A. A. Kirillov

Elements of the Theory of Representations

Translated from the Russian by Edwin Hewitt

Grundlehren der mathematischen Wissenschaften 220

A Series of Comprehensive Studies in Mathematics

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Springer-Verlag
Berlin Heidelberg New York 1976

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Title of the Russian Original Edition:
Elementy Teorii Predstavleniĭ
Publisher: Nauka, Moscow, 1972

AMS Subject Classifications (1970):
Primary 43-xx Secondary 20Cxx, 22A25, 22Dxx, 22Exx, 81A17, 81A78

ISBN 3-540-07476-7 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-07476-7 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Kirillov, Aleksandr Aleksandrovich, 1936—. Elements of the theory of representations (Grundlehren der mathematischen Wissenschaften; 220). Translation of Elementy teorii predstavleniĭ. Bibliography: p. Includes index. 1. Representations of groups. I. Title. II. Series: Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellung; 220. QA171. K5213. 512'.22. 75-31705

Printed in Germany
Typesetting and printing: Zechnersche Buchdruckerei, Speyer
Bookbinding: Konrad Triltsch, Würzburg

Translator's Preface

The translator of a mathematical work faces a task that is at once fascinating and frustrating. He has the opportunity of reading closely the work of a master mathematician. He has the duty of retaining as far as possible the flavor and spirit of the original, at the same time rendering it into a readable and idiomatic form of the language into which the translation is made. All of this is challenging. At the same time, the translator should never forget that he is not a creator, but only a mirror. His own viewpoints, his own preferences, should never lead him into altering the original, even with the best intentions. Only an occasional translator's note is permitted.

The undersigned is grateful for the opportunity of translating Professor Kirillov's fine book on group representations, and hopes that it will bring to the English-reading mathematical public as much instruction and interest as it has brought to the translator. Deviations from the Russian text have been rigorously avoided, except for a number of corrections kindly supplied by Professor Kirillov. Misprints and an occasional solecism have been tacitly taken care of. The translation is in all essential respects faithful to the original Russian.

The translator records his gratitude to Linda Sax, who typed the entire translation, to Laura Larsson, who prepared the bibliography (considerably modified from the original), and to Betty Underhill, who rendered essential assistance.

Seattle, June 1975 Edwin Hewitt

Preface

The author of this book has, over a number of years, given courses and directed a seminar at Moscow State University on the theory of group representations.

The majority of the students in these courses and participants in the seminar have been university students of the first two years (and also occasionally graduate students and gifted secondary school pupils).

The membership of the seminar has constantly renewed itself. There have been new participants, not overly burdened with knowledge, but ready to study things new to them and to solve a huge number of problems.

For each new group of participants, it was necessary to organize a "primer" on the parts of mathematics needed for the theory of representations and also on the foundations of the theory of representations.

The author quickly got the idea of replacing himself by a book which should be on the one hand not too thick (which always frightens the reader) but which on the other hand should contain all of the needed information.

Through various circumstances, the realization of this idea required far more time than was originally contemplated. Nevertheless, through the moral support of friends and of my teacher I. M. Gel'fand, the book has now finally been written. The author begs forgiveness of his readers for the facts that the book is thicker than one would like and also contains only a part of what it ought to.

The first part of the book (§§ 1–6) are not directly connected with the theory of representations. Here we give facts needed from other parts of mathematics, with emphasis on those that do not appear in the prescribed curricula of elementary university courses. A reader familiar with this material may begin at once with the second part (§§ 7–15). This part contains the principal concepts and methods of the theory of representations. In the third part (§§ 16–19), we illustrate the general constructions and theorems of the second part by concrete examples.

The historical sketch found at the end of the book reflects the author's view of the development of the theory of representations, and makes no claim to be a reference work on the history of mathematics. At the end of the essay, we describe the present state of the theory of representations and give references to the periodical literature.

