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Recent Mathematical Methods in Nonlinear Wave Propagation

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Lectures given at the 1st Session of the Centro Internazionale Matematico Estivo (C.I.M.E.), held in Montecatini Terme, Italy, May 23–31, 1994

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PREFACE

The book contains the text of the lectures presented at the first session of the Summer School 1994 organized in Montecatini Terme by the C.I.M.E. Foundation.
The aim of the School was the presentation of the state of the art on recent mathematical methods arising in Nonlinear Wave Propagation.
The lecture notes presented in this volume were delivered by leading scientists in these areas and deal with *Nonlinear Hyperbolic Fields and Waves* (by Professor G. Boillat of Clermont University), *The Theory of Hyperbolic Conservation Laws* (by Professor C. M. Dafermos of Brown University), *Outline of a Theory of the KdV Equation* (by Professor P. D. Lax of Courant Institute NYU), *Nonlinear Waves for Quasilinear-Hyperbolic-Parabolic Partial Differential Equations* (by Professor T.-P. Liu of Stanford University).
About fifty people (including research students and senior scientists) participated actively in the course. There were also several interesting contributions from the seminars on specialized topics.
We feel that the volume gives a coherent picture of this fascinating field of Applied Mathematics.

Tommaso Ruggeri

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Non Linear Hyperbolic Fields and Waves

Guy Boillat

En face de la nature, il faut étudier toujours,
mais à la condition de ne savoir jamais.

R. Töpffer

Introduction

Nonlinearity and hyperbolicity are essential features of Mechanics and Relativity. In the last decennials much work has been done leading to a better understanding of the systems in conservative form. Physics, however, as Infeld and Rowlands remark[0], still widely ignores the interesting properties of nonlinear theories. The topics presented in these lectures with physical examples include: discontinuity waves and shocks with particular emphasis on exceptional waves and characteristic shocks; symmetrization of conservative systems compatible with an entropy law, subluminal velocities in relativistic theories, systems with involutive constraints, new field equations by means of generators with special attention to extended thermodynamics and nonlinear electrodynamics. It is our hope that the applications proposed in these lectures will awake a large interest in the nonlinearity of Nature.

1. Hyperbolicity, Conservative form

The N components of the column vector $u(k^\alpha)$ $(\alpha = 0, 1, ..., n)$ satisfy the N partial differential equations of the quasi-linear system

(1) $$A^\alpha(u)u_\alpha = f(u), \quad u_\alpha = \partial u/\partial x^\alpha.$$

The matrix A^0 is supposed to be regular so that the system is usally written

$$u_t + A^i(u)u_i = f(u), \quad x^0 = t, \quad i = 1, 2, ..., n.$$

It is hyperbolic if, for any space vector $\vec{n} = (n_i)$ the matrix $A_n := A^i n_i$ has a complete set of (i.e., N) real eigenvectors [1]-[8]. If all eigenvalues are distinct hyperbolicity is *strict*. In three-dimensional space this will not be possible for all $\vec{n}$ when $N = \pm 2, \pm 3, \pm 4 \pmod 8$ [9],[10]. Therefore, there must correspond $m^{(j)}$ eigenvectors to the eigenvalue $\lambda^{(j)}$ of multiplicity $m^{(j)}$. We denote them with the initials of the Latin words *laevus* (left) and *dexter* (right). Two indices are needed, one for the eigenvalue, the other one for the multiplicity : $d_J^{(j)}$. However, the subscript suffices if we agree that $d_J, d_{J'}, (J, J' = 1, 2, ...m^{(j)})$ are eigenvectors corresponding to $\lambda^{(j)}, d_K, d_{K'}$ to $\lambda^{(k)}$ etc. Hence

(2) $$\ell_J(A_n - \lambda^{(j)} A^0) = 0, \quad (A_n - \lambda^{(k)} A^0)\, d_K = 0.$$

It can also be assumed that $\ell_J A^0 d_{J'} = \delta_{JJ'}$.

For the system of *balance laws* encountered in Mechanics and Physics

(3) $$\partial_\alpha f^\alpha(u) = f(u)$$

the matrices A^α are equal to the gradient of the vectors f^α with respect to u

(4) $$A^\alpha = \nabla f^\alpha,$$

so that with the choice $u := f^0, A^0 = I$. Although (3) expresses *conservation laws* only when there is no second member, this system is nevertheless said to have *conservative form* in reference to its first member.

