SOLID MECHANICS AND ITS APPLICATIONS

D.A. Hills, P.A. Kelly, D.N. Dai
and A.M. Korsunsky

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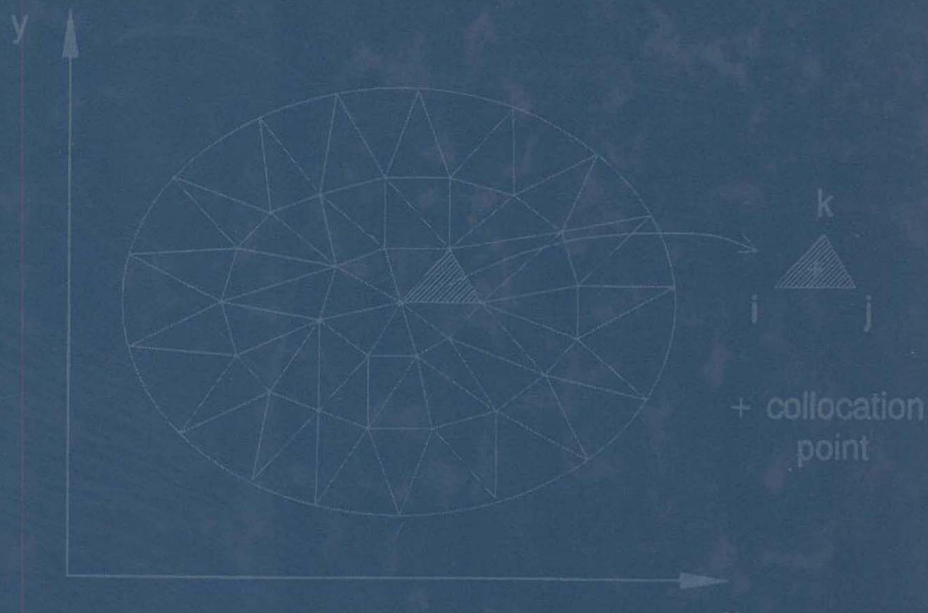
The Distributed Dislocation Technique



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Solution of Crack Problems

The Distributed Dislocation Technique

by

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to Edwin Hills and Alphonsus Kelly

Preface

This book is concerned with the numerical solution of crack problems. The techniques to be developed are particularly appropriate when cracks are relatively short, and are growing in the neighbourhood of some stress raising feature, causing a relatively steep stress gradient. It is therefore practicable to represent the geometry in an idealised way, so that a precise solution may be obtained. This contrasts with, say, the finite element method in which the geometry is modelled exactly, but the subsequent solution is approximate, and computationally more taxing.

The family of techniques presented in this book, based loosely on the pioneering work of Eshelby in the late 1950's, and developed by Erdogan, Keer, Mura and many others cited in the text, present an attractive alternative. The basic idea is to use the superposition of the stress field present in the unflawed body, together with an unknown distribution of 'strain nuclei' (in this book, the strain nucleus employed is the dislocation), chosen so that the crack faces become traction-free. The solution used for the stress field for the nucleus is chosen so that other boundary conditions are satisfied. The technique is therefore efficient, and may be used to model the evolution of a developing crack in two or three dimensions. Solution techniques are described in some detail, and the book should be readily accessible to most engineers, whilst preserving the rigour demanded by the researcher who wishes to develop the method itself.

We have endeavoured to base our solutions on a clear physical interpretation of the quantities involved in the solution. When once the integral equation needed to solve the crack is obtained, we apply more abstract mathematics to obtain the most efficient numerical solution possible, in particular to abstract crack tip stress intensity factors.

The book grew out of work by the authors conducted at Oxford during the past five years or so, but we would not wish to claim that our own efforts contribute more than a small fraction to the ideas contained within it, and which have evolved over the past 30 years, at many universities throughout the world. Our task has been to show the relationship between these methods, and to summarise the current state of knowledge. Very little information has been published in texts hitherto, so that the analyst wishing to pursue any of these approaches would have had to follow the original research papers.

