A. G. HAMILTON

# Numbers, sets and axioms

## the apparatus of mathematics

# *Numbers, sets and axioms*

*THE APPARATUS OF MATHEMATICS*



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# PREFACE

The mathematician's work proceeds in two directions: outwards and inwards. Mathematical research is constantly seeking to pursue consequences of earlier work and to postulate new sorts of entities, seeking to demonstrate that they have consistent and useful properties. And on the other hand, part of mathematics consists of introspection, of a process backwards in the logical sequence, of the study of the nature and the basis of the subject itself. This inward-looking part is what is normally called the foundations of mathematics, and it includes study of set theory, the number systems, (at least some) mathematical logic, and the history and philosophy of mathematics. This book is not intended to cover all of these areas comprehensively. It is intended to convey an impression of what the foundations of mathematics are, and to contain accessible information about the fundamental conceptual and formal apparatus that the present-day working mathematician relies upon. (This apt description is due to G. T. Kneebone.) The concepts directly involved are: numbers (the various number systems), sets, orderings of sets, abstract mathematical structures, axiomatic systems, and cardinal and ordinal numbers.

The book presupposes no knowledge of mathematical logic. It does presuppose a certain amount of experience with mathematical ideas, in particular the algebra of sets and the beginnings of mathematical analysis and abstract algebra. Its structure is in many ways a compromise between what is appropriate for a course text and what is appropriate for a more readable background text. Different teachers would place different emphases on the topics here dealt with, and there is no clear logical sequence amongst the topics. It is intended, therefore, that the chapters (at least the earlier ones) should have few interdependences. There are forward and backward references where appropriate, but these are generally illustrative rather than essential. Each chapter begins with a brief summary which includes an indication of its relationships with the other chapters. The level of mathematical sophistication does increase as the book proceeds, but throughout the book there are exercises of a routine nature as well as more taxing ones. These exercises are an integral part of the presentation, and they are designed to help the reader to consolidate the material.

The content is definitely mathematics, not logic or philosophy. Nevertheless, no book on this sort of subject matter can avoid involvement with philosophical issues. As readers will discover, underlying the whole of the presentation is a definite philosophical viewpoint, which may be summarised in the following remarks. Mathematics is based on human perceptions, and axiom systems for number theory or set theory represent no more than a measure of the agreement that exists (at any given time) amongst mathematicians about these perceptions. Too much prominence has been given to formal set theory as a foundation for mathematics and to sets as 'the' fundamental mathematical objects. Mathematics as a whole is *not* a formal axiomatic system. Mathematics as practised is not merely the deduction of theorems from axioms. Basing all of mathematics on set theory begs too many questions about the nature and properties of sets. Thus, while axiomatic set theory is treated in some detail in Chapter 4, it is treated as a *part* of mathematics, and not as a basis for it.

Following each chapter there is a list of references for further reading with some comments on each one. The details of these titles will be found in the full bibliography at the end of the book. Also provided at the end of the book are a glossary of symbols and hints and answers to selected exercises. In the text the symbol ► is used to denote the resumption of the main exposition after it has been broken by a theorem, example, remark, corollary or definition.

During the development of a text and its presentation it is very useful for an author to have the work read by another person. A more detached reader can spot deficiencies which, whether through over-familiarity or carelessness, are not apparent to the writer. I have been fortunate, in this respect, to have been kindly and conscientiously assisted by Dr G. T. Kneebone and Dr John Mayberry. Both have read several versions of the various chapters and have made numerous suggestions for improvement, most of which have been incorporated in the book. I am most grateful to them for their effort, and I can only hope that they will derive some satisfaction from the final version which is my response to their comments.

May Abrahamson made a marvellous job of typing (and patiently re-typing). Cambridge University Press has again been most efficient and helpful. The University of Stirling has assisted by allowing me leave during which the bulk of the writing was done. My sincere thanks go to all of these.

A.G.H. January 1982

convey mental impression.

# *CONTENTS*

# 1

# NUMBERS

**Summary**

First we consider what are the basic notions of mathematics, and emphasise the need for mathematicians to agree on a common starting point for their deductions. Peano's axioms for the natural numbers are listed. Starting with a system of numbers satisfying Peano's axioms, we construct by algebraic methods the systems of integers, rational numbers, real numbers and complex numbers. At each stage it is made clear what properties the system constructed has and how each number system is contained in the next one. In the last section there is a discussion of decimal representation of rational numbers and real numbers.

