (p_s \times p_s) \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1s} \\ A_{21} & A_{22} & \ldots & A_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ A_{s1} & A_{s2} & \ldots & A_{ss} \end{bmatrix}$$

and let $B, C, \ldots$ be partitioned in exactly the same manner. Then sums, differences, and products may be formed using the matrices $A_{11}, A_{12}, \ldots; B_{11}, B_{12}, \ldots; C_{11}, C_{12}, \ldots$.

SOLVED PROBLEMS

1. (a) $\begin{bmatrix}1 & 2 & -1 & 0\\ 4 & 0 & 2 & 1\\ 2 & -5 & 1 & 2\end{bmatrix} + \begin{bmatrix}3 & -4 & 1 & 2\\ 1 & 5 & 0 & 3\\ 2 & -2 & 3 & -1\end{bmatrix} = \begin{bmatrix}1+3 & 2+(-4) & -1+1 & 0+2\\ 4+1 & 0+5 & 2+0 & 1+3\\ 2+2 & -5+(-2) & 1+3 & 2+(-1)\end{bmatrix} = \begin{bmatrix}4 & -2 & 0 & 2\\ 5 & 5 & 2 & 4\\ 4 & -7 & 4 & 1\end{bmatrix}$

(b) $\begin{bmatrix}1 & 2 & -1 & 0\\ 4 & 0 & 2 & 1\\ 2 & -5 & 1 & 2\end{bmatrix} - \begin{bmatrix}3 & -4 & 1 & 2\\ 1 & 5 & 0 & 3\\ 2 & -2 & 3 & -1\end{bmatrix} = \begin{bmatrix}1-3 & 2+4 & -1-1 & 0-2\\ 4-1 & 0-5 & 2-0 & 1-3\\ 2-2 & -5+2 & 1-3 & 2+1\end{bmatrix} = \begin{bmatrix}-2 & 6 & -2 & -2\\ 3 & -5 & 2 & -2\\ 0 & -3 & -2 & 3\end{bmatrix}$

(c) $3\begin{bmatrix}1 & 2 & -1 & 0\\ 4 & 0 & 2 & 1\\ 2 & -5 & 1 & 2\end{bmatrix} = \begin{bmatrix}3 & 6 & -3 & 0\\ 12 & 0 & 6 & 3\\ 6 & -15 & 3 & 6\end{bmatrix}$

(d) $-\begin{bmatrix}1 & 2 & -1 & 0\\ 4 & 0 & 2 & 1\\ 2 & -5 & 1 & 2\end{bmatrix} = \begin{bmatrix}-1 & -2 & 1 & 0\\ -4 & 0 & -2 & -1\\ -2 & 5 & -1 & -2\end{bmatrix}$

2. If $A = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$ and $B = \begin{bmatrix}-3 & -2\\ 1 & -5\\ 4 & 3\end{bmatrix}$, find $D = \begin{bmatrix}p & q\\ r & s\\ t & u\end{bmatrix}$ such that $A + B - D = 0$.

If $A + B - D = \begin{bmatrix}1-3-p & 2-2-q\\ 3+1-r & 4-5-s\\ 5+4-t & 6+3-u\end{bmatrix} = \begin{bmatrix}-2-p & -q\\ 4-r & -1-s\\ 9-t & 9-u\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$, $-2-p = 0$ and $p = -2$, $4-r = 0$ and $r = 4$, ... Then $D = \begin{bmatrix}-2 & 0\\ 4 & -1\\ 9 & 9\end{bmatrix} = A + B$.