A particular feature of the book, and one which has reduced its size enormously, is the large number of problems. These problems, and the remarks appended to them, are printed in separate paragraphs. Nevertheless, one must not ignore them, since they play an essential rôle in the main text. In particular, a majority of the proofs are given in the form of a cycle of mutually connected problems. Almost all problems are supplied with remarks, which as a rule enable one to reconstruct the solution without difficulty. All the same it is useful to try to solve a problem independently and to turn to the remark appended only in case of failure.

We point out certain peculiarities in our choice of subject matter. Very little mention is made in the book of finite-dimensional representations of semisimple Lie groups and Lie algebras. The fact is that there are already available in the Russian language a sufficient number of good expositions of this part of the theory of representations (see [47], [46], [57]), and the author did not wish to repeat them.

The rôle of the theory of group representations in the theory of special functions is completely ignored in the present book. The monograph of N. Ja. Vikenkin [53] may serve as a good introduction to this topic.

The author's task has also not included a description of the manifold applications of the theory of representations in mathematical physics. At the present time there is a wide literature dealing with these applications (see for example [3], [37], [40], [28]).

We have apportioned a large space to the method of orbits, which has up to now not made its way into textbooks, and which by its simplicity and perspicuity without doubt belongs to the fundamentals of the theory of representations. A certain incompleteness in § 15, which deals with the method of orbits, is explained by the current state of knowledge in this area. Many important theorems have been proved only in special cases, or indeed exist only as conjectures.

At the present time, many mathematicians are working in this field both in the Soviet Union and in other countries. Beyond any peradventure our knowledge of the connections between orbits and representations will be much greater within a few years than it is now. The author hopes that some of the readers of this book may bring their contributions to the development of the theory of orbits.

A. Kirillov

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First Part. Preliminary Facts

§ 1. Sets, Categories, Topology

1.1. Sets

The background in the theory of sets needed to read this book is amply supplied by what is given in ordinary university courses. (See for example the first chapters of the textbooks of A. N. Kolmogorov and S. V. Fomin [38] and of G. E. Šilov [49].) One can find more penetrating treatments (including an exact definition of the concept of set) in the books of Fraenkel and Bar-Hillel [17] and P. Cohen [12].

For the reader's convenience we list here some of the notation that we shall use and also recall certain definitions.

$\emptyset$ denotes the void set;

$x \in X$ means that the element x belongs to the set X;

$x \notin X$ means that the element x does not belong to the set X;

$X \subset Y$ means that the set X is contained in the set Y (and possibly coincides with it);

$\bigcup_{\alpha \in A} X_\alpha$ denotes the union of the system of sets X_α, which are indexed by the set A; if A is finite, then we also use the notation $X \cup Y \cup \cdots \cup Z$;

$\bigcap_{\alpha \in A} X_\alpha$ denotes the intersection of the system of sets X_α;

$X \setminus Y$ denotes the complement of the set X in the set Y;

$\prod_{\alpha \in A} X_\alpha$ denotes the Cartesian product of the sets X_α, that is, the collection of all functions $\{x_\alpha\}_{\alpha \in A}$, where $x_\alpha \in X_\alpha$;

$f\colon X \to Y$ or $X \xrightarrow{f} Y$ denotes the mapping f of the set X into the set Y;

$f\colon x \mapsto y$ or $x \xrightarrow{f} y$ means that the mapping f carries the element x into the element y;

X^Y is the set of all mappings of the set Y into the set X;

card X is the cardinal number of the set X; for finite sets X we also use the notation $|X|$;

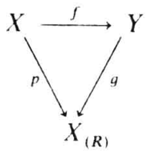
$\{x; A\}$ denotes the set of all x that satisfy the condition A.

We say that a *binary relation* is defined in a set X if we have specified a certain subset R of $X \times X$. Instead of the relation $(x, y) \in R$ we also write xRy and we say that *x and y stand in the relation R* or that *x and y are connected by the relation R*. The *inverse relation* R^{-1} is defined as the set of all pairs (x, y) for which $(y, x) \in R$. The *product* $R_1 \cdot R_2$ is the set of all pairs (x, y) for which there exists an element z such that $(x, z) \in R_1$ and $(z, y) \in R_2$.