The reader is presumed to have some experience of working with sets and functions, and to be familiar with the ideas of bijection, equivalence relation and equivalence class.

## 1.1 Natural numbers and integers

It is fashionable nowadays at all levels of study from elementary school to university research, to regard the notion of set as the basic notion which underlies all of mathematics. The standpoint of this book is that the idea of set is something that no modern mathematician can be without, but that it is first and foremost a *tool* for the mathematician, a helpful way of dealing with mathematical entities and deductions. As such, of course, it becomes also an object of study by mathematics. It is inherent in the nature of mathematics that it includes the study of the methods used in the subject; this is the cause of much difficulty and misunderstanding, since it apparently involves a vicious circle. The trouble is that most people (mathematicians included) try to regard mathematics as a *whole* – a logical system for proving true theorems based on indubitable principles. The present author believes that this is a misleading picture. Mathematics is rather a mixture of intuition, analogy and logic – a body of accepted knowledge based on perceived reality, together with tools and techniques for drawing analogies, making conjectures and providing logical justification for conclusions drawn.

The fundamental notions of mathematics now are the same as they were a hundred years ago, namely, numbers or, to be more specific, the number systems. Modern abstract mathematics (with the exception, perhaps, of geometry and topology) is based almost entirely on analogies drawn with properties of numbers. Here are some simple examples. The algebraic theory of fields arises from a generalisation of the properties of addition and multiplication of numbers. Real analysis is just the study of functions from real numbers to real numbers. Functional analysis applies the methods of algebra (themselves derived from methods used in concrete numerical situations) to mathematical systems which are generalisations of three-dimensional physical space (which can be represented, of course, via coordinate geometry, by means of ordered triples of real numbers).

Our knowledge of the number systems derives from our perception of the physical world. We count and we measure, and the origins of mathematics lie in these activities. Modern methods can help in writing down and working out properties of numbers and in clarifying relationships between these properties. Indeed, this process has reached an advanced stage. Most mathematicians now agree on what are the principles which it is proper to use in order to characterise the number systems. This is very significant, for it provides a common starting point for logical deductions. If all mathematicians based their deductions on their own personal intuitions then communication would be very difficult and the subject would not be very coherent. One purpose of this book is to expound and explain the common starting point. In the first chapter we deal with the number systems out of which mathematics develops, and in subsequent chapters we shall investigate some of the tools (notably set theory) and try to explain what modern 'foundations of mathematics' is all about.

Counting is the first mathematical activity that we learn. We learn to associate the objects in a collection with words (numbers) which mark them off in a sequence and finally indicate 'how many' there are in the collection. This experience gives us an intuition about an unending sequence of numbers which can be used to count in this way any finite collection of objects. It is assumed that the readers of this book will have a well-developed intuition about natural numbers, so we shall not go into the psychology behind it (this is not to imply that the psychology of mathematical intuition is not worthy of study – just that it is outside the scope of this book).

*Notation* The set of natural numbers will be denoted by $\mathbb{N}$. $\mathbb{N}$ is the collection $\{0, 1, 2, \ldots\}$. Notice that we include 0 in $\mathbb{N}$. This is merely a convention. It is common but not universal.

Let us list some properties of these numbers which accord with intuition.

***Examples 1.1***

(a) There is an addition operation (a two-place function on $\mathbb{N}$) which is commutative and associative.
(b) $0+n=n$, for every $n \in \mathbb{N}$.
(c) In the list $\{0, 1, 2, \ldots\}$, the number following $n$ is $n+1$, for each $n$.
(d) $n+1 \neq n$, for every $n \in \mathbb{N}$.
(e) $m+1=n+1$ implies $m=n$, for every $m, n \in \mathbb{N}$.
(f) There is a multiplication operation (also a two-place function on $\mathbb{N}$) which is commutative and associative, and which distributes over addition.
(g) $0 \times n = 0$, and $1 \times n = n$, for every $n \in \mathbb{N}$.
(h) $m \times n = p \times n$ implies $m = p$, for every $m, n, p \in \mathbb{N}$ $(n \neq 0)$.

