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Edited by A. Dold and B. Eckmann

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INTRODUCTION

In these notes our goal has been to develop the algebraic machinery for the study of the Witt group $W(k,I)$ of degree k mapping structures and the Witt group of asymmetric inner product spaces. We are particularly interested in their relationship which arises in an exact octagon which is studied both for a field F and for the integers Z. We show that this octagon is the appropriate generalization of the Scharlau transfer sequence [Lm 201].

We have tried to develop the properties of these Witt groups in a self-contained and complete manner in order to make this accessible to a larger audience. When references are given they generally are specific including page numbers. However, we should mention two general references. First, The Algebraic Theory of Quadratic Forms by T.Y. Lam [Lm] develops the Witt group over a field of characteristic not equal to 2. Second, Notes on the Witt Classification of Hermitian Innerproduct Spaces over a Ring of Algebraic Integers by P.E. Conner [C] discusses Hermitian forms and the Witt group over an algebraic number field and ring of integers therein. Together these should provide any background material the reader might need.

Although the viewpoint we take is entirely algebraic, we would be remiss if we did not mention its topological motivation. Much of this work originated in our efforts to explain and exposit the important work of N.W. Stoltzfus, Unravelling the Integral Knot Concordance Group [Sf-1]. We shall describe this topological connection momentarily. First however we need to describe the algebraic objects at hand.

Our object is to define and study a Witt group $W(k,I)$ of degree k maps, where k is an integer, D is the underlying ring, and I is a D-module. This group consists of Witt equivalence classes of triples (M,B,ℓ) satisfying:

(1) $B: M \times M \to I$ is an I-valued inner product defined on the D-module M.

(2) $\ell: M \to M$ is a D-module homomorphism of M satisfying $B(\ell x,\ell y) = kB(x,y)$. We refer to ℓ as a map of degree k.

A triple (M,B,ℓ) is metabolic (Witt equivalent to zero) if there is an ℓ-invariant submodule $N \subset M$ with $N = N^{\perp}$. Here $N^{\perp}$ is the orthogonal complement. This enables us to define the Witt equivalence relation $\sim$ by:

$$(M,B,\ell) \sim (M_1,B_1,\ell_1) \quad \text{iff} \quad (M \oplus M_1,\ B \oplus -B_1,\ \ell \oplus \ell_1)$$

is metabolic. The Witt equivalence class of (M,B,ℓ) is denoted $[M,B,\ell]$. The operation of direct sum $\oplus$ makes this collection of equivalence classes into a group $W(k,I)$. The identity is the Witt equivalence class of metabolic triples.

We also develop the basic properties of asymmetric inner product spaces (M,B) where no symmetry requirement is placed on the inner product B. The key to understanding these asymmetric inner products is a "symmetry operator" s satisfying $B(x,y) = B(y,sx)$

for all $x,y \in M$. As above we define the notion of metabolic by (M,B) is metabolic if there is an s-invariant subspace $N \subset M$ with $N = N^{\perp}$. This leads us to define Witt equivalence as before, and there results the asymmetric Witt group.

There is a very interesting relationship between this asymmetric Witt group and the Witt group of degree k maps. This comes from the squaring map $S: W(k,D) \to W(k^2,D)$ given by $[M,B,\ell] \to [M,B,\ell^2]$. For $D = F$ a field an eight term exact sequence is developed from S. This octagon involves the Witt group of asymmetric inner products $A(F)$ just described.

As a special case of the octagon we obtain the transfer sequence of Scharlau, Elman and Lam [Lm 201] and [E,L 23-25]. This appears as an exact octagon in which several terms vanish. We are able to reinterpret the kernel and cokernel of this transfer sequence. Thus, these Witt group $W(k,F)$ and $A(F)$ are an appropriate generalization of the classical Witt group $W(F)$, at least in so far as relating to and explaining the Scharlau transfer sequence.

We should like to prove exactness in the octagon over Z. In order to continue we have a boundary sequence which relates $W(k,Z)$ to $W(k,Q)$

$$0 \to W(k,Z) \to W(k,Q) \xrightarrow{\partial} W(k,Q/Z)$$

When $k = \pm 1$ this sequence is short exact. Using this we are able to prove exactness in the octagon over Z when $k = \pm 1$ by comparing the octagon to the octagon over Q.

As we have said, in these notes we are developing the algebraic machinery to study the Witt group of degree k maps and the asymmetric Witt group. We should discuss the topological motivation for this work. When $k = +1$ the Witt group $W(+1,Z)$ is the crucial element determining the bordism groups Δ_n of orientation preserving diffeomorphisms of n-dimensional closed oriented smooth manifolds. Medrano introduced the appropriate Witt invariant as follows. Let $f\colon M^{2n} \to M^{2n}$ be an orientation preserving diffeomorphism of a closed, oriented, smooth $2n$-dimensional manifold M^{2n}. We consider then the pair (M^{2n},f) in Δ_{2n}. The degree $+1$ mapping triple associated to this pair is

$$(H^n(M^{2n};Z)/_{\text{torsion}},\ B,\ f^*)$$

where $B(x,y) = \varepsilon_*((x \cup y) \cap [M^{2n}])$, ε_* is augmentation, $[M^{2n}]$ is the fundamental class, $\cup$ is cup product, $\cap$ is cap product, f^* is the induced map on cohomology. If (M^{2n},f) bounds, this triple is metabolic. Thus there is an induced homomorphism

$$I\colon \Delta_{2n} \to W(+1,Z).$$

The task of computing the bordism groups Δ_{2n}, was completed by Kreck [K] who showed that this Witt invariant was essentially the only invariant for bordism of diffeomorphisms.

More generally, given a closed oriented $2n$-dimensional manifold together with a map ℓ of degree k, the corresponding Witt triple (M,B,ℓ) as above satisfies $B(\ell x,\ell y) = kB(x,y)$. We are thus led to examine the Witt group W(k,Z). The Witt group of asymmetric inner product spaces arises in Quinn's work [Q] on open book decomposition. The relation of this to the Witt group arising in Kreck's work above is discussed by Stoltzfus in The Algebraic Relationship Between Quinn's Invariant of Open Book Decomposition and the Isometric Structure of the Monodromy [Sf-2]. In this he gives a geometric application of the exact octagon obtained in Chapter X.

The exact octagon (renamed "the eight fold way") has also been extended to the setting of the surgery obstruction groups by A. Ranicki, L. Taylor, and B. Williams. For a further discussion of the use of quadratic forms and the Witt group in topology the reader is referred to Alexander, Conner, Hamrick, Odd Order Group Actions and Witt Classification of Inner Products [ACH], and Stoltzfus Unravelling the Integral Knot Concordance Group [Sf-1].

We now describe briefly the organization of these notes. Chapter I lays out the inner product spaces and Witt groups we will be studying. We continue our study of the Wiit group in Chapter II by describing Witt invariants which will be used to compute the Witt group in many cases. These invariants include rank mod 2, discriminant, and signatures.

In Chapter III we study the characteristic and minimal polynomial of the degree k map ℓ in a degree k mapping structure (M,B,ℓ). This study is used in Chapter IV where we compute the Witt group $W(k,F)$ for F a field. This is done by decomposing $W(k,F)$ according to the characteristic polynomial of ℓ, and $A(F)$ according to the characteristic polynomial of s.

In Chapter V we develop an 8 term exact octagon which relates $W(k,F)$ to $A(F)$.

$$\begin{array}{ccccccccc} & \nearrow & W^{+1}(k,F) & \overset{S}{\to} & W^{+1}(k^2,F) & \to & W^{+1}(-k,F) & \searrow & \\ A(F) & & & & & & & & A(F) \\ & \nwarrow & W^{-1}(-k,F) & \leftarrow & W^{-1}(k^2,F) & \overset{S}{\leftarrow} & W^{-1}(k,F) & \swarrow & \end{array}$$

S is the squaring map $[M,B,\ell] \to [M,B,\ell^2]$. We prove again the Scharlau, Elman, Lam transfer sequence, and see that the exact octagon we develop is its appropriate generalization. In order to study this octagon over Z, we relate $W(k,Z)$ to $W(k,Q)$ by a boundary sequence in Chapter VI.