2. Wave velocities

Discontinuities $[u_\alpha]$ of the first order derivatives u_α may occur across some characteristic surface (wave front) $\varphi(x^\alpha) = 0$. In fact by a well known result of Hadamard [11]

$$[u_\alpha] = \varphi_\alpha \delta u$$

and taking the jump of (1) results in

$$A^\alpha \varphi_\alpha \delta u = 0$$

or, by introducing *normal wave velocity* $\lambda \vec{n}$

(5) $$\lambda = -\varphi_t / |\text{grad } \varphi|, \quad n_i = \varphi_i / |\text{grad } \varphi|$$

$$(A_n - \lambda I)\ \delta u = 0.$$

The physical meaning of the eigenvalue $\lambda^{(i)}$ is thus quite simple : it is the velocity of the wave front propagating the weak discontinuity

(6) $$\delta u = \pi^I d_I, \quad I = 1, 2, ..., m^{(i)}.$$

It was Courant who suggested to Peter Lax that he study the evolution of these discontinuities [12] and they showed [1],[13] how they propagate along the bicharacteristic curves

(7) $$dx^\alpha / d\sigma = \partial \psi^{(i)} / \partial \varphi_\alpha, \quad d\varphi_\alpha / d\sigma = -\partial \psi^{(i)} / \partial x^\alpha,$$

(8) $$\psi^{(i)} := \varphi_t + |\text{grad } \varphi|\ \lambda^{(i)}(u, \vec{n}) = 0.$$

The velocity of propagation $\partial \psi / \partial \varphi_i$ is the *ray velocity*

(9) $$\vec{\Lambda} = \lambda\ \vec{n} + \partial \lambda / \partial \vec{n} - \vec{n}(\vec{n}.\partial \lambda / \partial \vec{n}), \quad \lambda = \vec{\Lambda}.\vec{n}.$$

In the non-linear case the components π^I satisfy along the rays a Bernoulli system of differential equation [44], [14]

(10) $$d\pi^I / d\sigma + a^I_{I'} \pi^{I'} + |\text{grad } \varphi|\ \pi^{I'}\ \pi^I\ \nabla \lambda^{(i)} d_{I'} = 0$$

where the coefficients depend on the solution in the unperturbed state u_0 and on the geometry of the wave front. (Another approach [15] involving discontinuities in the derivatives of φ, leads to an equivalent though different system [16]).

For *asymptotic waves* [1], [17], [18]

$$u = u_0(x^\alpha) + u_1(x^\alpha;\xi)/\omega + u_2(x^\alpha;\xi)/\omega^2 + ..., \qquad \xi = \omega\varphi$$

the equations of evolution given by Y. Choquet-Bruhat [19]-[22]

$$(11) \qquad \partial u^I/\partial\sigma + |\text{grad }\varphi|\ u^{I'}(\nabla\lambda^{(i)}d_{I'})\ \partial u^I/\partial\xi + a^I_{I'}u^{I'} = 0,\ u_1 = u^I d_I$$

yields also (10) as a special solution : $u^I = \xi\ \pi^I$. When u_0 depends on ξ see D. Serre [23].

When the disturbance propagates into a constant state u_0 equation (7) shows that the points M of the wave front S at time σ are related to those M_0 of the initial wave surface S_0 by

$$\vec{M}(\sigma) = \vec{M}_0 + \vec{\Lambda}(u_0, \vec{n}_0)\ \sigma, \quad \vec{n}(M) = \vec{n}(M_0) = \vec{n}_0.$$

It follows that S and S_0 are obtained by translation (or are parallel surfaces) if $\vec{\Lambda}$ (or λ) do not depend on $\vec{n}$.

In the absence of a source term a *simple wave* [24]-[26] solution $u = u(\varphi)$ satisfies the ordinary differential system [27]

$$(12) \qquad du/d\varphi = \alpha^J(\varphi)\ d_J(u, \vec{n})$$

where $\varphi(t, x^i)$ is explicitly defined by

$$g(\varphi) = x^i n_i - \lambda^{(j)}(u, \vec{n})t, \quad \vec{n} = \text{const},$$

and $g = \varphi$ or 0 (for centred waves).