Chapter 1 reviews the fundamentals of fracture mechanics, although the reader is assumed to be familiar with the basic concepts. The stress intensity factor is shown to quantify crack growth under a wide range of conditions. The rest of the text is concerned with finding these quantities under plane, axi-symmetric and general conditions; in Chapter 2 the basic ideas needed for the solution of two dimensional problems are developed *ab initio*, and more sophisticated developments are given in Chapters 3 and 4. The formulation appropriate for an axi-symmetric crack is considered in Chapter 5 and three dimensional problems are studied in Chapters 6 and 7. The three basic forms of the solution are treated in a uniform way, and the relationships between them highlighted, so that the book retains a coherent theme.

We would like to thank Dr. John O'Connor for his patience in giving one of us (PAK) time to work on the book when it should have been used for other things, and for making available the computing facilities at the Oxford Orthopaedic Engineering Centre.

We would particularly like to thank the series editor, Prof. Graham Gladwell, for his constructive and encouraging comments, together with meticulous attention to detail, both mathematical and grammatical. Nevertheless, the final text is entirely our responsibility, and we would be very pleased if readers would report any errors to us.

D.A.H.
P.A.K.
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Oxford, September 1995.

Contents

Chapter 1

Introduction to Fracture Mechanics

1.1 Designing Against Failure

When an engineer designs a structure or piece of equipment, there are two basic forms of mechanical failure which must be considered. These are brittle fracture and, if the material employed is metallic, yielding. The second is by far the easier to take into account: it is necessary to know only the state of stress at every point in order to assemble the appropriate yield parameter. The maximum value of this quantity is found within the structure and set equal to the yield stress, which is taken as a true material property, i.e. it is independent of geometry. Usually the loading or stresses are reduced by a so-called factor of safety, which allows for unexpected overloads during the life of the structure. The load level corresponding to the onset of first yield is known as the elastic limit. There are considerable reserves of strength in any real structure if the elastic limit is moderately exceeded, partly because most real structures exhibit a high degree of redundancy, partly because cyclic loading will induce beneficial residual stresses, promoting shakedown, and partly because most common metals and alloys exhibit work hardening to a greater or lesser degree. By far the most important characteristics of yield from the point of view of design are:

(a) that the yield stress is a highly repeatable true material property, being very insensitive to the geometry of the component under consideration.

(b) that given the yield stress under uniaxial loading, the combination of stresses which will cause local failure under multiaxial conditions is well defined - assumptions of isotropy, independence of yield from hydrostatic stress and convexity of the yield surface (Paul, 1968) being necessary to obtain excellent bounding values for physically acceptable yield criteria.

By contrast, it is quite difficult to predict the load at which brittle fracture will occur. Historically, it was once thought that, by analogy with the yield criteria, a critical state of stress alone would be sufficient to predict fracture. Indeed, it is certainly true that, under a wide range of conditions, it is the greatest principal stress which governs the fracture process. However, early experiments showed that the fracture load of a component or specimen, even one suffering a simple uniaxial state of tensile stress, was not a very repeatable quantity, but varied over a wide range from test to test. It became clear that the brittle fracture strength of a component was also dependent on both its geometry and the quality of material from which it was made.

