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L.G. Kovács (Ed.)

Groups – Canberra 1989

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This volume contains some of the papers presented (in one case unfortunately only *in absentia*) during the Australian National University Group Theory Program 1989. The focal point of the Program was the Third International Conference on the Theory of Groups and Related Topics; all but three of the papers here were in fact given during that Conference.

Each paper in this volume has been assessed by at least one referee; the editor is deeply indebted to those who have given so freely of their time.

All authors have declared that their paper is in final form and no similar paper has been or is being submitted elsewhere.

INTRODUCTION

The Third International Conference on the Theory of Groups and Related Topics was held at the Australian National University in Canberra from 25 to 29 September, 1989. In many respects it continued the tradition of the earlier conferences held in 1965 and 1973. The total number of participants was 103. While this is less than the 1973 figure, it is pleasing to be able to report that the number of overseas participants was greater, and their geographical distribution wider, than at either of the earlier conferences. There were in fact 54 overseas participants, and they came from 17 countries: Canada, People's Republic of China, France, Federal Republic of Germany, Hungary, Italy, Japan, Republic of Korea, New Zealand, Pakistan, Philippines, Singapore, Thailand, UK, USA, USSR, Zimbabwe.

The conference was more concentrated than its predecessors (one week instead of two). Nevertheless, the talks ranged widely over the whole field of group theory, paying somewhat greater attention than before to cognate areas where group theory finds its applications. There were some 40 talks in all, of which half were invited lectures. The generally high standard of the talks contributed greatly to the success of the conference.

The majority of the visitors were accommodated in Ursula College, which was the centre for the social activities. We are greatly indebted to the College authorities for the friendly, efficient, and tactful way in which they looked after us. Both the initial reception by the Vice-Chancellor of the Australian National University, Professor L. Nichol, and the highly successful dinner were held at the College. Other highlights of the social program were a party hosted by Bernhard and Dorothea Neumann at their home and an excursion to the Tidbinbilla nature reserve.

Financing a conference in which many participants come from distant countries inevitably poses difficulties. Apart from the registration fees, three major sources of support may be identified. First, the conference was officially sponsored by the International Mathematical Union, the Australian Mathematical Society, and the Australian National University. The considerable financial contribution from these institutions at an early stage was vital to forward planning. Second, the conference was held in conjunction with the Australian National University Group Theory Program 1989, whose other activities included miniconferences on group representations, on computation in groups, on soluble groups, and on Burnside questions. The Program, which attracted over 30 overseas visitors as well as a number of Australian mathematicians, greatly enhanced the value of the conference by enabling overseas mathematicians to extend their stay in Australia. Third, the conference is indebted to the many institutions,

both in Australia and overseas, who by one means or another provided travel and living expenses for individual participants. The valuable support of the Universities of Melbourne, La Trobe, Sydney, and New South Wales should be specially mentioned.

Many people worked behind the scenes to ensure the success of the conference. My warmest thanks are extended to all of them, particularly to the members of the Organising and Local Organising Committees and to the Proceedings Editor.

Bernhard Neumann was the driving force behind the first two conferences. Indeed, Bernhard has always been the personification of mathematics in action. What better way, then, to celebrate his impending eightieth birthday (on October 15) than by holding a conference? Members of Bernhard's immediate family and many members of his extended family of colleagues and students were present at the conference. Speaker after speaker paid tribute to his contribution to mathematics and his influence on their lives. A photographic display featuring his mathematical career was mounted at the conference dinner and an evening was devoted to reminiscences. We wish Bernhard many more active and productive years.

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CONTENTS

A LIE APPROACH TO FINITE GROUPS

J. L. Alperin

A couple of years ago I received a postal card from Bernhard Neumann bearing the message, "Happy one-half birthday." I replied, thanking him, recalling that I had been in Canberra a few years before on the occasion of his three-quarters birthday, and I let him know that I had just accepted an invitation to speak at the celebration of his four-fifths birthday. Now I would like to add that I am looking forward to his first birthday, in the year 2009.

Group theory is a subject that comes in many different variations: finite groups, infinite groups, Lie groups, transformation groups, algebraic groups and so on. It cuts across algebra, analysis and geometry. Nevertheless, all the different types of groups are closely related, either directly or by analogy. The two oldest kinds of groups, finite groups (where the subject began) and Lie groups are particularly close. We wish to add to that relationship here.