This simple wave solution is important because it describes the state adjacent to a constant state [26]. It singles out among the solutions for which the direction of u_x is submitted to some restrictions [28].

The velocity varies according to

$$(13) \qquad d\lambda^{(j)}/d\varphi = \alpha^J \nabla\lambda^{(j)} d_J.$$

The characteristic equations of covariant field equations appear in covariant form as

$$(14) \qquad \psi := G^{\alpha\beta...\gamma}\varphi_\alpha\varphi_\beta...\varphi_\gamma = 0$$

where G is a completely symmetric tensor. The ray velocity is given by

$$(15) \qquad \Lambda^\alpha = \partial\psi/\partial\varphi_\alpha$$

and, in a relativistic theory, must not exceed the velocity of light i.e., the ray velocity must be a time-like (or null) vector

$$(16) \qquad g_{\alpha\beta}\Lambda^\alpha\Lambda^\beta \geq 0$$

while the wave surface, by $\Lambda^\alpha\varphi_\alpha = 0$, satisfies

$$g^{\alpha\beta}\varphi_\alpha\varphi_\beta \leq 0.$$

The Rarita-Schwinger wave fronts, on the contrary, propagate faster than light [29].

3. Exceptional waves

The evolution equations (10) and (11) are nonlinear unless

$$\nabla\lambda^{(i)} d_I \equiv 0, \quad I = 1, 2, ..., m^{(i)}. \tag{17}$$

In this case we say with Lax [25], [26] that the wave is *exceptional*, for there is no reason, in general, for the gradient of $\lambda^{(i)}$ to be orthogonal to its eigenvectors. However, since it is clearly the case for linear fields ($\nabla\lambda^{(i)} \equiv 0$), the velocity is also said to be *linearly degenerated*. Instead, when $\nabla\lambda.d \neq 0$ the characteristic field is *genuinely nonlinear*. By (6) the condition (17) is simply written

$$\delta\lambda^{(i)} \equiv 0 \tag{18}$$

or by (9)

$$\vec{n}.\delta\vec{\Lambda}^{(i)} = 0. \tag{19}$$

The disturbance of the ray velocity, the only physically meaningful vector, one can derive from (1), without further information on the field equations, is orthogonal to the wave normal. Therefore, such a wave may also be called *transverse wave* [2].

The corresponding simple wave, by (13), moves with a constant velocity and first integrals of (12) can then easily be found for conservation laws. In fact, since

$$d(f_n - \lambda^{(j)}u)/d\varphi + u d\lambda^{(j)}/d\varphi = 0$$

$$f_n(u) - \lambda^{(j)}(u)u = f_n(u_0) - \lambda^{(j)}(u_0)\, u_0, \ \lambda^{(j)}(u) = \lambda^{(j)}(u_0), \quad u_0 = u(0).$$

The explicit solution of these equations will be given below [30].

In a covariant formulation exceptionality of (14) is expressed by

$$\delta\psi = \varphi_\alpha\varphi_\beta...\varphi_\gamma\delta G^{\alpha\beta...\gamma} = 0, \quad \varphi_\alpha\delta\Lambda^\alpha = 0. \tag{20}$$

Only in this case does the disturbance of a tensor depending on φ_α have a covariant meaning [31].

Actually, in spite of their name, exceptional waves are rather common and can be encountered for instance, in the equations of Einstein for gravitation, of the fluids (entropy and Alfvén waves [2]), of Monge-Ampère, of nonlinear electromagnetism, of Nambu [32]. Also *multiple waves of conservative systems are exceptional* [33], [34]. To see this take the derivative of (2) in the direction of $d_{K'}$,

$$(\nabla A_n - \lambda^{(k)}\nabla A^0)\, d_{K'}d_K + (A_n - \lambda^{(k)}A^0)\, \nabla d_K d_{K'} - A^0 d_K(\nabla\lambda^{(k)}d_{K'}) = 0,$$

exchange K and K' and substract. The first terms drop out, due to (4), and the result follows. As a consequence the system of two equations

$$u_t + w^i(u,v)\, u_i = 0, \quad v_t + w^i(u,v)\, v_i = 0$$

which has the double velocity $\lambda = w^i n_i$ cannot be put in a conservative form unless the w^i's are constant (hence λ exceptional).