3. (a) $[4\ \ 5\ \ 6]\begin{bmatrix}2\\ 3\\ -1\end{bmatrix} = [4(2) + 5(3) + 6(-1)] = [17]$

(b) $\begin{bmatrix}2\\ 3\\ -1\end{bmatrix}[4\ \ 5\ \ 6] = \begin{bmatrix}2(4) & 2(5) & 2(6)\\ 3(4) & 3(5) & 3(6)\\ -1(4) & -1(5) & -1(6)\end{bmatrix} = \begin{bmatrix}8 & 10 & 12\\ 12 & 15 & 18\\ -4 & -5 & -6\end{bmatrix}$

(c) $[1\ \ 2\ \ 3]\begin{bmatrix}4 & -6 & 9 & 6\\ 0 & -7 & 10 & 7\\ 5 & 8 & -11 & -8\end{bmatrix}$

$= [1(4) + 2(0) + 3(5) \quad 1(-6) + 2(-7) + 3(8) \quad 1(9) + 2(10) + 3(-11) \quad 1(6) + 2(7) + 3(-8)]$

$= [19\ \ 4\ \ -4\ \ -4]$

(d) $\begin{bmatrix}2 & 3 & 4\\ 1 & 5 & 6\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix} = \begin{bmatrix}2(1) + 3(2) + 4(3)\\ 1(1) + 5(2) + 6(3)\end{bmatrix} = \begin{bmatrix}20\\ 29\end{bmatrix}$

(e) $\begin{bmatrix}1 & 2 & 1\\ 4 & 0 & 2\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & 5\\ -2 & 2\end{bmatrix} = \begin{bmatrix}1(3) + 2(1) + 1(-2) & 1(-4) + 2(5) + 1(2)\\ 4(3) + 0(1) + 2(-2) & 4(-4) + 0(5) + 2(2)\end{bmatrix} = \begin{bmatrix}3 & 8\\ 8 & -12\end{bmatrix}$

4. Let $A = \begin{bmatrix}2 & -1 & 1\\ 0 & 1 & 2\\ 1 & 0 & 1\end{bmatrix}$. Then

$A^2 = \begin{bmatrix}2 & -1 & 1\\ 0 & 1 & 2\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ 0 & 1 & 2\\ 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}5 & -3 & 1\\ 2 & 1 & 4\\ 3 & -1 & 2\end{bmatrix}$ and $A^3 = A^2 \cdot A = \begin{bmatrix}5 & -3 & 1\\ 2 & 1 & 4\\ 3 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ 0 & 1 & 2\\ 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}11 & -8 & 0\\ 8 & -1 & 8\\ 8 & -4 & 3\end{bmatrix}$

The reader will show that $A^3 = A \cdot A^2$ and $A^2 \cdot A^3 = A^3 \cdot A^2$.

5. Show that:

(a) $\sum_{k=1}^{2} a_{ik}(b_{kj}+c_{kj}) = \sum_{k=1}^{2} a_{ik}b_{kj} + \sum_{k=1}^{2} a_{ik}c_{kj}$,

(b) $\sum_{i=1}^{2}\sum_{j=1}^{3} a_{ij} = \sum_{j=1}^{3}\sum_{i=1}^{2} a_{ij}$,

(c) $\sum_{k=1}^{2} a_{ik}(\sum_{h=1}^{3} b_{kh}c_{hj}) = \sum_{h=1}^{3}(\sum_{k=1}^{2} a_{ik}b_{kh})c_{hj}$.

(a) $$\sum_{k=1}^{2} a_{ik}(b_{kj}+c_{kj}) = a_{i1}(b_{1j}+c_{1j}) + a_{i2}(b_{2j}+c_{2j}) = (a_{i1}b_{1j}+a_{i2}b_{2j}) + (a_{i1}c_{1j}+a_{i2}c_{2j})$$
$$= \sum_{k=1}^{2} a_{ik}b_{kj} + \sum_{k=1}^{2} a_{ik}c_{kj}.$$

(b) $$\sum_{i=1}^{2}\sum_{j=1}^{3} a_{ij} = \sum_{i=1}^{2}(a_{i1}+a_{i2}+a_{i3}) = (a_{11}+a_{12}+a_{13}) + (a_{21}+a_{22}+a_{23})$$
$$= (a_{11}+a_{21}) + (a_{12}+a_{22}) + (a_{13}+a_{23})$$
$$= \sum_{i=1}^{2} a_{i1} + \sum_{i=1}^{2} a_{i2} + \sum_{i=1}^{2} a_{i3} = \sum_{j=1}^{3}\sum_{i=1}^{2} a_{ij}.$$

This is simply the statement that in summing all of the elements of a matrix, one may sum first the elements of each row or the elements of each column.