A relation R is called *reflexive* if R contains the diagonal $\Delta = \{(x, x); x \in X\}$, *symmetric* if $R = R^{-1}$, and *transitive* if $R \cdot R \subset R$. A relation with all three of these properties is called an *equivalence relation*. In this case, instead of $(x, y) \in R$, we say "x and y are R-equivalent" (or simply equivalent, if it is clear what relation R we have in mind).

The set of elements equivalent to a given $x \in X$ is the *equivalence class* containing x. The set $X_{(R)}$ of equivalence classes is called the *factor set* of the set X by the relation R. Assigning each $x \in X$ to the class containing it, we obtain the *canonical mapping* $p: X \to X_{(R)}$.

It is clear that a mapping $f: X \to Y$ can be embedded in the commutative diagram[1]



if and only if f is constant on each equivalence class. In this case, g is defined uniquely from f and is called a *factor mapping*.

An *order relation* on a set X is a transitive binary relation R that is antisymmetric in the following sense: $R \cap R^{-1} \subset \Delta$. For an order relation R, we usually write $x \succ y$ instead of $(x, y) \in R$.

A set that is equipped with an order relation is said to be *ordered* (sometimes *partially ordered*). An ordered set is said to be *linearly ordered* if $R \cup R^{-1} = X \times X$ (i.e., if every pair of elements are comparable). The following assertions are equivalent:

1 (Zermelo's axiom of choice). *The product of an arbitrary family of nonvoid sets is nonvoid.*

2 (Zorn's lemma). *Suppose that X is an ordered set in which every linearly ordered subset Y is bounded* (that is, there is an element $x \in X$ such that $x \succ y$ for all $y \in Y$). *Then the set X contains at least one maximal element* (that is, an element x_0 such that if $x \succ x_0$, then $x = x_0$; maximality of x_0 does not imply that $x_0 \succ x$ for all $x \in X$).

Zorn's lemma is a generalization of the well-known principle of mathematical induction, and replaces this principle in situations where we are considering uncountable sets.

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[1] See footnote 2 on page 4.

A *directed set*[1] is a set A with an order relation R defined in it satisfying the following additional condition:

for arbitrary $\alpha, \beta \in A$, there exists an element $\gamma \in A$ such that $\alpha \prec \gamma$, $\beta \prec \gamma$.

Let (A, R) be a directed set and X an arbitrary set. A mapping of A into X is called a *net* or a *direction* in X. Clearly this notion is a generalization of the notion of a sequence in X (to which it reduces if A is the set of natural numbers with the ordinary order relation).

As a rule one considers in mathematics sets which are endowed with one or another *structure* (for example, ordered sets, groups, topological spaces, and so on). We can give an exact meaning to this notion.

A *tower of sets* over X is any set obtained from X and auxiliary sets $S, T, \ldots$ by the elementary operations listed above (see page 1). To define a structure on X is to fix an element of a certain tower of sets over X. (For the foregoing examples of structures, the corresponding towers have the form $(2)^{X \times X}$, $X^{X \times X}$, $(2)^{(2)^X}$, where (2) is an auxiliary set consisting of exactly 2 elements.)

1.2. Categories and Functors

The language of categories, which will be used in this book, is so simple and natural that it presents no difficulties even for a reader unfamiliar with it. Here we give only a few basic definitions. More information can be found, for example, in the book of A. Grothendieck [25, Ch. I] or in the appendix of D. Buchsbaum to the book [10]. See also the Appendix "The language of categories" to the lecture notes of Ju. I. Manin on algebraic geometry (Publishing House of Moscow State University, 1970), from which we borrow the first sentence.

"The language of categories embodies a "sociological" approach to a mathematical object: a group or a space is considered not as a set with an inherent structure by itself, but as a member of the society of objects similar to it."