Incidentally, when two velocities crossed for some value $\vec{n}_0(u)$ of $\vec{n}$, a possibility evoked in the first paragraph, thus creating a variable multiplicity, the important criterion is the exceptionality (for $\vec{n}_0$) of the difference of these two velocities [35].

As an illustration consider the Euler equations of a fluid

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \vec{v}) &= 0 \\ \partial_t(\rho\vec{v}) + \partial_i(\rho\, v^i \vec{v}) + \overrightarrow{grad}\, p &= 0 \\ \partial_t(\rho S) + \operatorname{div}(\rho S \vec{v}) &= 0. \end{aligned} \tag{21}$$

To compute the perturbations and velocities one makes the substitutions

$$\partial_t \to -\lambda\delta,\ \partial_i \to n_i\delta,\ \overrightarrow{\text{grad}} \to \vec{n}\delta,\ \text{curl}\ \to \vec{n} \times \delta \tag{22}$$

or, simply in a covariant theory

$$\partial_\alpha, \quad \nabla_\alpha \longrightarrow \varphi_\alpha \delta. \tag{23}$$

Here one immediately obtains, with $p = p(\rho, S)$,

$$\begin{aligned} (v_n - \lambda)\ \delta\rho + \rho\delta v_n &= 0 \\ \rho(v_n - \lambda)\ \delta\vec{v} + \vec{n}\delta p &= 0 \\ (v_n - \lambda)\ \delta S &= 0. \end{aligned}$$

The entropy wave ($\delta S \neq 0$) moves with the fluid, $\lambda = v_n$, is exceptional, $\delta v_n = \delta p = 0$ and has a multiplicity equal to three (the number of arbitrary disturbances). The ray velocity is $\vec{v}$ and $\delta\vec{v} \neq 0$.

For the sonic wave $\delta S = 0, \delta p = p'\delta\rho,\ \lambda = v_n \pm \sqrt{p'}$. The ray velocity is $\vec{\Lambda} = \vec{v} \pm \sqrt{p'}\ \vec{n}$.

Further

$$\delta\lambda = \vec{n}.\delta\vec{\Lambda} = \pm\sqrt{p'}\ (\frac{1}{\rho} + \frac{p''}{2p'})\ \delta\rho. \tag{24}$$

The nonlinear d'Alembert equation derives from a variational principle with the Lagrangian [46]

$$L = L(Q),\ Q = -\frac{1}{2}\ g^{\alpha\beta}\partial_\alpha\ u\partial_\beta u,\ g^{\alpha\beta} = \text{diag}\ (1, -1, -1, -1)$$

and reads

$$\partial_\alpha\ (L' u^\alpha) = 0,\ (L'\ g^{\alpha\beta} - L''\ u^\alpha u^\beta)\ \partial_{\alpha\beta} u = 0,\ u^\alpha = g^{\alpha\beta}\ \partial_\beta u. \tag{25}$$

With the substitution (23) the wave front is easily obtained

$$\psi = L'\ g^{\alpha\beta}\varphi_\alpha\varphi_\beta - L''\ (u^\alpha\varphi_\alpha)^2 = 0$$

since

$$\delta u_\alpha = \varphi_\alpha\delta^2 u.$$

According to (15) the ray velocity is

$$\Lambda^\alpha = L' g^{\alpha\beta}\varphi_\beta - L''(u^\beta\varphi_\beta)\, u^\alpha$$

and, by (16),

(26) $$g_{\alpha\beta}\Lambda^\alpha\Lambda^\beta = -(u^\alpha\varphi_\alpha)^2\,(L' + 2QL'')\,L'' \geq 0.$$

The expression of its covariant perturbation

(27) $$\delta\Lambda^\alpha/\delta^2 u = -2(u^\beta\varphi_\beta)\,L''\Lambda^\alpha/L' + (u^\beta\varphi_\beta)^2 u^\alpha(L''' - 3L''^2/L')$$

is valid, as mentioned above, only in the exceptional case when $\varphi_\alpha\delta\Lambda^\alpha = 0$ i.e.,

