Lecture Notes in Mathematics

Laure Saint-Raymond

Hydrodynamic Limits of the Boltzmann Equation

1971

$$f(v) = \frac{1}{(2\pi T)^{3/2}} \exp\left(-\frac{|v-U|^2}{2T}\right)$$

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Laure Saint-Raymond
Département de Mathématiques et Applications
Ecole Normale Supérieure
45 rue d'Ulm
75005 Paris
France
laure.saint-raymond@ens.fr

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Preface

The material published in this volume comes essentially from a course given at the Conference on "Boltzmann equation and fluidodynamic limits", held in Trieste in June 2006. The author is very grateful to Fabio Ancona and Stefano Bianchini for their invitation, and their encouragements to write these lecture notes.

The aim of this book is to present some mathematical results describing the transition from kinetic theory, and more precisely from the Boltzmann equation for perfect gases to hydrodynamics. Different fluid asymptotics will be investigated, starting always from solutions of the Boltzmann equation which are only assumed to satisfy the estimates coming from physics, namely some bounds on mass, energy and entropy. In particular the present survey does not consider convergence results requiring further regularity. However, for the sake of completeness, we will give in the first chapter some rough statements and bibliographical references for these smooth asymptotics of the Boltzmann equation, as well as for the transition from Hamiltonian systems to hydrodynamics.

Our starting point in the second chapter is some brief presentation of the Boltzmann equation, including its fundamental properties such as the formal conservations of mass, momentum and energy and the decay of entropy (for further details we refer to the book of Cercignani, Illner and Pulvirenti [31] or to the survey of Villani [106]). We then introduce the physical parameters characterizing the qualitative behaviour of the gas, and we derive formally the various hydrodynamic approximations obtained in the fast relaxation limit, i.e. when the collision process is dominating. We finally introduce the main existing mathematical frameworks dealing with the Cauchy problem for the Boltzmann equation, which can be useful for the study of hydrodynamic limits : we will particularly focus on the notion of renormalized solution defined by DiPerna and Lions [44], which will be used in all the sequel.

The third chapter is devoted to some technical results which are crucial tools for the mathematical derivation of hydrodynamic limits. Note that the general strategy to rigorously justify the formal asymptotics is to proceed by analogy, that is to recognize the structure of the expected limiting hydrodynamic model in the corresponding scaled Boltzmann equation. These tools will therefore not be equally used in all fluid regimes. The first point to be discussed is the implications of the entropy inequality, which provides some bound on the (relative) entropy, as well as some control on the entropy dissipation, and possibly some estimates on a boundary term known as the Darrozès-Guiraud information, depending on the scaling to be considered. The second point is to understand how these bounds, especially that on the entropy dissipation, allow to control the relaxation mechanism, and which consequences this implies on the distribution function. Note that, for fluctuations around a global equilibrium, such a study goes back to Hilbert [65] and Grad [59]. The last point to be investigated is the balance between this relaxation process due to collisions, and the other important physical mechanism, namely the free transport : in viscous regime the global structure of the scaled Boltzmann equation is actually of hypoelliptic type, and one can exhibit some regularizing effect of the free transport (extending for instance the velocity averaging lemma due to Golse, Lions, Perthame and Sentis [53]).

The incompressible Navier-Stokes limit, studied extensively in the fourth chapter, is therefore the only hydrodynamic asymptotics of the Boltzmann equation for which we are actually able to implement all the mathematical tools presented in Chapter 3, and for which an optimal convergence result is known. By "optimal", we mean here that this convergence result

- holds globally in time;
- does not require any assumption on the initial velocity profile;
- does not assume any constraint on the initial thermodynamic fields;
- takes into account boundary conditions, and describes their limiting form.

We start by recalling some basic facts about the limiting system, in particular its weak stability established by Leray [70]. We then explain the general strategy used to establish the convergence result of the renormalized solutions to the suitably scaled Boltzmann equation (which is very similar to the weak compactness argument of Leray), as well as the main difficulties to be overcomed.