1. Structure of Lie type groups

The Lie type groups, the classical groups and the other analogs of Lie groups, are at the heart of finite group theory. Much of their structure is entirely similar with the structure of Lie groups. We shall examine some of the most basic aspects of the structure of Lie type groups from this point of view. For the sake of simplicity of exposition, we shall restrict ourselves to the case of the general linear group $G = GL(n, q)$ over a finite field with $q = p^e$ elements; as is often the case, nothing will be lost in doing this.

The Borel subgroups are the most common and useful of the subgroups. These are the group B of upper triangular matrices and its conjugates. The parabolic subgroups are the conjugates of the subgroups which contain B and these are easily described. Each parabolic containing B is a group of upper block-triangular matrices, that is, the group of all non-singular matrices which, for a fixed sequence of positive integers $k_1, \ldots, k_r$ which sum to n, have square matrices of sizes $k_1, \ldots, k_r$ along the main diagonal and are zero on the remaining entries below this diagonal. Each of these parabolic subgroups is consequently an extension of a p-group (the block-triangular

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matrices with identity matrices for the square matrices) by a direct product of general linear groups of dimensions $k_1, \ldots, k_r$.

Having examined a bit of the subgroup structure, let us now turn to the geometry associated with the general linear group. If V is the natural n-dimensional module for $GL(n,q)$ then the parabolic subgroups are the stabilizers of flags of subspaces of V, that is, of strictly ordered sequences of non-zero proper subspaces of V. In particular, B is the stabilizer of a complete flag, that is a sequence $V_1, V_2, \ldots, V_{n-1}$, where V_i has dimension i. It is a truly great idea of Tits' that the uniform way to study the different geometries of the Lie and Lie type groups, for example, unitary, symplectic or orthogonal geometries, is to introduce a simplicial complex, that is, a triangulated space, called the building, on which the group acts. In the case of the general linear group the description is quite simple. The vertices of this space, that is, the zero-dimensional simplices, are the non-zero proper subspaces of V. Two vertices are joined by an edge, a one-simplex, if one of the subspaces is properly contained in the other. Three vertices are joined to form a triangle, a two-simplex, if one of the subspaces is strictly contained in a second which in turn is properly contained in the third, that is, the three subspaces form a flag. We now have an immediate and close connection between the subgroup structure we have discussed and the geometry: the parabolic subgroups are the stabilizers of the simplices in the building!

Having discussed subgroup structure and geometry, let us now turn to representation theory. One of the common types of structures that arise here are the Hecke algebras. If P is a parabolic subgroup of G, then the coset space G/P is a G-set and we can form a vector space over a field with this set as basis, so that we get a module for G. The endomorphism algebra of this module is known as a Hecke algebra and these algebras play a central role in the representation theory of G. Notice that we have, at once, a connection with the subgroup structure. The representation theory of groups of Lie type like G, over an algebraically closed field k of characteristic p, where $q = p^e$, is extremely analogous with the finite-dimensional representation theory of Lie groups. In particular, there is a well-developed theory of weights [**3**]. If S is a simple kG-module and U is the group of upper triangular matrices with ones on the main diagonal (the p-subgroup in the description of the way a parabolic subgroup is an extension in the case of B) then the subspace of S of elements left fixed by U is one-dimensional and the stabilizer of this subspace is a parabolic subgroup containing B (one of the upper block-triangular groups). In this way we have attached a one-dimensional module for a parabolic subgroup to each simple kG-module. It is then true that this defines a one-to-one correspondence (modulo isomorphism and conjugacy, of course) between the simple modules and the one-dimensional representations, the weights, of the parabolic subgroups. Again, the representation theory is closely related to subgroup structure.

Finally, in this survey of some of the most basic ideas that arise in studying Lie type groups, we turn to a result which ties together the subgroup structure, the geometry and the representation theory. Let C_0 be the kG-module which has the 0-simplices, the vertices, of the building as a basis, let C_1 be the kG-module similarly constructed for the 1-simplices, the edges, and so on up to C_{n-2}, the module corresponding to the $n-2$ simplices, that is the complete flags on V. Since the building is a simplicial complex there are boundary maps which are kG-homomorphisms of each C_i to C_{i-1}, for i positive. Moreover, we can let C_{-1} be the one-dimensional trivial module for G, with the empty set as a basis element, and define a boundary map, the augmentation, from C_0 to C_{-1} which sends each vertex to the empty set. In this way we can define a sequence of modules and maps, the augmented chain complex of the building with coefficients in k, as follows:

$$C_{n-2} \to C_{n-3} \to \cdots \to C_1 \to C_0 \to C_{-1} \to 0$$

The image of each boundary map is contained in the kernel of the next one and the successive quotients are the (reduced) homology groups of the building with coefficients in k, a sequence of kG-modules. The theorem of Solomon-Tits [**6**] states that all these homology groups are zero, with the one exception of H_{n-2} and this is a simple module, which is also projective at the same time, and is known as the Steinberg module, one of the most basic of all the modules. In this way, we have an example of how three different aspects of the structure of groups of Lie type, subgroup structure, geometry and representation theory, are all very closely related.