► Clearly, we can continue writing down such properties indefinitely. These are the kind of things we learn in elementary school. We learn them, discover that they work, and come to believe them as truths which do not require justification. However, the mathematician who is working in the theory of numbers needs a starting point in common with other mathematicians. *Peano's axioms* (listed first in 1888 by Dedekind, and not originating with Peano) are such a common starting point. They are five basic properties, all of them intuitively true, which serve as a basis for logical deduction of true theorems about numbers. They are as follows.

(P1) There is a number 0.
(P2) For each number $n$, there is another number $n'$ (the *successor* of $n$).
(P3) For no number $n$ is $n'$ equal to 0.

(P4) If $m$ and $n$ are numbers and $m' = n'$, then $m = n$.
(P5) If $A$ is a set of numbers which contains 0 and contains $n'$ for every $n \in A$, then $A$ contains all numbers.

(Note that we have used the word 'number' here as an abbreviation for 'natural number'.)

***Remarks 1.2***

(a) (P1) and (P2) provide the process for generating the sequence of natural numbers corresponding to the intuitive counting procedure. (P3) reflects the fact that the sequence has a beginning.

(b) (P4) is a more complicated property of the sequence of numbers: different numbers have different successors.

(c) (P5) is the principle of mathematical induction. This is the most substantial of the five, and is the basis of most proofs in elementary number theory. It may be more familiar as a method of proof rather than an axiom, and in a slightly different form: if $P(n)$ is a statement about a natural number $n$ such that $P(0)$ holds, and $P(k+1)$ holds whenever $P(k)$ holds, then $P(n)$ holds for every natural number $n$. This can be seen to be equivalent to (P5) if we think of the set $A$ and the statement $P(n)$ related by:

$n \in A$ if and only if $P(n)$ holds.

Thus, given a set $A$, a statement $P(n)$ is determined and vice versa.

► There is no mention in Peano's axioms of the operations of addition and multiplication. This is because these can be defined in terms of the other notions present.

The common starting point, therefore, *need not* mention these operations. However, it is quite difficult to carry out the procedure of defining them and justifying their existence (see Section 4.3) and, for our present purposes, it is certainly unnecessary. For our purposes we can broaden the common starting point, that is to say, we can include amongst our basic intuitive properties the following.

(A) There is a two-place function (denoted by +) with the properties:

$m + 0 = m$, for every number $m$.

$m + n' = (m + n)'$, for all numbers $m, n$.

(M) There is a two-place function (denoted by $\times$) with the properties:

$m \times 0 = 0$, for every number $m$.

$m \times n' = (m \times n) + m$, for all numbers $m, n$.

(Following the usual mathematical practice we shall usually omit multiplication signs, and write $mn$ rather than $m \times n$. The exceptions will be when special emphasis is being placed on the operation of multiplication.)

We could also include amongst our basic intuitive properties the assertions that these operations satisfy the commutative, associative and distributive laws, but it is not difficult to prove, from the properties given above, that these hold. Let us carry out one such proof, as an example.

***Theorem 1.3***

Addition on $\mathbb{N}$ is commutative.

*Proof*

This is an exercise in proof by induction. We require two preliminary results:

(i) $0 + m = m$, for all $m \in \mathbb{N}$.

(ii) $m' + n = (m + n)'$, for all $m, n \in \mathbb{N}$.

For (i), we use induction on $m$. By property (A) we have $0 + 0 = 0$. Suppose that $0 + k = k$. Then $0 + k' = (0 + k)' = k'$ (using property (A) again).

Hence, by the induction principle, $0 + m = m$ holds for all $m \in \mathbb{N}$.

For (ii), we apply (P5) to the set

$$A = \{n \in \mathbb{N}: m' + n = (m + n)', \text{ for every } m \in \mathbb{N}\}.$$

First, $0 \in A$, since $m' + 0 = m'$ (by property (A)) and $(m + 0)' = (m)'$ (again by property (A)), and so $m' + 0 = (m + 0)'$, for any $m \in \mathbb{N}$. Second, suppose that $k \in A$, i.e. $m' + k = (m + k)'$ for every $m \in \mathbb{N}$. Then

$$\begin{aligned} m' + k' &= (m' + k)' && \text{(by property (A))}, \\ &= ((m + k)')' && \text{(by our supposition that } k \in A), \\ &= (m + k')' && \text{(by property (A))}. \end{aligned}$$

This holds for every $m \in \mathbb{N}$, so $k' \in A$. We can therefore apply (P5) to deduce that $A = \mathbb{N}$, i.e., (ii) holds for all $m, n \in \mathbb{N}$.

Now to complete the proof of the theorem we need a further induction. Let $B$ be the set $\{n \in \mathbb{N}: m + n = n + m, \text{ for every } m \in \mathbb{N}\}$.

First, $0 \in B$ since $m+0=m$ (by property (A)), and $0+m=m$ (by (i) above). Second, let us suppose that $k \in B$, i.e., $m+k=k+m$ for every $m \in \mathbb{N}$. Now

$$\begin{aligned} m+k' &= (m+k)' \quad \text{(by property (A))}, \\ &= (k+m)' \quad \text{since } k \in B, \\ &= k'+m \quad \text{(by (ii) above)}. \end{aligned}$$

This holds for every $m \in \mathbb{N}$, so $k' \in B$. Applying (P5) to $B$ we conclude that $B=\mathbb{N}$, i.e. $m+n=n+m$ for all $m, n \in \mathbb{N}$.

► It is not our purpose to develop elementary number theory, but there are some basic results which we should at least mention.

***Remark 1.4***

For every natural number $n$, either $n=0$ or $n=m'$ for some natural number $m$.

This may be proved using Peano's axioms. It is left as an exercise for the reader, with the hint that (P5) should be applied to the set $A=\{n \in \mathbb{N}: \text{either } n=0 \text{ or } n=m' \text{ for some } m \in \mathbb{N}\}$.

***Theorem 1.5***

Every non-empty set of natural numbers has a least member.

Before we prove this we require to give an explanation of the term 'least member'. Again this is an intuitive notion, but its properties can be derived from the definitions and properties of numbers already given. For $m, n \in \mathbb{N}$ we write $m<n$ if there is $x \in \mathbb{N}$, with $x \neq 0$, such that $m+x=n$. (We also use the notation $m \leqslant n$, with the obvious meaning.) A set $A$ of natural numbers has a *least member* if there is an element $m \in A$ such that $m<n$ for every other element $n \in A$. The result of Theorem 1.5 is intuitively true when we think of the normal sequence $\{0, 1, 2, \ldots\}$ of natural numbers and note that the relation $<$ corresponds to the relation 'precedes'.

*Proof* (of Theorem 1.5)

Let $A$ be a set of natural numbers which contains no least member. We show that $A$ is empty. We apply (P5) to the set $B=\{x \in \mathbb{N}: x \leqslant n \text{ for every } n \in A\}$. Certainly $0 \in B$, since $0 \leqslant n$ for every $n \in A$. Suppose that $k \in B$. Then $k \leqslant n$ for every $n \in A$. But $k$ cannot belong to $A$, since if it did it would be the least element of $A$. Hence

$k < n$ for every $n \in A$, and consequently $k+1 \leqslant n$ for every $n \in A$, i.e. $k+1 \in B$. By (P5), then, we have $B = \mathbb{N}$. By the definition of $B$, this means that $x \leqslant n$ holds for every $x \in \mathbb{N}$ and every $n \in A$. This is impossible unless $A$ is empty, in which case it is vacuously true. The proof is now complete.

► The above theorem can be used to justify a slightly different version of the principle of mathematical induction.

(P5*) If $A$ is a set of natural numbers which contains 0 and contains $n'$ whenever $0, 1, \ldots, n$ all belong to $A$, then $A$ contains all natural numbers.

***Theorem 1.6***

(P5*) holds (as a consequence of (P5), through Theorem 1.5).

*Proof*

Let $A \subseteq \mathbb{N}$, with $0 \in A$ and such that $n' \in A$ whenever $0, 1, \ldots, n \in A$. We require to show that $A = \mathbb{N}$. Consider the set $\mathbb{N} \backslash A$ (the set of all elements of $\mathbb{N}$ which do not belong to $A$). Suppose that $\mathbb{N} \backslash A$ is not empty. Then by Theorem 1.5 it contains a least member, $n_0$, say. We have therefore $n_0 \notin A$, and $x \in A$ for every $x$ with $x < n_0$. Now $n_0 \neq 0$, since $0 \in A$, by our original hypothesis. Hence, $n_0 = m'$ for some $m \in \mathbb{N}$ (by the result of Remark 1.4). So we have $m' \notin A$, but we have also $0, 1, \ldots, m \in A$. This is a contradiction since our hypothesis says that we have $n' \in A$ whenever $0, 1, \ldots, n \in A$. It follows that $\mathbb{N} \backslash A$ is empty, and consequently $A = \mathbb{N}$.

► The last of our basic results is one that we shall refer to when we discuss properties of the other number systems. It is the result which is commonly known as the *division algorithm*. Its proof is given here for the sake of completeness, and the reader may omit it.

***Theorem 1.7***

Let $a \in \mathbb{N}$, $b \in \mathbb{N}$, $b \neq 0$. There exist $q \in \mathbb{N}$, $r \in \mathbb{N}$ with

$$a = qb + r \quad \text{and} \quad r < b.$$

Moreover, the numbers $q$ and $r$ are uniquely determined.

*Proof*

Let $S = \{y \in \mathbb{N} : y + xb = a, \text{ for some } x \in \mathbb{N}\}$. ($S$ may be thought of as the set of differences $a - xb$ for all those $x \in \mathbb{N}$ such that $a \geqslant xb$.)

$S$ is not empty, since $a \in S$ (corresponding to $x = 0$). Hence, by Theorem 1.5, $S$ contains a least element, say $r$. Since $r \in S$, there is $q \in \mathbb{N}$ such that $r + qb = a$, i.e. $a = qb + r$. We must have $r < b$, for otherwise $b \leqslant r$, so that $r = b + r_1$, say, with $r_1 \in \mathbb{N}$, and $r_1 < r$, necessarily. Then $r + qb = a$ gives $r_1 + b + qb = a$, i.e. $r_1 + (q+1)b = a$, and this implies that $r_1 \in S$. This contradicts the choice of $r$ as the least element of $S$.

It remains to show that $q$ and $r$ are unique. Suppose that $a = qb + r = q'b + r'$, with $r < b$, $r' < b$, and $r \leqslant r'$, say. Then there is $t \in \mathbb{N}$ such that $r + t = r'$, and we have $qb + r = q'b + r + t$, and hence $qb = q'b + t$. It follows that $q'b \leqslant qb$, so $q' \leqslant q$. Let $q = q' + u$, say, with $u \in \mathbb{N}$. Then $q'b + ub = qb = q'b + t$, giving $ub = t$. Since $r + t = r'$, then, we have $r + ub = r'$. Consequently $ub \leqslant r'$, which contradicts $r' < b$, unless $u = 0$. Thus we must have $u = 0$, and this implies that $r = r'$, $t = 0$, and $q = q'$, as required.

In the above proof we have used, besides Theorem 1.5, a few properties of addition, multiplication and inequalities which have not been explicitly derived from our basic assumptions. The most apparent, perhaps, is the cancellation law for inequalities: if $ax \leqslant bx$ and $x \neq 0$, then $a \leqslant b$. This may be treated as an exercise.

► Natural numbers are a product of intuition. There is no need for a mathematical *definition* of natural numbers. Peano's axioms may be seen as an attempt to define, but they are in fact merely an attempt to characterise natural numbers. But immediately two questions arise. First, are Peano's axioms true of our intuitive natural numbers? And second, is there any collection of objects, essentially different from the set of natural numbers, for which Peano's axioms also hold true? The answer to the first question is clearly (intuitively) in the affirmative. The answer to the second is much harder to find, for it involves the mathematical abstractions: 'collection of objects for which Peano's axioms hold true', and 'essentially different'. We shall see in due course that the second answer is negative, but before that we must explain the abstractions.

Consider the set $2\mathbb{N} = \{2n : n \in \mathbb{N}\}$ of even natural numbers, and denote $k + 2$ by $k^*$, for each $k \in 2\mathbb{N}$. Then the following are true:

(1) $0 \in 2\mathbb{N}$.

(2) For each $k \in 2\mathbb{N}$, $k^* \in 2\mathbb{N}$.

(3) For no $k \in 2\mathbb{N}$ is $k^*$ equal to 0.

(4) If $k, l \in 2\mathbb{N}$, then $k^* = l^*$.

(5) If $A \subseteq 2\mathbb{N}$ is such that $0 \in A$ and $k^* \in A$ whenever $k \in A$, then $A = 2\mathbb{N}$.

In other words, Peano's axioms 'hold' for the set $2\mathbb{N}$ (together with the operation *). It is not difficult to conceive of other structures (i.e. sets together with unary operations) for which Peano's axioms also hold. We can say precisely what this means in general.

***Definition***

A *model* of Peano's axioms is a set $N$, together with a function $f$ and an object $e$ (a triple $(N, f, e)$) such that

(P1*) $e \in N$.

(P2*) The domain of $f$ is $N$, and for each $x \in N$, $f(x) \in N$.

(P3*) If $x \in N$, then $f(x) \neq e$.

(P4*) If $x, y \in N$ and $f(x) = f(y)$, then $x = y$.

(P5*) If $A$ is a subset of $N$ which contains $e$ and contains $f(x)$ for every $x \in A$, then $A = N$.

The function $f$ is to act like the successor function and $e$ is to act like 0. The reader should compare these conditions (P1*), . . . , (P5*) carefully with (P1), . . . , (P5).

► The model $(2\mathbb{N}, *, 0)$ given above, by its very existence, tells us that Peano's axioms do not characterise the set of natural numbers uniquely. But this new model has a *structure* which is identical to the structure of $(\mathbb{N}, ', 0)$. The two models are *isomorphic*, that is to say there is a bijection $\varphi : \mathbb{N} \to 2\mathbb{N}$ such that $\varphi(n') = (\varphi(n))^*$ for all $n \in \mathbb{N}$, and $\varphi(0) = 0$. (The function $\varphi$ is given by $\varphi(n) = 2n$.) In general we can make the following definition.

***Definition***

Two models $(N_1, f_1, e_1)$ and $(N_2, f_2, e_2)$ of Peano's axioms are *isomorphic* if there is a bijection $\varphi : N_1 \to N_2$ such that

(i) $\varphi(f_1(x)) = f_2(\varphi(x))$, for all $x \in N_1$,

and

(ii) $\varphi(e_1) = e_2$.

Such a function is said to be an *isomorphism*.

► Models of Peano's axioms exist which are different from, but isomorphic to, $(\mathbb{N}, ', 0)$. Mathematically, such models are essentially the same, and for mathematical purposes it really does not matter whether natural numbers are taken to be the elements of $\mathbb{N}$ or the elements of a different but isomorphic model. This will form the basis of our construction of natural numbers within set theory in Section 4.3. In a sense it is only a matter of labelling. If two models are isomorphic then their mathematical characteristics are the same but their elements may be objects of different sorts.

What makes the overall situation sensible, however, is the result of Corollary 1.9 below. It implies that there is no model of Peano's axioms which is not isomorphic to $(\mathbb{N}, ', 0)$. In other words, Peano's axioms do characterise the *structure* of $(\mathbb{N}, ', 0)$ completely.

***Theorem 1.8*** (definition by induction)

Let $(N, f, e)$ be any model for Peano's axioms. Let $X$ be any set, let $a \in X$ and let $g$ be any function from $X$ to $X$. Then there is a unique function $F$ from $N$ to $X$ such that

$$F(e) = a,$$

and

$$F(f(x)) = g(F(x)), \quad \text{for each } x \in N.$$

► Theorem 1.8 legitimises what is probably a familiar process for defining functions with domain $\mathbb{N}$. This process was used on page 4 above in the properties (A) and (M). First specify the value of $F(0)$, and then, on the assumption that $F(n)$ has been defined, specify $F(n+1)$ in terms of $F(n)$. Here, of course, we are dealing with an arbitrary model of Peano's axioms, rather than $\mathbb{N}$. The proof of Theorem 1.8 is lengthy and technical, so we shall omit it at this stage. Theorem 4.15 is a particular case of Theorem 1.8, concerning that model of Peano's axioms (the set of abstract natural numbers) which is constructed in Section 4.3. The proof given there can be generalised in a straightforward way to apply to an arbitrary model, as required here.

***Corollary 1.9***

Any two models of Peano's axioms are isomorphic.

*Proof*

Let $(N_1, f_1, e_1)$ and $(N_2, f_2, e_2)$ be models of Peano's axioms. By Theorem 1.8, there is a unique function $F: N_1 \to N_2$ such that

$$F(e_1) = e_2,$$