Before proceeding to examine the applied mechanics of fracture, it is perhaps worth examining the underlying reasons for the great variability in the practically realisable strength of a metal. The theoretical strength in either tension or shear of a perfect crystalline lattice may be estimated by calculating the forces required to overcome the binding forces between atoms. This is the Frenkel calculation (see Dieter, 1961). The force required to shear a row of atoms over an adjacent row is estimated to be $\mu/2\pi$ where μ is the modulus of rigidity; this value is a material property. The same calculation may be carried out for tensile loading, and the theoretical cleavage strength is of the order $E/2\pi$, where E is Young's modulus. However, both calculations predict strengths ranging from 10^2 to 10^4 times the practically measured values. For shear failure, manifested as plastic flow, the discrepancy is made good by inferring the existence of *dislocations*. Instead of one entire layer of material slipping uniformly relative to another, it is assumed that slip is initiated at one point and then propagates through the slip plane with a certain velocity; the dislocation is the boundary between the slipped and unslipped regions. The stress induced by the dislocation is sufficient to disrupt the lattice and promote flow. The making and breaking of rows of bonds is not a conservative process as it requires the expenditure of energy. This is manifested physically as a resistance to the motion of the dislocation, which must therefore be propelled by a force (the Peierls Nabarro force). If the external force exceeds the Peierls Nabarro force the dislocation will glide, and hence flow will occur. The force providing the resistance is a characteristic of a particular material, and hence so is the yield strength.

For brittle fracture, Griffith (1921) explained the discrepancy between theoretical predictions and experimental results by assuming the existence of cracks or defects in a material. The sharp flaws or defects act as concentrators of the applied load so that the *degree* of concentration is critically geometry-dependent. However, whilst the presence of lattice defects (dislocations) has a consistent effect on the strength of real materials, the influence of macroscopic defects will vary from flaw to flaw, depending on their size, shape, and orientation; it is this variability which gives rise to the lack of repeatability of the brittle fracture strength of a material.

Brief mention might be made of the classical approaches to brittle fracture strength associated with the names Mohr and Coulomb (see Paul, 1968 or Jaeger, 1969). In these theories it is assumed that the state of stress at a point *is* sufficient to quantify brittle fracture, and that the failure surface in stress space is dependent on the hydrostatic component of stress. In particular, it is observed that failure will occur under conditions of hydrostatic tension, and it is also recognised that, under loading conditions where all three principal stresses are negative, failure must occur by a shearing process. Nevertheless, although these ideas have relevance even today in rock and soil mechanics, they are of limited applicability in the study of brittle fracture of metals.

The two starting points for the development of a rigorous theory of brittle fracture are:

(a) Energetic considerations of failure, based on the celebrated Griffith's hypothesis (Griffith, 1921).

(b) Stress analysis for sharply notched, i.e. cracked, bodies, and the definition of the stress intensity factor.

These two elements may be brought together to produce a unified theory, which is well explained in many final year undergraduate textbooks such as those by Ewalds and Wanhill (1986), Broek (1985, 1988), Gdoutos (1993), or Kanninen and Popelar (1985). However, the essentials will be reviewed here, in order to establish the stress intensity factor as the pre-eminent quantity needed to assess all kinds of brittle fracture, whether under monotonic loading or repeated fatigue loading.

1.2 Review of Linear Elastic Fracture Mechanics

Griffith (1921) argued that, for a perfectly brittle material, crack extension could not occur unless the release of elastic energy stored within the system accompanying crack extension is at least as great as the energy needed to form the new free surfaces of the crack, i.e.

$$G \geq 2\gamma\,, \tag{1.1}$$

where G is the release of energy per unit area of the crack, usually known as the strain energy release rate (or sometimes the generalized crack extension force), and γ is the surface energy of the material, with a factor of 2 included as two new surfaces are created. It is seen that the left hand side of inequality (1.1) is the *driving* force propelling the crack, and is dependent on the load and geometry, whilst the term on the right hand side is a material property which, in principle, may be determined by a number of experiments.