(c) $$\sum_{k=1}^{2} a_{ik}(\sum_{h=1}^{3} b_{kh}c_{hj}) = \sum_{k=1}^{2} a_{ik}(b_{k1}c_{1j} + b_{k2}c_{2j} + b_{k3}c_{3j})$$
$$= a_{i1}(b_{11}c_{1j} + b_{12}c_{2j} + b_{13}c_{3j}) + a_{i2}(b_{21}c_{1j} + b_{22}c_{2j} + b_{23}c_{3j})$$
$$= (a_{i1}b_{11} + a_{i2}b_{21})c_{1j} + (a_{i1}b_{12} + a_{i2}b_{22})c_{2j} + (a_{i1}b_{13} + a_{i2}b_{23})c_{3j}$$
$$= (\sum_{k=1}^{2} a_{ik}b_{k1})c_{1j} + (\sum_{k=1}^{2} a_{ik}b_{k2})c_{2j} + (\sum_{k=1}^{2} a_{ik}b_{k3})c_{3j}$$
$$= \sum_{h=1}^{3}(\sum_{k=1}^{2} a_{ik}b_{kh})c_{hj}.$$

6. Prove: If $A = [a_{ij}]$ is of order $m \times n$ and if $B = [b_{ij}]$ and $C = [c_{ij}]$ are of order $n \times p$, then $A(B+C) = AB + AC$.

The elements of the ith row of A are $a_{i1}, a_{i2}, \ldots, a_{in}$ and the elements of the jth column of $B+C$ are $b_{1j}+c_{1j}, b_{2j}+c_{2j}, \ldots, b_{nj}+c_{nj}$. Then the element standing in the ith row and jth column of $A(B+C)$ is $a_{i1}(b_{1j}+c_{1j}) + a_{i2}(b_{2j}+c_{2j}) + \ldots + a_{in}(b_{nj}+c_{nj}) = \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$, the sum of the elements standing in the ith row and jth column of AB and AC.

7. Prove: If $A = [a_{ij}]$ is of order $m \times n$, if $B = [b_{ij}]$ is of order $n \times p$, and if $C = [c_{ij}]$ is of order $p \times q$, then $A(BC) = (AB)C$.

The elements of the ith row of A are $a_{i1}, a_{i2}, \ldots, a_{in}$ and the elements of the jth column of BC are $\sum_{h=1}^{p} b_{1h}c_{hj}$, $\sum_{h=1}^{p} b_{2h}c_{hj}, \ldots, \sum_{h=1}^{p} b_{nh}c_{hj}$; hence the element standing in the ith row and jth column of $A(BC)$ is

$$a_{i1}\sum_{h=1}^{p} b_{1h}c_{hj} + a_{i2}\sum_{h=1}^{p} b_{2h}c_{hj} + \ldots + a_{in}\sum_{h=1}^{p} b_{nh}c_{hj} = \sum_{k=1}^{n} a_{ik}(\sum_{h=1}^{p} b_{kh}c_{hj})$$
$$= \sum_{h=1}^{p}(\sum_{k=1}^{n} a_{ik}b_{kh})c_{hj} = (\sum_{k=1}^{n} a_{ik}b_{k1})c_{1j} + (\sum_{k=1}^{n} a_{ik}b_{k2})c_{2j} + \ldots + (\sum_{k=1}^{n} a_{ik}b_{kp})c_{pj}$$

This is the element standing in the ith row and jth column of $(AB)C$; hence, $A(BC) = (AB)C$.

