PLAYING WITH INFINITY

MATHEMATICAL EXPLORATIONS AND EXCURSIONS

Rózsa Péter

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BY

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Dedicated to my brother, Dr. Nicolas Politzer,
who perished at Colditz in Saxony, 1945

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PREFACE

THIS book is written for intellectually minded people who are not mathematicians. It is written for men of literature, of art, of the humanities. I have received a great deal from the arts and I would now like in my turn to present mathematics and let everyone see that mathematics and the arts are not so different from each other. I love mathematics not only for its technical applications, but principally because it is beautiful; because man has breathed his spirit of play into it, and because it has given him his greatest game—the encompassing of the infinite. Mathematics can give to the world such worthwhile things—about ideas, about infinity; and yet how essentially human it is—unlike the dull multiplication table, it bears on it for ever the stamp of man's handiwork.

The popular nature of the book does not mean that the subject is approached superficially. I have endeavoured to present concepts with complete clarity and purity so that some new light may have been thrown on the subject even for mathematicians and certainly for teachers. What has been left out is the systematization which can so easily become boring; in other words, only technicalities have been omitted. (It is not the purpose of the book to teach anyone mathematical techniques.) If an interested pupil picks up this book it will give him a picture of the whole of mathematics. In the beginning I did not mean the book to be so full; the material expanded itself as I was writing it and the number of subjects which could be omitted rapidly decreased. If there were parts to which memories of boredom previously attached I felt that I was picking up some old piece of furniture and blowing the dust off in order to make it shine.

It is possible that the reader may find the style a little naïve in places, but I do not mind this. A naïve point of view in relation to simple facts always conjures up the excitement of new discovery.

I shall tell the reader in the Introduction how the book originated. The writer of whom I speak there is Marcell

Benedek. I began by writing to him about differentiation and it was his idea that a book could grow out of these letters.

I do not refer to any sources. I have learned a lot from others but today I can no longer say with certainty from whence each piece came. There was no book in front of me while I was writing. Here and there certain similes came to my mind with compelling force, the origins of which I could sometimes remember; for example, the beautiful book by Rademacher and Toeplitz,* or the excellent introduction to analysis by Beke.† Once a method had been formed in my mind I could not really write it in any other way just to be more original. I chiefly refer, in this connexion, to the ideas I gained from László Kalmár. He was a contemporary of mine as well as my teacher in mathematics. Anything I write is inseparably linked with his thoughts. I must mention, in particular, that the 'chocolate example', with the aid of which infinite series are discussed, originated with him, as well as the whole idea of the building up of logarithm tables.

I shall have to quote my little collaborators in the schoolroom by their christian names; they will surely recognize themselves. Here I must mention my pupil Kató, who has just finished the fourth year at the grammar school and contributed to the book while it was being written. It is to her that I must be grateful for being able to see the material with the eyes of a gifted pupil.

The most important help I received was from those people who have no mathematical interests. My dear friend Béla Lay, theatrical producer, who had always believed that he had no mathematical sense, followed all the chapters as they were being written; I considered a chapter finished only when he was satisfied with it. Without him the book perhaps would never have been written.

Pál Csillag examined the manuscript from the point of view of the mathematician; also László Kalmár found time, at the last minute, for a quick look. I am grateful to them for the certainty I feel that everything in the book is right.

Budapest
Autumn, 1943 RÓZSA PÉTER

* Rademacher and Toeplitz: *The Enjoyment of Mathematics*.
† Manó Beke: *Introduction to the Differential and Integral Calculus*.
[I mention this here for those who might be eager to follow it up.]

PREFACE TO THE ENGLISH EDITION

SINCE 1943, seventeen eventful years have passed. During this time my mathematician friend, Pál Csillag, and my pupil, Kató (Kató Fuchs), have fallen victims of Fascism. The father of my pupil Anna, who suffered imprisonment for seventeen years for illegal working-class activity, has been freed. In this way perhaps even in Anna's imagination the straight lines forever approaching one another will meet. (See page 218.) No book could appear during the German occupation; a lot of existing copies were destroyed by bombing, the remaining copies appeared in 1945—on the first free book-day.

I am very grateful to Dr. Emma Barton, who took up the matter of the English publication of my book, to Professor Dr. R. L. Goodstein, who brought it to a head, to Dr. Z. P. Dienes for the careful translation and to Messrs. G. Bell & Sons for making possible the propagation of the book in the English-speaking world.

The reader should remember that the book mirrors my methods of thinking as they were in 1943; I have hardly altered anything in it. Only the end has been altered substantially. Since then, László Kalmár and I have proved that the existence of absolutely undecidable problems follows from Gödel's Theorem on relatively undecidable problems, but of course in no circumstances can a consequence be more important than the Theorem from which it follows.