Now we place an additional requirement on the degree k mapping structure $[M,B,\ell]$, namely:

(*) ℓ satisfies the monic integral irreducible polynomial $f(t)$.

The resulting Witt group of triples satisfying the additional requirement (*) is denoted $W(k,Z;f)$.

The action of ℓ induces a $Z[t]/(f(t))$ - module structure on M. To simplify the notation let $S = Z[t]/(f(t))$. Note that S is only an order in the Dedekind ring of integers $O(E)$ of the algebraic number field $E = Q[t]/(f(t))$. This order S may not be the maximal order $O(E)$.

The first step to understanding $W(k,Z;f)$ or $W(k,Z;S)$ (the same thing) is to study the group $W(k,Z;D)$ for $D = O(E)$ the maximal order. This group consists of Witt equivalence classes of inner product spaces (M,B) in which M is a finitely generated torsion free D-module. This is in contrast to $W(k,Z;S)$ in which we only insist that the module structure of M lifts to the order $S = Z[t]/(f(t))$.

In Chapter VI we are interested only in $W(k,Z;D)$ and the resultant boundary sequence for the maximal order. We read this boundary sequence on the Hermitian level, where the - involution on E is given by $t \to kt^{-1}$ and $t^{-1} \to k^{-1}t$. One uses the following commutative diagram.

$$\begin{array}{ccccccc} 0 & \to & H(\Delta^{-1}(D/Z)) & \to & H(E) & \xrightarrow{\partial(D)} & H(E/\Delta^{-1}(D/Z)) \\ & & \downarrow t & & \downarrow t & & \downarrow t \\ 0 & \to & W(k,Z;D) & \to & W(k,Q;D) & \xrightarrow{\partial(D)} & W(k,Q/Z;D) \end{array}$$

$\Delta^{-1}(D/Z)$ denotes the inverse different of D over Z. The vertical isomorphisms denoted by t are induced by the trace of E over Q.

Thus the method employed for computing $W(k,Z;D)$ is to study the corresponding boundary sequence in the isomorphic Hermitian case. The image of $\partial(D)$ is the group $H(E/\Delta^{-1}(D/Z))$ which is computed as follows:

$$H(E/\Delta^{-1}(D/Z)) \xrightarrow{t} W(k,Q/Z;D) \xrightarrow{g} \bigoplus_{P=\overline{P}} W(k,F_p;D/P) \xleftarrow{tr} \bigoplus_{P=\overline{P}} H(D/P)$$

Here we sum over all - invariant maximal ideals P in D. The isomorphism g is induced by selecting a generator $1/p$ for the p-torsion in Q/Z. The trace map on finite fields from D/P to F_p induces the isomorphism tr with the Hermitian groups $\bigoplus_{P=\overline{P}} H(D/P)$.

We use the letter M to denote - invariant maximal ideals in S. In order to study $W(k,Z;S)$ one must use the following commutative diagram:

$$\begin{array}{ccccccccccccc} 0 & \to & H(\Delta^{-1}(D/Z)) & \to & H(E) & \xrightarrow{\partial(D)} & H(E/\Delta^{-1}(D/Z)) & \longrightarrow & & \longrightarrow & & & \\ & & \downarrow t & & \downarrow t & & \downarrow t & & & & & & \\ 0 & \to & W(k,Z;D) & \to & W(k,Q;D) & \xrightarrow{\partial(D)} & W(k,Q/Z;D) & \xrightarrow{g} & \bigoplus W(k,F_p;D/P) & \xleftarrow{tr} & \bigoplus H(D/P) \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & & & \downarrow tr \\ 0 & \to & W(k,Z;S) & \to & W(k,Q;S) & \xrightarrow{\partial(S)} & W(k,Q/Z;S) & \xrightarrow{g} & \bigoplus W(k,F_p;S/M) & \xleftarrow{tr} & \bigoplus H(S/M) \end{array}$$

First one computes $W(k,Z;D)$ for the maximal order by going to the Hermitian level and reading $\partial(D)$ in the group $\bigoplus_{P=\overline{P}} H(D/P)$. Then one forgets the D-module structure and remembers only the S-module structure via the maps f_i to gain a computation for $W(k,Z;S)$.