2. Lie structure of finite groups

It has long been observed, in the classification work on simple groups and in the representation theory of arbitrary finite groups, that there are subgroups that arise in the work that appear analogous with the parabolic subgroups of Lie type groups. In fact, there are many aspects of Lie type groups that appear in the study of general finite groups. Our thesis here is that these are not an unrelated set of interesting and provocative accidents, but evidence of an important unifying principle which should be taken very seriously and which is easy to enunciate:

If G is an arbitrary finite group and p is any prime divisor of its order, then there exist interesting and important analogs of all the aspects of the structure of Lie type groups whose natural characteristic is p.

In particular, to take a random example, say, let G be the symmetric group Σ_9 and $p = 3$: then there should be analogs of parabolic subgroups, weights and so on. We shall now illustrate this principle for the ideas we have just discussed in the preceding section. We begin by tabulating the results of this part of the paper where we shall discuss analogs for all these ideas. On the left are the Lie type concepts and on the right the analogous ideas for arbitrary finite groups.

Structure of Lie type groups	*Lie structure of finite groups*
Borel subgroups	Sylow normalizers
Parabolic subgroups	p-parabolic subgroups
Buildings	Brown complexes
Hecke algebras	Centralizer rings
Weights	Weights
Steinberg modules	Steinberg complexes

The Borel subgroup B is the normalizer of the subgroup U which is, in turn, a Sylow p-subgroup of $GL(n,q)$. The normalizer of a Sylow p-subgroup is the obvious analog for an arbitrary group and it is certainly clear that such subgroups play a central role in group theory, even appearing in a critical way in the most elementary uses of the Sylow theorems which are taught in first courses in algebra. A subgroup L of an arbitarary group G is called a p-parabolic if L is the normalizer in G of the largest normal p-subgroup $O_p(L)$ of L; in the Lie type case this property has long been known to characterize parabolic subgroups, by a theorem and Borel and Tits. The p-parabolic subgroups occur in many places; for example, the normalizer of a defect group of a p-block is a p-parabolic subgroup.

Turning to geometry, there is indeed a very good analog of the building for our arbitrary group G, the Brown complex due to K. Brown [**2**]. Consider the poset (partially ordered set) $S_p(G)$ of all non-identity p-subgroups of G. We define a simplicial complex $|S_p(G)|$ by letting the vertices be these non-identity p-subgroups, joining two vertices if one of the two subgroups is properly contined in the other, forming a triangle from three vertices that are a strictly linearly ordered set of three subgroups, and so on in the same way as we constructed the building but using a different poset, not the poset of non-zero proper subspaces of a vector space. The connection with buildings is quite direct: if G is of Lie type and characteristic p then the Brown complex and the building are of the same homotopy type [4]. (In fact, there is a closer connection: Let $|B_p(G)|$ be the Bouc complex which is the simplicial complex formed from the poset $B_p(G)$ of non-identity p-subgroups Q of G such that $N(Q)$ is a p-parabolic subgroup of G with $Q = O_p(N(Q))$; $|B_p(G)|$ is then also of the same homotopy type as the Brown complex and, if G is of Lie type and characteristic p, is homeomorphic with the building being, in fact, isomorphic with the first barycentric subdivision of the building). The Brown complex and related complexes are now of great interest to a number of group theorists and it is clear that they are basic objects of study in the theory of finite groups.