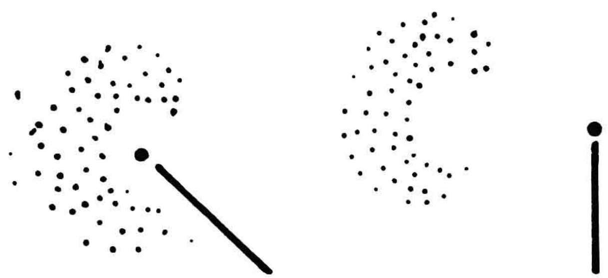
Budapest
1960

RÓZSA PÉTER

INTRODUCTION

A CONVERSATION I had a long time ago comes into my mind. One of our writers, a dear friend of mine, was complaining to me that he felt his education had been neglected in one important aspect, namely he did not know any mathematics. He felt this lack while working on his own ground, while writing. He still remembered the co-ordinate system from his school mathematics, and he had already used this in similes and imagery. He felt that there must be a great deal more such usable material in mathematics, and that his ability to express himself was all the poorer for his not being able to draw from this rich source. But it was all, so he thought, quite hopeless, as he was convinced of one thing: he could never penetrate right into the heart of mathematics.

I have often remembered this conversation; it has always suggested avenues of thought to me and plans. I saw immediately that there was something to do here, since in mathematics for me the element of atmosphere had always been the main factor, and this was surely a common source from which the writer and the artist could both draw. I remember an example from my schooldays: some fellow students and I were reading one of Shaw's plays. We reached the point where the hero asked the heroine what was her secret by means of which she was able to win over and lead the most unmanageable people. The heroine thought for a moment and then suggested that perhaps it could be explained by the fact that she really kept her distance from everyone. At this point the student who was reading the part suddenly exclaimed: 'That is just the same as the mathematical theorem we learnt today!' The mathematical question had been: Is it possible to approach a set of points from an external point in such a way that every point of the set is approached simultaneously? The answer is yes, provided that the external point is far enough away from the whole set:



From here you cannot; while you approach some of the points you go farther away from others.

From here you can.

I did not wish to believe the writer's other statement, namely that he could never penetrate right into the heart of mathematics, that for instance he would never be able to understand the notion of the differential coefficient. I tried to analyse the introduction of this notion into the simplest possible, obvious steps. The result was very surprising; the mathematician cannot even imagine what difficulties the simplest formula can present to the layman. Just as the teacher cannot understand how it is possible that a child can spell c–a–t twenty times, and still not see that it is really a *cat*; and there is more to this than to a cat!

This again was an experience that caused me to do a great deal of thinking. I had always believed that the reason why the public was so ill-informed about mathematics was simply that nobody had written a good popular book for the general public about, say, the differential calculus. The interest patently exists, as the public snaps up everything of this kind that is available to it; but no professional mathematician has so far written such a book. I am thinking of the real professional who knows exactly to what extent things can be simplified without falsifying them, who knows that it is not a question of serving up the usual bitter pill in a pleasanter dish (since mathematics for most is a bitter memory); one who can clarify the essential points so that they hit the eye, and who himself knows the joy of mathematical creation and writes with such a swing that he carries the reader along with him. I am now beginning to believe that for a lot of people even the really popular book is going to remain inaccessible.

Perhaps it is the decisive characteristic of the mathematician that he accepts the bitterness inherent in the path he is travelling. 'There is no royal road to mathematics', Euclid said to an interested potentate; it cannot be made comfortable even for kings. You cannot read mathematics superficially; the inescapable abstraction always has an element of self-torture in it, and the one to whom this self-torture is joy is the mathematician. Even the simplest popular book can be followed only by those who undertake this task to a certain extent, by those who undertake to examine painstakingly the details inherent in a formula until it becomes clear to them.

I am not going to write for these people. I am going to write mathematics without formulae. I want to pass on something of the feel of mathematics. I do not know if such an undertaking can succeed. By giving up the formula, I give up an essential mathematical tool. The writer and the mathematician alike realize that form is essential. Try to imagine how you could express the feel of a sonnet without the form of the sonnet. But I still intend to try. It is possible that, even so, some of the spirit of real mathematics can be saved.

One way of making things easy I cannot allow; the reader must not omit, leave for later reading, or superficially skim through, any of the chapters. Mathematics can be built up only brick by brick; here not one step is unnecessary, for each successive part is built on the previous one, even if this is not quite as obviously so as in a boringly systematic book. The few instructions must be carried out, the figures must really be studied, simple drawings or calculations must really be attempted when the reader is asked to do so. On the other hand I can promise the reader that he will not be bored.

I shall not make use of any of the usual school mathematics. I shall begin with counting and I shall reach the most recent branch of mathematics, mathematical logic.