The moment method, introduced by Bardos, Golse and Levermore [5] requires indeed to understand how one recovers the local conservation laws in the limit, and to determine the asymptotic behaviour of the flux terms, especially of the convection terms which are quadratic functions of the moments. In order to do so, the moments are actually proved to be regular with respect to the space variables x by a refined version of the velocity averaging result due to Golse and the author [56]. Furthermore the high frequency oscillating parts of the moments, known as acoustic waves, are filtered out by a compensated compactness argument due to Lions and Masmoudi [76]. One therefore gets a global weak convergence result ([54] or [55]) which does not require a precise study of the relaxation or oscillation phenomena.

In the case of a domain with boundaries, one has further to take into account the interactions between the gas and the wall, which leads to a braking condition if the kinetic condition is a diffuse reflection, and a slipping condition if the kinetic condition is a specular reflection.

The state of the art about the incompressible Euler limit, which is the main matter of the fifh chapter, is not so complete as for the incompresible Navier-Stokes limit. Due to the lack of regularity estimates for inviscid incompressible models, the convergence results describing the incompressible Euler asymptotics of the Boltzmann equation require additional regularity assumptions on the solution to the target equations.

Furthermore, the relative entropy method leading to these stability results controls the convergence in a very strong sense, which imposes additional conditions either on the solution to the asymptotic equations ("well-prepared initial data"), or on the solutions to the scaled Boltzmann equation (namely some additional non uniform a priori estimates giving in particular the local conservation of momentum and energy).

Under these additional a priori estimates, it is indeed possible to improve the relative entropy method, so as to take into account the acoustic waves and the Knudsen layers.

The last chapter of this survey is devoted to the compressible Euler limit, and is actually a series of remarks and open problems more than a compendium of results. The main challenge is of course to understand how the entropy dissipation concentrates on shocks and discontinuities, which should be studied in one space dimension.

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Laure Saint-Raymond

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Introduction

1.1 The Sixth Problem of Hilbert

1.1.1 The Mathematical Treatment of the Axioms of Physics

The sixth problem asked by Hilbert in the occasion of the International Congress of Mathematicians held in Paris in 1900 is concerned with the mathematical treatment of the axioms of Physics, by analogy with the axioms of Geometry. Precisely, it states as follows :

"Quant aux principes de la Mécanique, nous possédons déjà au point de vue physique des recherches d'une haute portée; je citerai, par exemple, les écrits de MM. Mach [81], Hertz [64], Boltzmann [14] et Volkmann [107]. Il serait aussi très désirable qu'un examen approfondi des principes de la Mécanique fût alors tenté par les mathématiciens. Ainsi le Livre de M. Boltzmann sur les Principes de la Mécanique nous incite à établir et à discuter au point de vue mathématique d'une manière complète et rigoureuse les méthodes basées sur l'idée de passage à la limite, et qui de la conception atomique nous conduisent aux lois du mouvement des continua. Inversement on pourrait, au moyen de méthodes basées sur l'idée de passage à la limite, chercher à déduire les lois du mouvement des corps rigides d'un système d'axiomes reposant sur la notion d'états d'une matière remplissant tout l'espace d'une manière continue, variant d'une manière continue et que l'on devra définir paramétriquement.

Quoi qu'il en soit, c'est la question de l'équivalence des divers systèmes d'axiomes qui présentera toujours l'intérêt le plus grand quant aux principes."

The problem, suggested by Boltzmann's work on the principles of mechanics, is therefore to develop "mathematically the limiting processes [. . .] which lead from the atomistic view to the laws of motion of continua", namely to obtain a unified description of gas dynamics, including all levels of description. In other words, the challenging question is whether macroscopic concepts such as the viscosity or the nonlinearity can be understood microscopically.