It is illuminating to consider a 'thought experiment' to determine how the value of G might be determined, for a particular geometry. Figure 1.1(a) shows the component, which may be of arbitrary shape, and made of perfectly elastic material, containing a pre-existing defect of length a, loaded under displacement-controlled conditions. As a displacement Δ is imposed, work is done by the force F on the specimen, equal to the area ABC under the force-extension curve, shown in Figure 1.1(b). Imagine now that the component is unloaded, the crack length is increased by some method (perhaps by using a razor blade or fine saw!) by length δa, and the

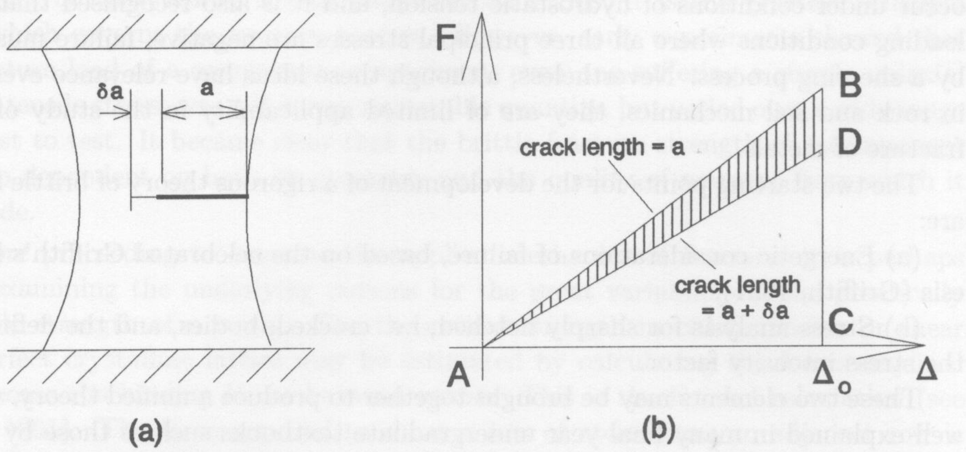

Figure 1.1: A 'thought experiment' to determine the value of G: (a) the cracked component (subject to mode I loading) and (b) the force-displacement history (under controlled displacement conditions)

displacement re-imposed. As the component is now clearly more compliant, trajectory AD might be followed, and hence the work done is given by area ADC. It follows that if the original component, containing a crack of length a, had suffered a crack extension δa, the loss of internal strain energy would be given by the shaded triangle ABD. This therefore represents the energy available to propel the crack. If the maximum displacement imposed during the experiment is Δ_o and the change in stiffness k $(= F/\Delta)$ associated with the crack extension is δk, the strain energy released, δU, is given by

$$\delta U = \frac{1}{2}\Delta_o^2 \delta k.$$

Dividing each side of the equation by δa, and taking the limit $\delta a \rightarrow 0$, we see that the strain energy release rate, G, is given by

$$G = \frac{\partial U}{\partial a} = \frac{1}{2}\Delta_o^2 \frac{dk}{da}. \tag{1.2}$$

This is an important result, as it means that the crack propelling force may be determined by measuring the change in stiffness of a cracked component with respect to crack length. The release of elastic energy arises because of unloading of the material adjacent to the crack tip as the crack grows, but it is important to note that the result described above does not rely on an understanding of the state of stress arising at the crack tip. The introductory texts cited earlier show that the result (1.2) also applies under imposed load (or so-called dead load) conditions. However, it should be emphasised that this result applies only when the force-displacement characteristic for the component is linear; cases where this does not hold are if the material has a non-linear stress-strain law, or when the external load-internal stress relation is not linear (for example, when pressing a sphere into a block of material).

The limitations of the approach just described are obvious, and a much more sophisticated way of determining the crack propelling term is needed. It should also be recognised that, for most engineering materials, the resistance to crack growth involves considerable dissipation of plastic work at the crack tip, which far exceeds the energy needed to create the new surfaces. A completely different approach to quantify the fracture process is therefore required, and this is provided by re-examining the problem starting from a consideration of the state of stress around a crack tip. In turn, this may be found by starting from a knowledge of the stresses arising near the apex of a large wedge of material, under arbitrary remote loading conditions. It is important to appreciate that the analysis which follows is strictly two-dimensional, and the results found therefore apply only under conditions of plane elasticity. A more general analysis is given later in the chapter.