We say that we are given a *category* K if

1) there is given a class $\operatorname{Ob} K$ of *objects* of the category K;

2) for every pair A, B of objects of K there is given a set $\operatorname{Mor}(A, B)$ of *morphisms* of the object A into the object B;

3) there is a *law of composition* defined for every triple A, B, C of objects in K, that is, a mapping

$$\operatorname{Mor}(A, B) \times \operatorname{Mor}(B, C) \to \operatorname{Mor}(A, C).$$

The composition of the morphisms $f \in \operatorname{Mor}(A, B)$ and $g \in \operatorname{Mor}(B, C)$ is denoted by $g \circ f$ and satisfies the following conditions:

a) $f \circ (g \circ h) = (f \circ g) \circ h$ for arbitrary $f \in \operatorname{Mor}(C, D)$, $g \in \operatorname{Mor}(B, C)$, $h \in \operatorname{Mor}(A, B)$;

b) for every $A \in \operatorname{Ob} K$ there exists an element $1_A \in \operatorname{Mor}(A, A)$ such that $1_A \circ f = f$, $g \circ 1_A = g$ for arbitrary $f \in \operatorname{Mor}(B, A)$, $g \in \operatorname{Mor}(A, B)$.

As an example, consider the category M where $\operatorname{Ob} M$ is the class of all sets and $\operatorname{Mor}(A, B) = B^A$. Many of the categories considered below are *subcategories*

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[1] Bourbaki uses the expression *filtering to the right*.

of M, that is, the objects of these categories are sets, the morphisms are mappings of sets, and composition of morphisms is composition of mappings[1].

If a morphism $f \in \mathrm{Mor}(A, B)$ admits an *inverse* morphism f^{-1} (i.e., a morphism such that $f \circ f^{-1} = 1_B$, $f^{-1} \circ f = 1_A$), then it is called an *isomorphism*, and the objects A and B are called *isomorphic*.

For every category K, we can define the *dual* category K°. By definition, we have $\mathrm{Ob}\, K^\circ = \mathrm{Ob}\, K$, $\mathrm{Mor}(A, B)^\circ = \mathrm{Mor}(B, A)$. The composition of f and g in K° is defined as the composition of g and f in K.

An object X is called *universally repelling (universally attracting)* if for every $Y \in \mathrm{Ob}\, K$, the set $\mathrm{Mor}(X, Y)$ ($\mathrm{Mor}(Y, X)$) consists of exactly one element. From this definition, it follows that if there are several universal objects in a category K, then they are all canonically isomorphic.

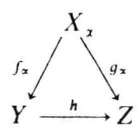
It is clear that in going from a category to its dual, universally repelling objects turn into universally attracting objects, and conversely.

The concept of universal object permits us to consider from a single point of view a great number of constructions that are used in mathematics. In particular, we shall see below that tensor products, enveloping algebras, induced representations, and cohomology of groups can be defined as universal objects in appropriately chosen categories.

By way of an example, we now give the definition of the sum and product of objects of an arbitrary category.

Let $\{X_\alpha\}_{\alpha \in A}$ be a family of objects of a category K. We shall consider a new category K_A. The objects of K_A are the collections $(Y, \{f_\alpha\}_{\alpha \in A})$, where Y is an object in K and $f_\alpha \in \mathrm{Mor}(X_\alpha, Y)$.

A morphism from $(Y, \{f_\alpha\}_{\alpha \in A})$ to $(Z, \{g_\alpha\}_{\alpha \in A})$ is defined as a morphism $h: Y \to Z$ such that for all $\alpha \in A$, the following diagram is commutative[2]:



We shall suppose that there is a universally repelling object $\{X, i_\alpha\}$ in the category K_A (all such objects are canonically isomorphic, as we mentioned above). Then the object X is called the *sum* of the family $\{X_\alpha\}$ and the morphism i_α is called the *canonical embedding* of the summand X_α in the sum X.

The definition of the *product* P of the family $\{X_\alpha\}_{\alpha \in A}$ and of the *canonical projection* $p_\alpha \in \mathrm{Mor}(P, X_\alpha)$ is obtained from the definition of sum by *reversing arrows*, that is, by replacing K_A by $(K_A^\circ)^\circ$.

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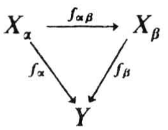
[1] The reader may wish to have an example of a category not of this type. Examples are the category of formal groups and the category of diagrams. The category K_A considered below is a special case of these.

[2] A diagram consisting of objects and morphisms of a category K is said to be *commutative* if the composition of morphisms along a path marked out by arrows of the diagram depends only upon the initial and terminal points of the path. In the following example, this means that $h \circ f_\alpha = g_\alpha$.

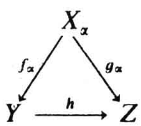
Problem 1. Show that the sum and product exist for an arbitrary family of objects in the category M.

Hint. Consider the operations of disjoint union and ordinary product of sets.

A small change in the definition of sum and product leads to the concepts of inductive and projective limits. Suppose that the set of indices A is a directed set and, if $\alpha \prec \beta$, then there exists a morphism $f_{\alpha\beta} \in \mathrm{Mor}(X_\alpha, X_\beta)$ such that $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$ for every triple $\alpha \prec \beta \prec \nu$. We consider the category whose objects are collections $(Y, \{f_\alpha\}_{\alpha \in A})$, $f_\alpha \in \mathrm{Mor}(X_\alpha, Y)$, such that for all $\alpha \prec \beta$ the following diagrams commute:



The morphisms of $(Y, \{f_\alpha\})$ into $(Z, \{g_\alpha\})$ are those morphisms $h \in \mathrm{Mor}(Y, Z)$ such that for all $\alpha \in A$, the following diagrams commute:



A universally repelling object in this category is called the *inductive limit* of the family $\{X_\alpha\}$. The definition of *projective limit* is obtained by reversing arrows.

Problem 2. Let A be the set of all natural numbers with the following definition of order: $m \prec n$ means that m divides n. Let X_n be the set of all integers, for every n. Let f_{mn} be the operation of multiplying by n/m. Prove that the inductive limit of the family $\{X_m\}$ can be identified in a natural way with the set of rational numbers, and the mapping f_m with division by m.

Let K_1 and K_2 be two categories. Suppose that to every object X in K_1, there corresponds an object $F(X)$ in K_2 and to every morphism $f \in \mathrm{Mor}(X, Y)$ there corresponds a morphism

$$F(f) \in \mathrm{Mor}(F(X), F(Y)).$$

Suppose further that the equalities

$$F(1_X) = 1_{F(X)}, \qquad F(f \circ g) = F(f) \circ F(g),$$

hold. Then we say that F is a *covariant functor* from K_1 into K_2. We obtain the notion of a *contravariant functor* if we replace the last condition by $F(f \circ g) = F(g) \circ F(f)$. (This is equivalent to replacing one of the categories K_1 and K_2 by its dual category.)

Covariant functors from K_1 into K_2 are themselves a category. The morphisms of F into G are the so-called *functorial morphisms*. These assign to each object X in K_1 a morphism $\phi(X)\colon F(X)\to G(X)$ such that the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\phi(X)} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\phi(Y)} & G(Y) \end{array}$$

is commutative for all $f\in \mathrm{Mor}(X, Y)$.

The category of contravariant functors is defined analogously.

We can also define functors of several variables, covariant in some variables and contravariant in the others.

Problem 3. Let K be an arbitrary category. Show that the mapping $(X, Y)\to \mathrm{Mor}(X, Y)$ can be completed to a functor from $K\times K$ into M, contravariant in the first variable and covariant in the second.

Hint. For $f\in \mathrm{Mor}(X_1, X)$, $g\in \mathrm{Mor}(Y, Y_1)$, $\phi\in \mathrm{Mor}(X, Y)$, set $F(f,g)\colon \phi\mapsto g\circ\phi\circ f$.

A covariant functor F from the category K into M is called *representable* if it is isomorphic to a functor $\mathrm{Mor}(X,\cdot)$, obtained from a *bifunctor*[1] Mor by fixing the first variable in an obvious way. The object X is called a *representing object* for the functor F.

Analogously, a contravariant functor has a representing object Y if F is isomorphic to $\mathrm{Mor}(\cdot, Y)$.

Many important functors are representable or become representable under suitable modification of the category.

1.3. The Elements of Topology

This section consists essentially of a list of terminology, concepts, and basic facts about topology. The fundamentals of topology can be found in the book [38, Ch. II]. A more detailed treatment is given in the book of J. L. Kelley [36].

A topological space is a set X in which we are given a family τ of subsets having the following properties:

1) the void set and X itself belong to τ;

2) the intersection of a finite number of elements of τ belongs to τ;

3) the union of an arbitrary family of elements of τ belongs to τ.

The system τ is called a *topology* on X.

A subsystem $\tau'\subset\tau$ is called a *base for the topology* τ if every element of τ is the union of a certain family of elements of τ'. Every system of subsets of X that satisfies 1) and 2) above is the base for a certain topology.

Sets belonging to the topology τ are called *open* relative to this topology. An open set containing a point $x\in X$ is called a neighborhood of that point. The complements of open sets are called *closed*. For every $Y\subset X$, there exists a smallest closed set containing Y. It is called the *closure* of Y. We say that a

[1] That is, a functor of two variables.

subset Y is dense in X if the closure of Y coincides with X. The sets that are obtained from open and closed sets by the operations of countable unions, countable intersections, and complementation are called *Borel sets*.

One can define the general concept of a *limit* in a topological space in the following way. Let $\{x_\alpha\}_{\alpha \in A}$ be a direction in X (that is, a family of points of X, numbered by the elements of a directed set A; see para. 1.1). A point x is said to be a limit of the direction $\{x_\alpha\}$ if for every neighborhood U of the point x, there exists an element $\alpha \in A$ such that $x_\beta \in U$ for all $\beta \succ \alpha$. We write this as $x_\alpha \underset{A}{\rightarrow} x$ or $\lim_{\alpha \in A} x_\alpha = x$. The reference to the set A is often omitted.

A mapping of one topological space into another is called *continuous* if the inverse image of every open set is open, and is called a *Borel mapping* if the inverse image of every open set is a Borel set.

Problem 1. Prove that a mapping f is continuous if and only if the condition $x_\alpha \underset{A}{\rightarrow} x$ implies that $f(x_\alpha) \underset{A}{\rightarrow} f(x)$ (that is, f commutes with the operation of passing to the limit).

In spaces with a countable basis, the general directions of problem 1 may be replaced by ordinary sequences.

Topological spaces and continuous mappings form a category T, in which the sum and the product of an arbitrary family of objects is defined.

Problem 2. Prove that the product of a family of topological spaces $\{X_\alpha, \tau_\alpha\}_{\alpha \in A}$ is the set $\prod_{\alpha \in A} X_\alpha$ with the topology τ, a basis for which is formed by sets of the form $\prod_{\alpha \in A_1} U_\alpha \times \prod_{\alpha \in A \setminus A_1} X_\alpha$, where A_1 is a finite subset of A and the sets U_α belong to τ_α.

Hint. First consider the product of two spaces.

Every subset Y of a topological space (X, τ) is itself a topological space if we define the open subsets of Y to be the intersections with Y of open subsets of X. A subset Y with this topology is called a *subspace of* X.

Let R be an equivalence relation on X. The factor set $X_{(R)}$ will be a topological space if we define as open those subsets whose inverse images are open in X. The set $X_{(R)}$ with this topology is called a *factor space* of the space X.

A topological space is called *compact* if every covering of it by open sets admits a finite subcovering.

Problem 3. The product of an arbitrary family of compact spaces is compact. The image of a compact space under a continuous mapping is compact.

A topological space is called *separated* or *Hausdorff* if every pair of distinct points admit disjoint neighborhoods. It is called *semiseparated*, or a T_0 *space*, if one of each pair of distinct points admits a neighborhood not containing the other. A compact Hausdorff space is called a *compactum*.

In Hausdorff spaces, every direction can admit only one limit. This property accounts for the fact that the majority of topological spaces used in mathematics and in its applications are Hausdorff.

Nevertheless, there are important classes of topological spaces for which Hausdorff separation does not in general hold. An example is provided by factor