L. Saint-Raymond, *Hydrodynamic Limits of the Boltzmann Equation*,
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1.1.2 From Microscopic to Macroscopic Equations

Classical dynamics for systems constituted of identical particles are characterized by a Hamiltonian

$$H(x,v) = \frac{1}{2}\sum_{i=1}^{N} |v_i|^2 + \sum_{i\neq j} V(x_i - x_j)$$

with V a two-body potential.

The corresponding Liouville equation is

$$\partial_t f_N(t,x,v) + \mathbf{L} f_N(t,x,v) = 0 \tag{1.1}$$

where f_N is the density with respect to the Lebesgue measure of the system at time t, and the Liouville operator is given by

$$\mathbf{L} = \sum_{i=1}^{N} \left[\frac{\partial H}{\partial v_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial v_i} \right].$$

For a given configuration $\omega(t) = (x(t), v(t))$ the empirical density and momentum (which rigorously speaking are measures) are then defined by

$$R_\omega(X) = \frac{1}{N}\sum_{i=1}^{N} \delta(X - x_i)$$

$$Q_\omega(X) = \frac{1}{N}\sum_{i=1}^{N} v_i \delta(X - x_i)$$

Macroscopic equations such as the Euler equations or the Navier-Stokes equations (which have been historically derived through a continuum formulation of conservation of mass, momentum and energy) are then expected to be obtained as some asymptotics of the equations governing these observable quantities.

1.2 Formal Study of the Transitions

The microscopic versions of density, velocity, and energy should actually assume their macroscopic, deterministic values through the *law of large numbers*. Therefore, in order the equations describing the evolution of macroscopic quantities to be exact, certain limits have to be taken, with suitably chosen scalings of space, time, and other macroscopic parameters of the systems. So the first step in the derivation of such equations is a choice of scaling.

1.2.1 Scalings

Denote coordinates by (x,t) in the microscopic scale, and by $(\tilde{x},\tilde{t})$ in the macroscopic scale. Let $\rho = N/L^3$ be the typical density in the microscopic unit, i.e. the number of particles per microscopic unit volume. Then, if ε is the ratio between the microscopic unit and the macroscopic unit, there are typically three choices of scalings :

- the Grad limit $\rho = \varepsilon, \quad (\tilde{x},\tilde{t}) = (\varepsilon x, \varepsilon t)$;
  (The typical number of collisions per particle is finite.)
- the Euler limit $\rho = 1, \quad (\tilde{x},\tilde{t}) = (\varepsilon x, \varepsilon t)$;
  (The typical number of collisions per particle is ε^{-1}.)
- the diffusive limit $\rho = 1, \quad (\tilde{x},\tilde{t}) = (\varepsilon x, \varepsilon^2 t)$;
  (The typical number of collisions per particle is ε^{-2}.)

The Euler and diffusive limits will be referred to as hydrodynamic limits.