8. Assuming A, B, C, D conformable, show in two ways that $(A+B)(C+D) = AC + AD + BC + BD$.

Using (e) and then (f), $(A+B)(C+D) = (A+B)C + (A+B)D = AC + BC + AD + BD$.

Using (f) and then (e), $(A+B)(C+D) = A(C+D) + B(C+D) = AC + AD + BC + BD$
$= AC + BC + AD + BD$.

9. (*a*) $$\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&2\\0&0&1&3\end{array}\right]\left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\\hline 3&1&2\end{array}\right] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}1\\2\\3\end{bmatrix}[3\ \ 1\ \ 2] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}3&1&2\\6&2&4\\9&3&6\end{bmatrix} = \begin{bmatrix}4&1&2\\6&3&4\\9&3&7\end{bmatrix}$$

(*b*) $$\left[\begin{array}{cc|cc|cc}1&0&0&0&0&0\\0&2&0&0&0&0\\\hline 0&0&3&0&0&0\\0&0&0&4&0&0\\\hline 0&0&0&0&5&0\\0&0&0&0&0&6\end{array}\right]\left[\begin{array}{cc|cc|cc}1&0&0&0&0&0\\0&1&0&0&0&0\\\hline 0&0&1&0&0&0\\0&0&0&3&0&0\\\hline 0&0&0&0&2&0\\0&0&0&0&0&3\end{array}\right] = \begin{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} & [0] & [0]\\ [0] & \begin{bmatrix}3&0\\0&4\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix} & [0]\\ [0] & [0] & \begin{bmatrix}5&0\\0&6\end{bmatrix}\begin{bmatrix}2&0\\0&3\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&3&0&0&0\\0&0&0&12&0&0\\0&0&0&0&10&0\\0&0&0&0&0&18\end{bmatrix}$$

(*c*) $$\left[\begin{array}{cc|ccc|c}1&1&0&0&0&0\\2&1&0&0&0&0\\\hline 0&0&3&1&2&0\\0&0&1&2&1&0\\0&0&0&1&1&0\\\hline 0&0&0&0&0&1\end{array}\right]\left[\begin{array}{cc|ccc|c}1&2&3&4&5&6\\2&3&4&5&6&7\\\hline 3&4&5&6&7&8\\4&5&6&7&8&9\\9&8&7&6&5&4\\\hline 8&7&6&5&4&1\end{array}\right] = \begin{bmatrix}\begin{bmatrix}1&1\\2&1\end{bmatrix}\begin{bmatrix}1&2\\2&3\end{bmatrix} & \begin{bmatrix}1&1\\2&1\end{bmatrix}\begin{bmatrix}3&4&5\\4&5&6\end{bmatrix} & \begin{bmatrix}1&1\\2&1\end{bmatrix}\cdot\begin{bmatrix}6\\7\end{bmatrix}\\ \begin{bmatrix}3&1&2\\1&2&1\\0&1&1\end{bmatrix}\begin{bmatrix}3&4\\4&5\\9&8\end{bmatrix} & \begin{bmatrix}3&1&2\\1&2&1\\0&1&1\end{bmatrix}\begin{bmatrix}5&6&7\\6&7&8\\7&6&5\end{bmatrix} & \begin{bmatrix}3&1&2\\1&2&1\\0&1&1\end{bmatrix}\cdot\begin{bmatrix}8\\9\\4\end{bmatrix}\\ [1]\cdot[8\ \ 7] & [1]\cdot[6\ \ 5\ \ 4] & [1]\cdot[1]\end{bmatrix}$$

$$= \begin{bmatrix}\begin{bmatrix}3&5\\4&7\end{bmatrix} & \begin{bmatrix}7&9&11\\10&13&16\end{bmatrix} & \begin{bmatrix}13\\19\end{bmatrix}\\ \begin{bmatrix}31&33\\20&22\\13&13\end{bmatrix} & \begin{bmatrix}35&37&39\\24&26&28\\13&13&13\end{bmatrix} & \begin{bmatrix}41\\30\\13\end{bmatrix}\\ [8\ \ 7] & [6\ \ 5\ \ 4] & [1]\end{bmatrix} = \begin{bmatrix}3&5&7&9&11&13\\4&7&10&13&16&19\\31&33&35&37&39&41\\20&22&24&26&28&30\\13&13&13&13&13&13\\8&7&6&5&4&1\end{bmatrix}$$