Thus in Chapter VII we study non-maximal orders. Let us be explicit here in describing the key problems involved.

At every prime P in D there exists a canonically defined element $\rho(P)$ in $E/\Delta^{-1}(D/Z)$ with the following properties.

(1) The map of $O(E) \to E/\Delta^{-1}(D/Z)$ given by $\lambda \to \lambda\,\rho(P)$ induces an embedding of the residue field.

(2) We also have the map $Z \to Q/Z$ given by $n \to n/p$ which induces an embedding of $F_p = Z/pZ \to Q/Z$.

The element $\rho(P)$ is canonical in the sense that it makes the following diagram commute.

$$\begin{array}{ccc} O(E)/P & \to & E/\Delta^{-1}(D/Z) \\ \downarrow tr & & \downarrow t \\ F_p & \to & Q/Z \end{array}$$

The horizontal maps were just described. tr again denotes the map induced by the finite field trace. t denotes the map induced by the number field trace.

Thus we see that it is precisely these elements $\rho(P)$ which determine the isomorphism $tr^{-1} \circ g \circ t$ identifying $H(E/\Delta^{-1}(D/Z))$ with $\bigoplus_{P = \overline{P}} H(D/P)$. If we wish to use the commutative diagram just discussed to compute $W(k,Z;S)$, we must therefore study those elements $\rho(P)$. For it is in terms of these elements that one reads the local boundary

$$\partial(D,P)\colon H(E) \to H(E/\Delta^{-1}(D/Z)) \to \bigoplus_{P = \overline{P}} H(D/P) \to H(D/P).$$

in such a way as to make our diagram commute. The last map is projection to the P^{th} coordinate.

The first two sections of Chapter VII are due to Conner. In these we present his theorems which develop the fundamental properties of these canonical localizers $\rho(P)$. We complete our study of $W(k,Z;S)$ for non-maximal orders by discussing the finite field trace tr, and the maps f_i.

In Chapter 8 we finish our discussion of the boundary homomorphism. This includes the notion of coupling from Stoltzfus [Sf-1] between various $\partial(D)$, and a proof that the boundary

$$\partial\colon W(k,Q) \to W(k,Q/Z) \text{ is onto when } k = \pm 1.$$

In Chapter IX the terms and maps in the octagon are studied in detail. This, together with the information about the boundary map enables us to prove exactness in the octagon over Z in Chapter X.

The idea to study this problem and the possibilities inherent in the program we have undertaken comes from Professor P.E. Conner. It is thus a pleasure to thank him for his help in this project without which the notes would ner have been written. The author feels fortunate to have had the opportunity to study under Professor Conner, whom we thank not only for his invaluable ideas, but also for his patience and understanding.
Further, for many of the ideas herein, he should receive credit.

The author is also grateful to Professor Stoltzfus for numerous conversations throughout this project. We also thank Professor Dan Shapiro at Ohio State University for his help and suggestions, not only on this paper but also when the author was just beginning to study mathematics. We also express our appreciation to Professors Cordes and Butts at Louisiana State University; to Professor A. Liulevicius at the University of Chicago; and to Professor A. Ross at Ohio State University for their interest in the author at various stages of his career.

Finally, the author wishes to express his gratitude to his wife, Hiroko, and parents Dr. and Mrs. Albert Warshauer who have given their constant support and encouragement.

CONVENTIONS

A complete list of symbols and notations, as well as an index, can be found in the back. We number theorems, propositions, and definitions consecutively in each chapter. We refer to a theorem in the same chapter as it is numbered. However, when referring to a theorem from another chapter, we use a Roman numeral to indicate the chapter from which the theorem is taken.

The end of a proof is designated by the symbol □ . Occasionally, this symbol is also used alone, without a proof, to indicate that certain Lemmas or Propositions follow in a straightforward manner from the preceding, or that the proof is not difficult.

References are usually given together with a page number, eg. [Lm 201] refers to page 201 of reference [Lm].