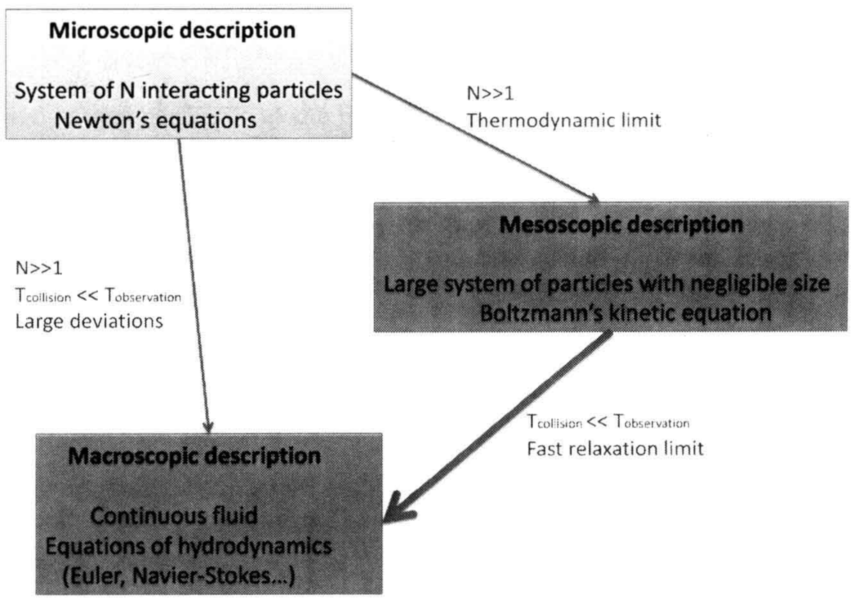

Fig. 1.1. Transitions between the different levels of description

1.2.2 Hydrodynamic Limits

To obtain hydrodynamic equations, we then differentiate the scaled empirical density and momentum and more precisely their integral agasinst any test function φ :

$$\int \varphi(\tilde{x}) R_{\omega(\tilde{t}/\varepsilon),\varepsilon}(\tilde{x}) d\tilde{x} = \frac{1}{N} \sum_{i=1}^{N} \varphi(\varepsilon x_i(\tilde{t}/\varepsilon)),$$

$$\int \varphi(\tilde{x}) Q_{\omega(\tilde{t}/\varepsilon),\varepsilon}(\tilde{x}) d\tilde{x} = \frac{1}{N} \sum_{i=1}^{N} v_i(\tilde{t}/\varepsilon)\varphi(\varepsilon x_i(\tilde{t}/\varepsilon)).$$

We get for instance

$$\frac{d}{d\tilde{t}} \frac{1}{N} \sum_{i=1}^{N} v_i(\tilde{t}/\varepsilon)\varphi(\varepsilon x_i(\tilde{t}/\varepsilon)) = -\frac{1}{N} \sum_{i=1}^{N} \varepsilon^{-1} \varphi(\varepsilon x_i) \frac{\partial H}{\partial x_i} + \frac{1}{N} \sum_{i=1}^{N} v_i \partial_i \varphi(\varepsilon x_i) \frac{\partial H}{\partial v_i}$$

$$= -\frac{1}{2N} \sum_{i=1}^{N} \nabla\varphi(\varepsilon x_i) \sum_{i \neq j} \frac{x_i - x_j}{\varepsilon} \cdot \nabla V\Big(\frac{x_i - x_j}{\varepsilon}\Big) + \frac{1}{N} \sum_{i=1}^{N} v_i \otimes v_i \nabla\varphi(\varepsilon x_i) + O(\varepsilon)$$

using Taylor's formula for φ, and symmetries to discard the main term.

In order to obtain the conservation of momentum in the **Euler equations** we then need to show that the microscopic current

$$-\frac{1}{2N} \sum_{i=1}^{N} \nabla\varphi(\varepsilon x_i) \sum_{i \neq j} \frac{x_i - x_j}{\varepsilon} \cdot \nabla V\left(\frac{x_i - x_j}{\varepsilon}\right)$$

converges to some macroscopic current $P = P(R, Q, E)$ depending on the macroscopic density, momentum and internal energy, in the limit $\varepsilon \to 0$. This convergence has to be understood in the sense of law of large numbers with respect to the density f_N (solution to the Liouville equation)

$$\frac{1}{N} \int f_N(t,\omega) \left| \sum_i \nabla\varphi(\varepsilon x_i) \left[\sum_{i \neq j} \frac{x_i - x_j}{\varepsilon} \cdot \nabla V\left(\frac{x_i - x_j}{\varepsilon}\right) - P(R, Q, E) \right] \right| d\omega \to 0 \tag{1.2}$$

The key observation, due to Morrey [86], is that (1.2) holds if we replace f_N by any Gibbs measure with Hamiltonian H, or more generally if "locally" f_N is a Gibbs measure of the Hamiltonian H.

The point is therefore to establish that "locally" $f_N(t)$ is a equilibrium measure with finite specific entropy. The conclusion follows then from the *ergodicity* of the infinite system of interacting particles : the translation invariant stationary measures of the dynamics such that the entropy per microscopic unit of volume is finite are Gibbs ($\exp(-\beta H)$).