10. Let $\begin{cases}x_1 = a_{11}y_1 + a_{12}y_2\\ x_2 = a_{21}y_1 + a_{22}y_2\\ x_3 = a_{31}y_1 + a_{32}y_2\end{cases}$ be three linear forms in y_1 and y_2 and let $\begin{cases}y_1 = b_{11}z_1 + b_{12}z_2\\ y_2 = b_{21}z_1 + b_{22}z_2\end{cases}$ be a linear transformation of the coordinates (y_1, y_2) into new coordinates (z_1, z_2). The result of applying the transformation to the given forms is the set of forms

$$\begin{cases}x_1 = (a_{11}b_{11} + a_{12}b_{21})z_1 + (a_{11}b_{12} + a_{12}b_{22})z_2\\ x_2 = (a_{21}b_{11} + a_{22}b_{21})z_1 + (a_{21}b_{12} + a_{22}b_{22})z_2\\ x_3 = (a_{31}b_{11} + a_{32}b_{21})z_1 + (a_{31}b_{12} + a_{32}b_{22})z_2\end{cases}$$

Using matrix notation, we have the three forms $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$ and the transformation $\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix}$. The result of applying the transformation is the set of three forms

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix}$$

Thus, when a set of m linear forms in n variables with matrix A is subjected to a linear transformation of the variables with matrix B, there results a set of m linear forms with matrix $C = AB$.

SUPPLEMENTARY PROBLEMS

11. Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$,

(*a*) Compute: $A+B = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$, $A-C = \begin{bmatrix} -3 & 1 & -5 \\ 5 & -3 & 0 \\ 0 & 1 & -2 \end{bmatrix}$

(*b*) Compute: $-2A = \begin{bmatrix} -2 & -4 & 6 \\ -10 & 0 & -4 \\ -2 & 2 & -2 \end{bmatrix}$, $0 \cdot B = 0$

(*c*) Verify: $A + (B-C) = (A+B) - C$.

(*d*) Find the matrix D such that $A+D = B$. Verify that $D = B - A = -(A-B)$.

12. Given $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, compute $AB = 0$ and $BA = \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$. Hence, $AB \neq BA$ generally.

13. Given $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$, show that $AB = AC$. Thus, $AB = AC$ does not necessarily imply $B = C$.

14. Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, show that $(AB)C = A(BC)$.

15. Using the matrices of Problem 11, show that $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$.

16. Explain why, in general, $(A \pm B)^2 \neq A^2 \pm 2AB + B^2$ and $A^2 - B^2 \neq (A-B)(A+B)$.

17. Given $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$,

(*a*) show that $AB = BA = 0$, $AC = A$, $CA = C$.

(*b*) use the results of (*a*) to show that $ACB = CBA$, $A^2 - B^2 = (A-B)(A+B)$, $(A \pm B)^2 = A^2 + B^2$.

18. Given $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, where $i^2 = -1$, derive a formula for the positive integral powers of A.

Ans. $A^n = I, A, -I, -A$ according as $n = 4p, 4p+1, 4p+2, 4p+3$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

19. Show that the product of any two or more matrices of the set $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$, $\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ is a matrix of the set.

20. Given the matrices A of order $m \times n$, B of order $n \times p$, and C of order $r \times q$, under what conditions on p, q, and r would the matrices be conformable for finding the products and what is the order of each: (*a*) ABC, (*b*) ACB, (*c*) $A(B+C)$?