1.2.1 Asymptotics of Plane Elasticity Problems

Figure 1.2(a) shows a notched component under arbitrary remote loading. Every engineer knows intuitively that the notch, or sharp *internal* corner, acts as a stress concentrator. We will show that, in many cases, the stress tends to infinity as the apex is approached. Infinite stresses are, of course, physically impossible; the predictions of the linear theory of elasticity are based on the assumption of small strains whereas, in reality, the material in the vicinity of the apex undergoes large strain deformation and attendant yielding. Nevertheless, by characterising the elastic strain field in the hinterland, which controls conditions at the apex, we can obtain useful information regarding fracture.

We can apply an asymptotic analysis to ascertain the nature of the apex stresses by considering only a region of material *very* close to the apex, as shown in Figure 1.2(b). This figure is *self-similar*, in that if we magnify it, or reduce it in size, it still looks the same. We can reduce it in size by as much as we please, which is equivalent to focusing attention closer and closer to the apex, so that the region of material can be considered as an infinite wedge of included angle 2α. As mentioned, we expect large stresses at internal corners, but we also expect bounded (zero) stresses at *external* corners ($2\alpha < 180^0$). The question arises: at what included angle does a bounded state of stress give way to a singular state? Is it simply at $2\alpha = 180^0$? These questions were answered by Williams (1952), who assumed that the stresses near the apex, $r \to 0$, could be written in a series expansion of the form

$$\sigma_{ij}(r,\theta) = \sum_k e_k r^{\lambda_k} f_k(\theta) = e_1 r^{\lambda_1} f_1(\theta) + e_2 r^{\lambda_2} f_2(\theta) + \ldots\ldots \quad (1.3)$$

where the constants e_k, the functions f_k and the exponents λ_k are to be found. If the λ_k are all positive, then the stresses at the apex are zero. We are more interested here

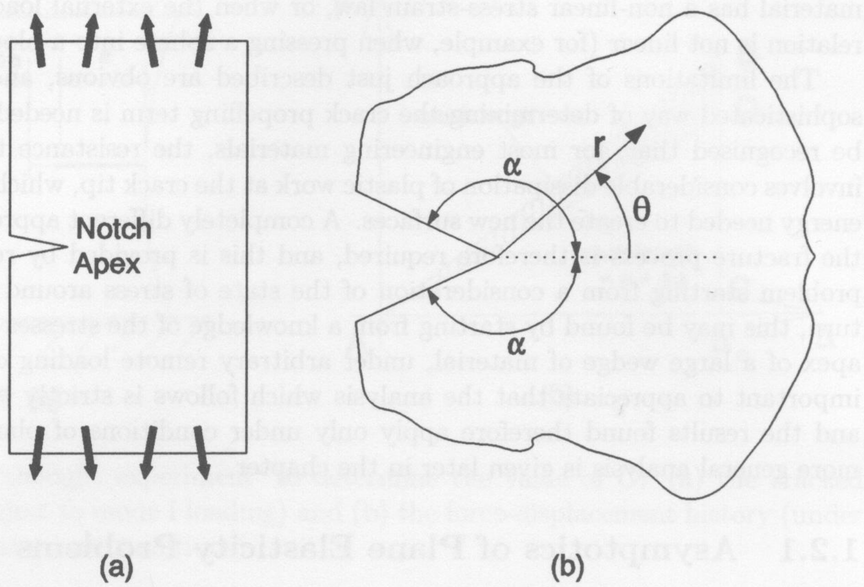

Figure 1.2: The state of stress at (a) the apex of a notch and (b) very close to the apex

in the case when the stresses are singular so that at least one of the λ_k is negative, in which case $\sigma_{ij} \to \infty$ as $r \to 0$. In fact, if $\lambda_1 < 0$ and we assume that $\lambda_1 < \lambda_2 < ...$, the stresses given by equation (1.3) are dominated by the first term, so that

$$\sigma_{ij}(r,\theta) \sim e_1 r^{\lambda_1} f_1(\theta) , \qquad r \to 0. \tag{1.4}$$