The **Navier-Stokes equations** are the next order corrections to the Euler equations. In order to derive them one needs to show that the microscopic current is well approximated up to order ε by the sum of the macroscopic current $P = P(R, Q, E)$ and a viscosity term $\varepsilon\nu\nabla Q$ (in the sense of law of large numbers).

Since there is an ε appearing in the viscosity term, proving such an asymptotics requires to understand the next order correction to Boltzmann's hypothesis. This difficulty, recognized long time ago by Dobrushin, Lebowitz and Spohn, has been overcome recently for simplified particle dynamics : the mathematical interpretation is indeed given by the fluctuation-dissipation equation which states

$$\begin{aligned} &-\frac{1}{2N}\sum_{i=1}^{N}\nabla\varphi(\varepsilon x_i)\sum_{i\neq j}\frac{x_i-x_j}{2\varepsilon}\cdot\nabla V\left(\frac{x_i-x_j}{\varepsilon}\right) \\ &= P(R_{\omega,\varepsilon},Q_{\omega,\varepsilon},E_{\omega,\varepsilon})+\varepsilon\nu\nabla Q_{\omega,\varepsilon}+\varepsilon\mathbf{L}g_{\omega,\varepsilon}+o(\varepsilon) \end{aligned} \tag{1.3}$$

for some function $g_{\omega,\varepsilon}$, where $\mathbf{L}$ is the Liouville operator. In other words, the expected asymptotics is correct only up to a quotient of the image of the Liouville operator. The image of the Liouville operator is understood as a fluctuation, negligible in the relevant scale *after time average* : for any bounded function g

$$\varepsilon\int_0^t ds f_N(s,\omega)(\varepsilon\mathbf{L}g)(\omega)d\omega = \varepsilon^2(f_N(t,\omega)-f_N(s,\omega))g(\omega)d\omega = O(\varepsilon^2)$$

and is thus negligible to the first order in ε.

In order to avoid the difficulties of the multiscale asymptotics, we may turn to the **incompressible Navier-Stokes equations** which are invariant under the incompressible scaling

$$(x,t,u,p)\mapsto(\lambda x,\lambda^2 t,\lambda^{-1}u,\lambda^2 p)$$

under which the fluctuation-dissipation equation becomes

$$\begin{aligned} &-\frac{1}{2N}\sum_{i=1}^{N}\nabla\varphi(\varepsilon x_i)\sum_{i\neq j}\frac{x_i-x_j}{\varepsilon}\cdot\nabla V\left(\frac{x_i-x_j}{\varepsilon}\right) \\ &= P(R_{\omega,\varepsilon},Q_{\omega,\varepsilon},E_{\omega,\varepsilon})+\nu\nabla Q_{\omega,\varepsilon}+\mathbf{L}g_{\omega,\varepsilon}+o(\varepsilon) \end{aligned} \tag{1.4}$$

where both the viscosity ν and the functions g are unknown. Notice that the solution to the fluctuation-dissipation equation requires inversion of the Liouville operator.

In the following two sections, we intend to describe briefly the different mathematical approaches which allow to obtain rigorous convergence results for these asymptotics. These results will be stated in a rather unformal way in order to avoid definitions and notations. We refer to the quoted publications for precise statements and proofs.

1.3 The Probabilistic Approach

The most natural approach for the mathematical understanding of hydrodynamic limits consists in using probabilistic tools such as the law of large numbers and some large deviations principle. Nevertheless the complexity of the problem is such that there is still no complete derivation of fluid models starting from the full deterministic Hamiltonian dynamics.

1.3.1 The Euler Limit

Concerning the derivation of the Euler equations, what has been proved by Olla, Varadhan and Yau [89] is the following result.

Theorem 1.3.1 *Consider a general Hamiltonian system with superstable pairwise potential, and the corresponding stochastic dynamics obtained by adding a noise term which exchanges the momenta of nearby particles. Suppose the Euler equation has a smooth solution in* $[0, T]$. *Then the empirical density, velocity and energy converge to the solution of the Euler equations in* $[0, T]$ *with probability one.*

The strength of the noise term is of course chosen to be very small so that it disappears in the scaling limit.