Ans. (*a*) $p = r$; $m \times q$ (*b*) $r = n = q$; $m \times p$ (*c*) $r = n$, $p = q$; $m \times q$

21. Compute AB, given:

(a) $A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 0 & 0 & 1 \end{array}\right]$ and $B = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 1 & 0 \end{array}\right]$ *Ans.* $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

(b) $A = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$ and $B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline -1 & 2 \end{array}\right]$ *Ans.* $\begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}$

(c) $A = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right]$ and $B = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$ *Ans.* $\begin{bmatrix} 0 & 0 & 4 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}$

22. Prove: (a) trace $(A+B)$ = trace A + trace B, (b) trace $(kA) = k$ trace A.

23. If $\begin{cases} x_1 = y_1 - 2y_2 + y_3 \\ x_2 = 2y_1 + y_2 - 3y_3 \end{cases}$ and $\begin{cases} y_1 = z_1 + 2z_2 \\ y_2 = 2z_1 - z_2 \\ y_3 = 2z_1 + 3z_2 \end{cases}$, verify $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$
$= \begin{bmatrix} -z_1 + 7z_2 \\ -2z_1 - 6z_2 \end{bmatrix}$.

24. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are of order $m \times n$ and if $C = [c_{ij}]$ is of order $n \times p$, show that $(A+B)C = AC + BC$.

25. Let $A = [a_{ij}]$ and $B = [b_{jk}]$, where $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p;\ k = 1, 2, \ldots, n)$. Denote by β_j the sum of the elements of the jth row of B, that is, let $\beta_j = \sum_{k=1}^{n} b_{jk}$. Show that the element in the ith row of $A \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_p \end{bmatrix}$ is the sum of the elements lying in the ith row of AB. Use this procedure to check the products formed in Problems 12 and 13.

26. A relation (such as parallelism, congruency) between mathematical entities possessing the following properties:

(i) Determinative Either a is in the relation to b or a is not in the relation to b.
(ii) Reflexive a is in the relation to a, for all a.
(iii) Symmetric If a is in the relation to b then b is in the relation to a.
(iv) Transitive If a is in the relation to b and b is in the relation to c then a is in the relation to c.

is called an **equivalence relation**.

Show that the parallelism of lines, similarity of triangles, and equality of matrices are equivalence relations. Show that perpendicularity of lines is not an equivalence relation.

27. Show that conformability for addition of matrices is an equivalence relation while conformability for multiplication is not.

28. Prove: If A, B, C are matrices such that $AC = CA$ and $BC = CB$, then $(AB \pm BA)C = C(AB \pm BA)$.

Chapter 2

Some Types of Matrices

THE IDENTITY MATRIX. A square matrix A whose elements $a_{ij} = 0$ for $i > j$ is called **upper triangular**; a square matrix A whose elements $a_{ij} = 0$ for $i < j$ is called **lower triangular**. Thus

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \text{is upper triangular and}$$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \quad \text{is lower triangular.}$$

The matrix $D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$, which is both upper and lower triangular, is called a **diagonal matrix**. It will frequently be written as

$$D = \text{diag}(a_{11}, a_{22}, a_{33}, \ldots, a_{nn})$$

See Problem 1.

If in the diagonal matrix D above, $a_{11} = a_{22} = \ldots = a_{nn} = k$, D is called a **scalar** matrix; if, in addition, $k = 1$, the matrix is called the **identity matrix** and is denoted by I_n. For example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When the order is evident or immaterial, an identity matrix will be denoted by I. Clearly, $I_n + I_n + \ldots$ to p terms $= p \cdot I_n = \text{diag}(p, p, p, \ldots, p)$ and $I^p = I \cdot I \ldots$ to p factors $= I$. Identity matrices have some of the properties of the integer 1. For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $I_2 \cdot A = A \cdot I_3 = I_2 A I_3 = A$, as the reader may readily show.