Following the original solution devised by Williams (1952), we choose an Airy stress function[1] of the form

$$\phi = r^{\lambda+2} \left[A\cos(\lambda+2)\theta + B\sin(\lambda+2)\theta + C\cos\lambda\theta + D\sin\lambda\theta \right] , \tag{1.5}$$

from which the stress components may be found by differentiation alone, using the relations (Timoshenko and Goodier, 1970)

$$\sigma_{rr}(r,\theta) = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{1.6}$$

$$\sigma_{\theta\theta}(r,\theta) = \frac{\partial^2\phi}{\partial r^2} \tag{1.7}$$

[1]It is not possible in this book to develop the theory of elasticity *ab initio*. A certain familiarity with the basic equations is therefore assumed. Suitable background texts are Timoshenko and Goodier (1970), or the more recent and up to date treatment by Barber (1992).

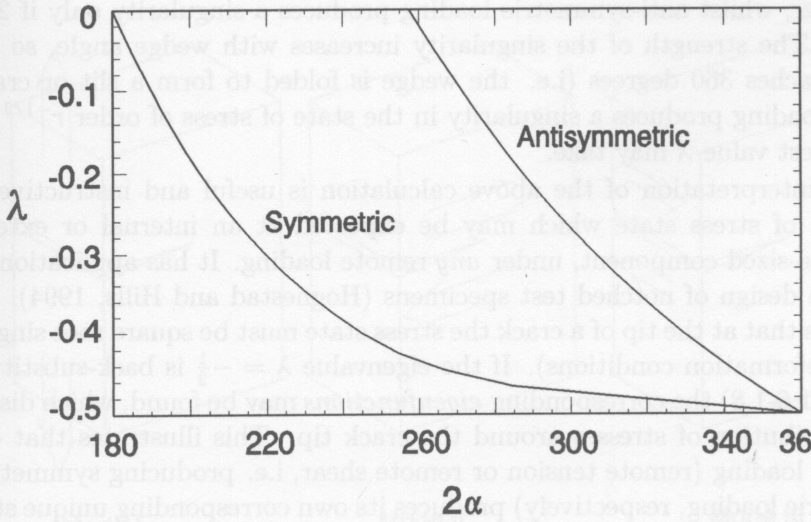

Figure 1.3: The strength of the singularity in the stress field at a re-entrant corner, $\sigma \sim r^{\lambda}$

$$\sigma_{r\theta}(r,\theta) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right). \tag{1.8}$$

Only equations (1.7) and (1.8) are relevant in the present context, and to these the following boundary conditions must be applied,

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0, \qquad \theta = \pm\alpha\,, \tag{1.9}$$

in order to ensure that the wedge faces are traction-free. Equation (1.9) yields a system of four simultaneous homogeneous equations in the constants A, B, C, D, and which have a non-trivial solution only when the determinant of their coefficients vanishes. This leads to the requirements that

$$(\lambda+1)\sin 2\alpha + \sin 2(\lambda+1)\alpha = 0 \tag{1.10}$$

for a symmetric load, and

$$(\lambda+1)\sin 2\alpha - \sin 2(\lambda+1)\alpha = 0 \tag{1.11}$$

corresponding to an antisymmetric load. Equations (1.10) and (1.11) may readily be solved for λ, for any given wedge included angle 2α, subject to the constraint $\lambda > -1$, which is a necessary condition for continuity of the displacements in the wedge (Williams, 1952). If the equations yield more than one value of λ, we choose the *minimum* value, i.e. λ_1 in (1.3, 1.4).

The solutions to equations (1.10, 1.11) are presented graphically in Figure 1.3. The symmetric loading gives rise to a singularity if $2\alpha > 180$ degrees, i.e. at *any*