The proof consists of two main ingredients. The first point is to establish the ergodicity of the system, and more precisely the following statement : if, under a stationary measure, the distribution of velocities conditioned to the positions is a convex combinations of gaussians, then the stationary measure is a convex combination of Gibbs. Noise is therefore added to the system in order to guarantee such information on the distributions. The second point is to prove that there is no spatial or temporal meso-scale fluctuation to prevent the convergence (1.2).

It is based on the *relative entropy method*, so-called because the fundamental quantity to be considered is the relative entropy defined by

$$H(f|g) = \int f \log(f/g) d\omega$$

for any two probability densities f and g.

If f_N is the solution to the Liouville equation (1.1) and ψ_t is any density, we have the following identity

$$\partial_t H(f_N(t)|\psi_t) = -\int f_N(t) \left(\psi_t^{-1}(\mathbf{L} - \partial_t)\psi_t\right) d\omega \, .$$

From Jensen's inequality, we then deduce that

$$\partial_t H(f_N(t)|\psi_t) \leq H(f_N(t)|\psi_t) + \log \int \psi_t \left(\psi_t^{-1}(\mathbf{L} - \partial_t)\psi_t\right) d\omega \, .$$

Thus, if we have

$$\frac{1}{N} \log \int \psi_t \left(\psi_t^{-1}(\mathbf{L} - \partial_t)\psi_t\right) d\omega \to 0 \tag{1.5}$$

the relative entropy can be controlled on the relevant time scale. The remaining argument can be summarized as showing that a weak version of (1.5) holds if and only if ψ_t is a local Gibbs state with density, velocity and energy chosen according to the Euler equations :

$$\begin{aligned}
&\partial_t R + \nabla_x \cdot (RU) = 0,\\
&\partial_t (RU) + \nabla_x \cdot (RU \otimes U + P) = 0,\\
&\partial_t (RE) + \nabla_x \cdot (REU - UP) = 0.
\end{aligned}$$

This is therefore a dynamical variational approach because the problem is solved by guessing a good test function.

1.3.2 The Incompressible Navier-Stokes Equations

Equation (1.4) is very difficult to solve as it requires inversion of the Liouville operator. It has been first studied by Landim and Yau [68] for the asymmetric exclusion process.

The rigorous derivation of the incompressible Navier-Stokes equations from particle systems has then been obtained in the framework of *stochastic lattice models* which are more manageable. Esposito, Marra and Yau [46] have established the convergence when the target equations have smooth solutions :

Theorem 1.3.2 *Consider a 3D lattice system of particles evolving by random walks and binary collisions, with "good" ergodic and symmetry properties. Suppose the incompressible Navier-Stokes equations have a smooth solution u in $[0, t^*]$. Then the rescaled empirical velocity densities u_ε converge to that solution u.*

Quastel and Yau [91] have then been able to remove the regularity assumption :

Theorem 1.3.3 *Consider a 3D lattice system of particles evolving by random walks and binary collisions, with "good" ergodic and symmetry properties. Let u_ε be the distributions of the empirical velocity densities. Then u_ε are precompact as a set of probability measures with respect to a suitable topology, and any weak limit is entirely supported on weak solutions of the incompressible Navier-Stokes equations satisfying the energy inequality.*

The method used to prove this last result differs from the relative entropy method, insofar as it considers more general solutions to the target equations, but - as a counterpart - gives a weaker form of convergence. One main step of the proof is to obtain the energy estimate for the incompressible Navier-Stokes equations directly from the lattice gas dynamics by implementing a renormalization group. A difficult point is to control the large fluctuation using the entropy method and logarithmic Sobolev inequalities.

It is important to note that such a derivation fails if the dimension of the physical space is less than three, meaning in particular that the 2D Navier-Stokes equations should be relevant only for 3D flows having some translation invariance.