A CATALOGUE OF SELECTED DOVER BOOKS
IN ALL FIELDS OF INTEREST

A CATALOGUE OF SELECTED DOVER BOOKS IN ALL FIELDS OF INTEREST

CONTENTS

PART I

THE SORCERER'S APPRENTICE

1. Playing with fingers

LET us begin at the beginning. I am not writing a history of mathematics; this could be done only on the basis of written evidence, and how far from the beginning is the first written evidence! We must imagine primitive man in his primitive surroundings, as he begins to count. In these imaginings, the little primitive man, who grows into an educated human being before our eyes, will always come to our aid; the little baby, who is getting to know his own body and the world, is playing with his tiny fingers. It is possible that the words 'one', 'two', 'three' and 'four' are mere abbreviations for 'this little piggie went to market', 'this little piggie stayed at home', 'this one had roast beef', 'this one had none' and so on; and this is not even meant to be a joke: I heard from a medical man that there are people suffering from certain brain injuries who cannot tell one finger from another, and with such an injury the ability to count invariably disappears. This connexion, although unconscious, is therefore still extremely close even in educated persons. I am inclined to believe that one of the origins of mathematics is man's playful nature, and for this reason mathematics is not only a Science, but to at least the same extent also an Art.

We imagine that counting was already a purposeful activity in the beginning. Perhaps primitive man wanted to keep track of his property by counting how many skins he had. But it is also conceivable that counting was some kind of magic rite, since even today compulsion-neurotics use counting as a magic prescription by means of which they regulate certain forbidden thoughts; for example, they must count from one to twenty and only then can they think of something else. However this may be, whether it concerns animal skins or successive time-intervals, counting always means that we go beyond what is there by one: we can even go beyond our ten fingers and so emerges man's first magnificent mathematical creation, the infinite sequence of numbers,

$$1, 2, 3, 4, 5, 6, \ldots$$

the sequence of natural numbers. It is infinite, because after any number, however large, you can always count one more. This creation required a highly developed ability for abstraction, since these numbers are mere shadows of reality. For example, 3 here does not mean 3 fingers, 3 apples or 3 heartbeats, etc., but something which is common to all these, something that has been abstracted from them, namely their number. The very large numbers were not even abstracted from reality, since no one has ever seen a billion apples, nobody has ever counted a billion heartbeats; we imagine these numbers on the analogy of the small numbers which do have a basis of reality: in imagination one could go on and on, counting beyond any so-far known number.

Man is never tired of counting. If nothing else, the joy of repetition carries him along. Poets are well aware of this; the repeated return to the same rhythm, to the same sound pattern. This is a very live business; small children do not get bored with the same game; the fossilized grown-up will soon find it a nuisance to keep on throwing the ball, while the child would go on throwing it again and again.

We go as far as 4? Let us count one more, then one more, then one more! Where have we got to? To 7, the same number that we should have got to if we had straight away counted 3 more. We have discovered addition

$$4 + 1 + 1 + 1 = 4 + 3 = 7$$

Now let us play about further with this operation: let us add to 3 another 3, then another 3, then another 3! Here we have added 3's four times, which we can state briefly as: four threes are twelve, or in symbols:

$$3 + 3 + 3 + 3 = 4 \times 3 = 12$$

and this is multiplication.

We may so enjoy this game of repetition that it might seem difficult to stop. We can play with multiplication in the same way: let us multiply 4 by 4 and again by 4, then we shall get

$$4 \times 4 \times 4 = 64$$

This repetition or 'iteration' of multiplication is called raising to a power. We say that 4 is the base, and we indicate by means of a small number written at the top right-hand corner of the 4 the number of 4's that we have to multiply; i.e. the notation is this:

$$4^3 = 4 \times 4 \times 4 = 64$$

As is easily seen, we keep getting larger and larger numbers: 4×3 is more than $4 + 3$, and 4^3 is a good deal more than 4×3. This playful repetition carries us well up amongst the large numbers; even more so, if we iterate raising to a power itself. Let us raise 4 to the power which is the fourth power of four:

$$4^4 = 4 \times 4 \times 4 \times 4 = 64 \times 4 = 256$$

and we have to raise 4 to this power:

$$4^{4^4} = 4^{256} = 4 \times 4 \times 4 \times 4 \ldots$$

I have no patience to write any more, since I should have to put down 256 4's, not to mention the actual carrying out of the multiplication! The result would be an unimaginably large number, so that we use our common sense, and, however amusing it would be to iterate again and again, we do not include the iteration of powers among our accepted operations.

Perhaps the truth of the matter is this: the human spirit is willing to play any kind of game that comes to hand, but only those of these mathematical games become permanent features that common sense decides are going to be useful.

Addition, multiplication and raising to powers have proved very useful in man's common-sense activities and so they have gained permanent civil rights in mathematics. We have determined all those of their properties which make calculations easier; for example, it is a great saving that 7×28 can be calculated not only by adding 28 7 times, but also by splitting it into two multiplication processes: 7×20 as well as 7×8 can quite easily be calculated and then it is readily determined how much $140 + 56$ will be. Also in adding long columns of numbers how useful it is to know that no amount of rearranging