(28) $$L''' - 3L''^2/L' = 0$$

and reduces to its first term. However, it is true, in general, that

(29) $$\varphi_\alpha\delta\Lambda^\alpha = (u^\alpha\varphi_\alpha)^3\,(L''' - 3L''^2/L')\,\delta^2 u.$$

Now appears the difference with (24). For a fluid the scalar $\nabla\lambda d$ does not change sign when $\vec{n}$ varies (cf. the cases of magnetohydrodynamics [44] and nonlinear electromagnetic media) [45]. Therefore, a spherical perturbation propagating in a constant state will either increase without limit or decrease (compression or rarefaction wave). On the contrary, if $Q > 0$ (29) changes sign across the critical cone [36]

$$\left|\frac{u_i}{u_0}\,n_i\right| = 1$$

and the behaviour of the perturbation will depend on the direction of propagation.

In one-dimensional propagation directional exceptionality cannot occur but it may happen that the condition of linear degeneracy be verified only for some value of the field u. This phenomenon appears in a rigid heat conductor and has been studied in detail by Ruggeri, Muracchini and Seccia. They find critical temperatures in agreement with those experimentally observed for the NaF and Bi crystals [42].

Of interest in microphysics [37] the scalar field also describes the static irrotational isentropic and supersonic flow. In this case ([24], p.26 sqq., [38], p.201)]

$$\vec{v} = \text{grad } u$$

and (25) can be rewritten

$$(c^2\delta_{ij} - u_i u_j)\,\partial_{ij}u = 0, \quad c^2(Q) = -L'/L'', \quad Q = \frac{1}{2}\,\vec{v}^2.$$

Although there is no velocity of propagation this equation is still hyperbolic provided that $v^2 > c^2$. This is the condition for the existence of the characteristic surface whose normal satisfies

$$(\vec{v}.\vec{n})^2 = c^2$$

implying

(30) $$(L' + 2QL'')L'' > 0.$$

4. Exceptionality as a principle of selection

The concept of linear degeneracy introduced by Peter Lax plays a fondamental role in the propagation of weak and, as we shall see later, of strong discontinuities. Therefore, when the field equations are not completely determined one can require the exceptionality of some wave(s) to solve the indeterminacy and then inquire about the result which is expected to have some special physical or mathematical meaning.

A) Fluid

For a fluid equation (24) yields immediately the equation of state

$$p = b(S) - a(S)/\rho \tag{31}$$

which is the well-known law of von Kármán-Tsien and frequently approximates the pressure in the theory of subsonic flow ([24], p.10), [40]). A similar law

$$p = b - a^2/(\rho + bc^{-2})$$

holds in the relativistic case (ρ is now the density of energy) but another solution exists [41]

$$p = \rho c^2$$

the equation of state of the incompressible fluid whose sound waves travel at the limit [90] speed of light c both solutions of [19], [31], [87]

$$(\rho + p/c^2)\ p'' + 2p'\ (1 - p'/c^2) = 0,\ p' = \partial p\ (\rho, S)/\partial \rho. \tag{32}$$

This last solution is the only possible one for a completely exceptional charged fluid [52].

B) Elastic tube

The velocity v of a fluid of constant density ρ filling an elastic tube of section s satisfies the equations

$$\partial_t s + \partial_x(sv) = 0$$

$$\partial_t v + \partial_x(\frac{1}{2}\ v^2 + p/\rho) = 0,\ p = p(s).$$

We see immediately that the wave speeds are

$$\lambda = v \pm \sqrt{sp'/\rho}$$

and

$$\delta\lambda = (\lambda - v)\ (\frac{1}{s} + \frac{1}{2}\ \frac{(sp')'}{sp'})\ \delta s.$$

Thus the exceptional case corresponds to the pressure

$$p(s) = -a/2s^2 + b$$

which is the law of a rubber-like material. The theoretical and experimental study shows indeed that no shock forms [59].

C) Scalar field

Instead the scalar field will be completely exceptional if (29) vanishes i.e., if the Lagrangian satisfies

$$L'L''' - 3L''^2 = 0$$

the nonlinear solution of which is

$$L = \sqrt{2k_1 Q + k_2} + k_3$$

with two superfluous constants. Since, by (26), k_1 and k_2 are of the same sign one can choose

$$L = \sqrt{2Q + k}, \quad k > 0. \tag{33}$$

This Lagrangian introduced by Max Born [47] has also been considered by Heisenberg [48]. In one space dimension the characteristic curves are isocline [46], ([38], p.579, 617-20), [49].

From (27) follows

$$\delta\Lambda^\alpha // \Lambda^\alpha.$$

so that this weak discontinuity is zero when the ray velocity is normalized in time ($\Lambda^0 = 1$) or in space-time ($g_{\alpha\beta}\Lambda^\alpha\Lambda^\beta = 1$).

This is general : *the (normalized) ray velocity is not disturbed for an exceptional wave of Euler's variational equations (with a convex density of energy)* [131].

We have just seen for the entropy wave that

$$\delta\lambda = \delta v_n = 0, \; \delta\vec{\Lambda} = \delta\vec{v} \neq 0.$$

Hence, the fluid equations do not derive (at least in three dimensions) from a unique (Lagrangian) function but as we shall see below from a potential vector.

On the other hand, the elliptic signature of the metric for the static irrotational fluid implies with (30), (33) $k < 0$ and

$$c^2 = v^2 - |k|.$$

The analogy between the von Kármán fluid and the Born theory has been observed very early by means of the hodograph transformation [50], [51].

D) Born-Infeld Theory

If we apply a variational principle to a Lagrangian depending on the electromagnetic invariants [53]-[55]

$$Q = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}, \; R = \frac{1}{4} F_{\alpha\beta} \overset{*}{F}{}^{\alpha\beta}$$

the corresponding Euler equations

$$\partial_\alpha (L_Q F^{\alpha\beta} + L_R \overset{*}{F}{}^{\alpha\beta}) = 0, \; F_{\alpha\beta} = \partial_\alpha \phi_\beta - \partial_\beta \phi_\alpha \tag{34}$$

yield the Maxwell equations when L is a linear function of Q and R with the result that the electric field of a spherically symmetric particle at rest decreases like the inverse square of the distance meaning also that the field grows without limit when the distance tends to zero. By taking a non-linear Lagrangian, Born and Infeld solved this difficulty and did much more. Max Born has told how the idea came [56] ; it all began in 1933 in Selva/Wolkenstein, South Tyrol...

There are two families of wave fronts satisfying the characteristic equation [57]

$$(\tau^{\alpha\beta} + \mu g^{\alpha\beta}) \, \varphi_\alpha \varphi_\beta = 0$$

where $\tau^{\alpha\beta}$ is the usual Maxwell tensor and $\mu(Q, R)$ takes on two values μ_1 and μ_2 determined by the knowledge of the Lagrangian.

It turns out that these two values coincide only when

$$L = \sqrt{-R^2 + k(2Q + k)}, \quad \mu = Q + k \tag{35}$$

which is just the Born-Infeld Lagrangian. Due to the multiplicity of the wave the related system (34) is therefore completely exceptional but it is not the only one and other Lagrangians share this property with (35). They have been determined. The general one depends on several constants and gives (35) when one of them, maybe related to the Planck constant, vanishes [57].

At distance r the electric field of a charged particle is just

$$E(r) = \sqrt{k}/\sqrt{1 + (r/r_0)^4}, \quad E(0) = \sqrt{k}$$

so that $E^2 \leq k$. More generally, for an admissible Lagrangian,

$$E^2 \leq \min(\zeta_1, \zeta_2), \quad \zeta = \mu - Q$$

and the limit value ζ is called the absolute field. The electric field reaches this limit in the frame moving with the ray velocities. For this reason, having also in mind the entropy wave of continuum mechanics, we proposed [57] :

A stable particle moves along an exceptional bicharacteristic.

Thus are determined the equations of motion of the particles in nonlinear electrodynamics. The Born-Infeld theory was the first attempt to deal with the difficulties of microphysics by means of nonlinear equations. Although its development was hindered by this very nonlinearity it reveals from the mathematical point of view an interesting structure. From the physical point of view it has been shown that the present relativistic strings and membranes are just particular solutions of Born-Infeld [32], [58].

E) Elasticity

Some waves propagating in elastic solids may be naturally exceptional at least in certain direction or for certain kind of deformation [60]-[62]. Instead the requirement of linear degeneracy [63]-[66] leads to the determination of classes of elastic potentials containing the potentials of Grioli [67], Hadamard, Hooke, Mooney-Rivlin and Tolotti [68].

F) Monge-Ampère equation

The nonlinear equation

$$u_{tt} + f(x, t, u, p, q, r, s) = 0$$
$$p = u_x, \quad q = u_t, \quad r = u_{xx}, \quad s = u_{xt}$$

becomes quasi-linear when derivated with respect to t.

Assuming therefore discontinuities of the third order across a wave surface the application of the operator (22) gives $\delta r = \delta^3 u$, $\delta s = -\lambda \delta^3 u$

$$\lambda^2 - \lambda f_s + f_r = 0.$$

The requirement that $\delta\lambda$ be zero for both velocities leads to

$$f_{rr} - f_r f_{ss} = 0, \quad f_s f_{ss} - 2f_{sr} = 0$$

the integration of which results in [69]

$$F := Hu_{tt} + 2Ku_{tx} + Lu_{xx} + M + N\,(u_{tt}u_{xx} - u_{tx}^2) = 0,$$

an equation due to Monge and Ampère (when $N \neq 0$) [70] and well suited for initial-value problems ([1], p.499).

Now it is natural to use this characteristic property to extend its form to n independent variables or to higher orders. It follows in the first case that the Monge-Ampère function F *is a linear function of the Hessian of* u *and of all its minors* [71] in accordance with the results found by direct calculation for $n = 3$ [72] or $n = 4$ [73]. In the second case introducing

$$X_k = \partial^n u / \partial t^k\ \partial x^{n-k}, \quad 0 \leq k \leq n$$

and the Hankel matrix $K(K_{ij} = X_{i+j}; i, j = 0, 1, 2, ...m)$ *the Monge-Ampère function* F *is a linear function of all minors of* K *(including* K*) if* $n = 2m$ *or of* K *without its last row if* $n = 2m - 1$ [74].

The Natan equation [55]

$$\sum_{i,j} b_{ij}\,(X_i X_{j+1} - X_j X_{i+1}) + \sum_0^n a_i X_i + a = 0$$

is of this type.

A long-forgotten equation hundreds of papers have appeared in the last years on the Monge-Ampère field. It is also applied. For instance, a one-dimensional von Kármán fluid is such a field [75] and the third order equation has recently been used for the thermodynamics of fluctuation [76], [77]. Sometimes the classical equation can even be explicitely integrated (generalized Born-Infeld Lagrangians are Monge-Ampère solutions [57]) and this is not the least interesting feature of this equation.

All these equations belong to the general class of the Monge-Ampère systems which, among the nonlinear partial differential equations, is the closest to the quasi-linear class sharing with it an important linear property [78].

5. Symmetric systems. Symmetrization.

If the matrices A^α are symmetric and if A^0 is positive definite the system (1) is called symmetric [1], [79]. It is always hyperbolic and the Cauchy problem is well-posed [80], [81]. But where are these systems to be found? In 1961 Godunov [82] discovered that the Euler equations of the fluid where the conservation of energy

$$\partial_t\ \{\rho(e + \frac{1}{2}\ v^2)\} + \partial_i\ \{\rho(e + \frac{1}{2}\ v^2 + p/\rho)\ v^i\} = 0$$

replaces the last equation (21) were part of this class as well as the Euler variational equations with the Lagrangian $L(\partial_\alpha q^s)$.

In fact introducing the new field variables

(36) $\check{u}' = \frac{1}{T}(G - \frac{1}{2}\vec{v}^2, \vec{v}, -1)$, $G = e + pV - TS$, $de = TdS - pdV$, $V = 1/\rho$,

(e is the internal energy and G the chemical potential) it appears that $f^0 = u$ and f^i are just the derivatives with respect to u' of the functions

$$h'^0 = p/T,\ h'^i = pv^i/T;\ f^\alpha = \check{\nabla}' h'^\alpha$$

so that (3) takes a special symmetric form. The same can be done for the variational equations (see below). Godunov then deduced for systems of this form the existence of an additional conservation law.

Precisely in mechanics and physics systems of balance laws are supplemented with a law of conservation of energy or entropy

(37) $$\partial_\alpha\ h^\alpha(u) = g(u)$$

that must be compatible with (3) and therefore [83]

(38) $$\nabla h^i = \nabla h^0 A^i.$$

This is the starting point of the fundamental paper of Friedrichs and Lax. Differentiating in the direction of an arbitrary vector v and multiplying by w one obtains

$$\check{v}\nabla\check{\nabla}h^i w - \nabla h^0 \nabla\nabla f^i vw = \check{v}HA^i w,\quad H := \nabla\check{\nabla}h^0.$$

Exchanging v and w does not change the first member. Hence $\overline{A}^i := HA^i$ are symmetric matrices and the system is symmetrized by multiplication

(39) $$Hu_t + \overline{A}^i u_i = Hf.$$

One can only wonder at the fact that long-established theories of natural philosophy [84], [85] possess this so propitious mathematical structure.

Let us introduce the field

(40) $$u' = \check{\nabla}h^0.$$

By (39) it satisfies the equation

$$u'_t + \check{A}^i u'_i = Hf$$

which is not specially interesting since it is neither symmetric nor conservative. We multiply it by H^{-1} that is we go back to the original system making the Le Gendre [91] transformation

(41) $$h'^0 = \check{u}'u - h^0$$

and defining

(42) $$h'^i = \check{u}'f^i - h^i$$

so that, by (38), (40),

$$dh'^\alpha = \check{f}^\alpha du'.$$

As a result [92], [93], ([22], p.42)

$$f^\alpha = \check{\nabla}' h'^\alpha$$

and (3) becomes

$$\partial_\alpha \check{\nabla}' h'^\alpha = f, \quad A'^\alpha u'_\alpha = f, \quad A'^\alpha = \nabla' \check{\nabla}' h'^\alpha \tag{43}$$

that is the symmetric conservative system found by Godunov. Multiplying (43) by u' he deduced (37) where the h^α are given by (41), (42). We shall see later that it is convenient to consider the components of u', called the *main field* by Ruggeri and Strumia [92], as Lagrange multipliers [83], [94].

Now, according to the result of Friedrichs and Lax wherever there is an additional law we may look for the main field u' and the generating functions h'^α. Also for complex fields [95].

A) Relativistic fluid

It is even simpler than the classical case. In a covariant formalism [92] it is easy to check that

$$Td(rSu^\alpha) = u_\beta dT^{\alpha\beta} - (G+1)\; d(ru^\alpha)$$

and therefore, since

$$dh^\alpha = \check{u}' df^\alpha$$

$$h^\alpha = -rSu^\alpha, \quad \check{f}^\alpha = (ru^\alpha, T^{\alpha\beta})$$

$$h'^\alpha = pu^\alpha/T, \quad \check{u}' = (G+1, -u_\beta)/T. \tag{44}$$

Convexity of h^0 is warranted provided that the quadratic form also considered by Friedrichs [101]

$$\mathcal{Q} = \delta\check{u}' \; \delta f^\alpha \xi_\alpha \tag{45}$$

is positive definite for some time-like vector ξ_α. This means here

$$C_P > 0, \; 0 < (\partial p/\partial\rho)_S \leq 1.$$

The important role played by the inverse of the absolute temperature has already been underlined. It is the *coldness* and u_β/T is the *coldness vector* [98], [99]. The components of the main field u' less familiar than the components of the physical field u are interpreted as the observables of the system [100].

B) Hyperelastic material

In Lagrangian coordinates

$$u = (\rho v_i, F_{ik}, \frac{1}{2}\,\rho \vec{v}^{\,2} + e), \;\; f^j = (T_{ij}, v_i\delta^j_k, v_i T_{ij})$$

where T_{ij} is the first Piola-Kirchhoff stress tensor, F_{ij} the displacement gradient tensor, e the internal energy, ρ the constant mass density of the reference configuration and

$$de = TdS - T_{ij} dF_{ij}.$$