Turning to representation theory, the analogs of Hecke algebras are well-known and precede the introduction of Hecke algebras; they are the centralizer rings of Schur. If G acts transitively on a set X then the endomorphism ring of the module, formed from linear combinations of elements of X, is a centralizer ring and is a useful object in the study of permutation groups and in applications to representation theory. The tie between these two concepts is so close that it is now customary to call all centralizer rings by the name Hecke algebras! It is also possible to define the concept of weight for our arbitrary group G; the concept does not just apply in the specialized Lie situation. A pair (Q, S) is called (see [**1**]) a weight of G if Q is a p-subgroup of G and S is a simple kG-module for $N(Q)$ which is also projective when regarded as a module for $N(Q)/Q$ (which may be done as Q must act trivially on S since S is simple). (Of course, changes due to conjugacy or isomorphism are not regarded as changing the weight.) In the Lie type case these weights correspond naturally with the Lie weights; in fact, the Steinberg module plays a role in this connection. Moreover, there is a conjectured relation with simple kG-modules: the number of simple kG-modules equals the number of weights. This conjecture and its refinements have attracted a great deal of attention recently and a solution would represent a real breakthrough in representation theory. A reformulation due to Knörr and Robinson [**5**] is particularly suggestive since it involves use of the Brown complex and puts the above weight conjecture and the Alperin-McKay conjecture, on characters of height zero in a block, in a very similar form.

Finally, we wish to again tie up subgroup structure, geometry and representation theory in a way analogous to the Solomon-Tits theorem on the Steinberg module. Here, the decisive result is one of Webb's [**7**]. We can form an augmented chain complex for the Brown complex just as we did for the building. Webb's result is that this chain complex of kG-modules is the direct sum of two complexes, one consisting of just projective modules and the other complex having the property that it is a long exact sequence which is also split (which is called contractible and is analogous to the geometric idea). One may assume that the complex consisting of projective modules has no summand which is a contractible complex and it is then unique up to isomorphism, by the Krull-Schmidt theorem applied to complexes. If G is of Lie type then the projective complex just described consists of a complex all of whose modules are zero, with just one exception, the Steinberg module which appears as the component in one dimension. For this reason, it is entirely appropriate to call the complex of projective modules appearing in Webb's theorem by the name of the Steinberg complex. Webb has also shown how these results may be used in an extremely effective way to calculate cohomology of modules, a problem that hitherto seemed much more difficult, so that these ideas are not only interesting as analogs but they are very useful and basic.

3. Homology of the Brown complex

We now wish to apply this Lie principle to show how it leads to new mathematics. There are a number of topics which would illustrate this but we shall discuss the question of the nature of the homology of the Brown complex. This should certainly be interesting if the Brown complex is a good analog of the building. Webb's theorem also suggests this topic since the structure of the Steinberg complex is quite unknown. We shall restrict ourselves to reduced homology with the integers $\mathbb{Z}$ as coefficients and write $H_n(S_p(G))$ for $H_n(|S_p(G)|, \mathbb{Z})$, so we are really studying the augmented chain complex for the Brown complex with $\mathbb{Z}$ as coefficients in place of k. It would be just as easy to just deal with k as coefficients but we shall use the coefficients that usually appear in topology.

A group with a normal p-subgroup has all the homology of its Brown complex zero; the complex is contractible and the reverse implication is conjectured [4]. In the case that G is of Lie type and of characteristic p the connection between the building and the Brown complex again yields that the homology of the Brown complex is non-zero in exactly one dimension. A theorem of Quillen [4] also gives the same result for a solvable group G with abelian Sylow p-subgroups and with $O_p(G)$ the identity. These results suggest that there is a paucity of non-zero homology for the Brown complex, but nothing of the sort is the case as we shall now illustrate with a couple of theorems.

THEOREM 1. *If the generalized Fitting subgroup $F^*(G)$ is a p'-group and E is a maximal elementary abelian p-subgroup of G and of order p^e, then $H_{e-1}(S_p(G)) \neq 0$.*

(The assumption on $F^*(G)$ is equivalent with the existence of a normal p'-subgroup of G which contains its own centralizer.)

THEOREM 2. *If $p > 3$ and E is a maximal elementary abelian p-subgroup of the symmetric group Σ_n and of order p^e, then $H_{e-1}(S_p(\Sigma_n)) \neq 0$.*

The proofs have a number of ideas in common. One is the use of the Quillen complex $|A_p(G)|$ formed from the poset $A_p(G)$ of non-identity elementary abelian p-subgroups of G which is also of the same homotopy type as the Brown complex and so has the same homology. Another is the use of a complex which appears to be quite unrelated. For any group G and a prime p, we form a complex $|C_p(G)|$ on the set $C_p(G)$ of subgroups of G of order p. The vertices are these subgroups, two are joined by an edge if they commute elementwise, three form a triangle if they again commute elementwise and so on. The 1-skeleton of this complex, the graph consisting of the vertices and edges, is a well-studied graph and occurs in work on the classification of simple groups and in the construction of the Fischer sporadic simple groups. A general result is as follows: