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THE KOUROVKA NOTEBOOK

Unsolved Problems in Group Theory

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Unsolved Problems in Group Theory

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КОУРОВСКАЯ ТЕТРАДЬ
(НЕРЕШЕННЫЕ ВОПРОСЫ ТЕОРИИ ГРУПП)
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Редакторы: В. Д. Мазуров, Ю. И. Мерзляков, В. А. Чуркин

Институт математики СО АН СССР

Translated from the Russian by D. J. Johnson
with the assistance of R. W. Carter, B. Hartley, R. C. Lyndon,
Yu. I. Merzlyakov and B. F. Wehrfritz

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PREFACE

To form an up-to-date picture of what is going on in a given area of mathematics, we usually consult a shelf of current periodicals or, to save time, the appropriate section of a reviewing journal. Thus we learn of new advances in the area, which problems have been solved, what progress has been made with others, while rarely, and then only in the context of the author's own results, we learn which problems the author failed to solve but considers interesting. In all this, a summary of current problems has no less a place in the development of a subject than a list of achievements, though the apparent connection between the two is often deceptive. Thus, it is desirable to publish from time to time a summary of important problems with the participation of a large circle of authors. *The Kourovka notebook* is such a collection of unsolved problems in group theory.

The current edition is the seventh, the first having appeared in 1965. Experience has shown that the idea of collecting problems of interest in a given area at a given time is fully justified. Of the 422 problems in the sixth edition, 151 have now been solved. Cast in the same mold are *The Dniester notebook*[1] and *The Sverdlovsk notebook*,[2] which are collections of unsolved problems in ring theory and semigroup theory, respectively.

This edition is augmented by Chapter 7. The first six chapters have been reproduced from the sixth edition with slight editorial changes. The comments on the problems have been reviewed and augmented. As before, the problems for which comments appear carry an asterisk.[3] The editors are grateful to all who have commented on earlier editions.

New problems and comments should be sent to the following address: 630090, Novosibirsk, 90, Institute of Mathematics, Kourovka Notebook. Any known results quoted must be provided with a precise reference to the literature or an alternative source.

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CHAPTER 1

The Problem Day at the First All-Union Symposium on Group Theory (Kourovka, 1965)

From 9–17 February, 1965 the First All-Union Symposium on Group Theory was held at the Kourovka resort near Sverdlovsk. It was organized by the Institute of Mathematics of the Siberian Division of the Academy of Sciences of the USSR, by the Sverdlovsk section of the Steklov Institute of Mathematics, and by the Ural State University. There were sixty-nine participants from seventeen cities in the Soviet Union.

The work of the Symposium was organized in a form close to winter schools on the one hand and to traditional algebra colloquia on the other. Courses of lectures were given on various trends in the theory of groups, and proofs were produced either in full or in outline for individual important or new results. In addition, several reports were given on new results. For an account of the Symposium, see Uspekhi Mat. Nauk **20** (1965), no. 4 (124), 213–215.

The program included a special "Problem Day", on 16 February, which was a success. Problems of current interest, some original and some well known from the literature, but considered important for some reason, were presented. They were accompanied by explanations and entered by the authors in a special notebook. The record of problems was also continued on other days during the entire Symposium and, naturally enough, the individual styles of the authors are reflected in the contents of this notebook.

As one of its resolutions, the Symposium commissioned the organizing committee to publish the "Kourovka Notebook" as a separate booklet. Of course, not all the problems are equally important; some are quite well known, and their difficulty has been verified well enough, while others have only recently appeared. In either case, these problems reflect the current interest of a considerable number of Soviet specialists in group theory and, hopefully, will provide a valuable stimulus for further research in this field.

Problems

1.1. Does there exist a nontrivial finitely generated divisible group? Equivalently, does there exist a nontrivial finitely generated divisible simple group?

Yu. A. Bogan

*1.2. Let G be a group, F a free group with free generators $x_1, \ldots, x_n$ and R the free product of G and F. By an equation (in unknowns $x_1, \ldots, x_n$) over G, we mean an expression of the form $v(x_1, \ldots, x_n) = 1$, where the left member is an element of R not conjugate in R to any element of G. We call G *algebraically closed* if every equation over G has a solution in G. Does there exist an algebraically closed group? If we require that $v(x_1, \ldots, x_n)$ contain only positive powers of the unknowns, then such groups exist [207].

L. A. Bokut'

†1.3 (I. Kaplansky). Can the group-ring of a torsionfree group contain divisors of zero?

L. A. Bokut'

*1.4 (A. I. Mal'tsev). Does there exist a ring without zero-divisors which is not embeddable in a division ring, while the multiplicative semigroup of its nonzero elements is embeddable in a group?

L. A. Bokut'

1.5 (Well-known problem). Does there exist a group whose group-ring contains no divisors of zero but is not embeddable in a division ring?

L. A. Bokut'

1.6 (A. I. Mal'tsev). Is the group-ring of a right-ordered group embeddable in a division ring?

L. A. Bokut'

†1.7. Is every abelian subgroup of a finite simple group different from its normalizer?

V. M. Busarkin

†1.8. Characterize those finite simple groups which have a subgroup that contains the centralizer of each of its nonidentity elements.

V. M. Busarkin

*1.9. Can the factor group of a locally normal group by the second term of its upper central series be embedded (isomorphically) in a direct product of finite groups?

Yu. M. Gorchakov

*1.10. An automorphism φ of a group G is called *splitting* if for any element $g \in G$ the relation $gg^{\varphi} \cdots g^{\varphi^{n-1}} = 1$ holds, where n is the order of φ. Is a soluble group which has a regular splitting automorphism of prime order necessarily nilpotent?

Yu. M. Gorchakov

*1.11 (E. Artin). The conjugacy problem for the braid group $\mathfrak{B}_n$, $n > 4$.

M. D. Grindlinger
[M. Greendlinger]

1.12 (W. Magnus). Solve the problem of isomorphism with the trivial group for all groups with n generators and n defining relations, $n > 2$.

M. D. Grindlinger

1.13 (J. Stallings). If a finitely presented group is trivial, is it always possible to replace at least one defining relator by a primitive element without disturbing the triviality of the group?

M. D. Grindlinger

*1.14 (B. H. Neumann). Does there exist an infinite simple finitely presented group?

M. D. Grindlinger

1.15. Solve the problem of isomorphism with the infinite cyclic group for all knot groups by group-theoretical methods. It has been solved using topological methods by Haken [78].

M. D. Grindlinger

1.16 (C. D. Papakyriakopoulos). For a given natural number n, find those subsets $I \subseteq \{1,\dots,n\}$ and those elements τ_i, $i \in I$, of the commutator group of

$$G = \left\langle a_1, b_1, \dots, a_n, b_n : \prod_{i=1}^{n} [a_i, b_i] = 1 \right\rangle,$$

for which the factor group of G modulo the normal closure of $\{[a_i, b_i\tau_i] \mid i \in I\}$ is torsionfree. For $|I| = 1$, Rapaport [181] proved this factor group is always torsionfree.

M. D. Grindlinger

*1.17. Write an explicit set of generators and defining relations for one of the universal finitely presented groups whose existence was proved nonconstructively by Higman [84].

M. D. Grindlinger

1.18 (A. Tarski). Does there exist an algorithm for determining the solubility of equations in a free group? Such an algorithm exists for equations in one unknown. Describe all solutions of an equation when it has at least one solution.

Yu. L. Ershov

1.19 (A. I. Mal'tsev). Which subgroups (subsets) can be defined by a formula in a free group? Which subgroups can be defined relatively elementarily in a free group? In particular, is the commutator group so definable, or in a relatively elementary way?

Yu. L. Ershov

1.20. For which groups (classes of groups) is the lattice of normal subgroups definable by a formula in the lattice of all subgroups? In particular, is this true for free groups?

Yu. L. Ershov

†1.21. Are there only finitely many finite simple groups of a given exponent n?

M. I. Kargapolov

1.22. Investigate a finite group having a subgroup A which is disjoint from its conjugates and is such that the index of A in its normalizer is "significantly" less than its order $|A|$.

M. I. Kargapolov

*1.23. Does there exist a locally finite infinite simple group of finite rank?

M. I. Kargapolov

*1.24. Does every infinite group possess an infinite abelian subgroup?

M. I. Kargapolov

*1.25 a) Is the universal theory of the class of finite groups decidable?
b) Is the universal theory of the class of finite nilpotent groups decidable?

M. I. Kargapolov

*1.26. Does isomorphism of finitely presented nilpotent groups follow from elementary equivalence?

M. I. Kargapolov

1.27. Describe the universal theory of free groups.

M. I. Kargapolov

1.28. Describe the universal theory of free nilpotent groups of class n.

M. I. Kargapolov

1.29 (A. Tarski). Is the elementary theory of free groups decidable?

M. I. Kargapolov

1.30. Is the universal theory of the class of soluble groups decidable?

M. I. Kargapolov

1.31. Is a residually finite group with the maximum condition a finite extension of a polycyclic group?

M. I. Kargapolov

*1.32. Is the Frattini subgroup of a finitely generated matrix group over a field nilpotent?

M. I. Kargapolov

1.33 (A. I. Mal'tsev). Describe the automorphism group of a free soluble group.

M. I. Kargapolov

*1.34. Has every orderable polycyclic group an isomorphic (integral) matrix representation?

M. I. Kargapolov

1.35. A group is called *pro-orderable* if every partial ordering of the group extends to a linear ordering.

*a) Is the wreath product of arbitrary pro-orderable groups again pro-orderable?

*b) (A. I. Mal'tsev). Is every subgroup of a pro-orderable group again pro-orderable?

c) (A. I. Mal'tsev and L. Fuchs). Do there exist pro-orderable simple groups?

M. I. Kargapolov

1.36. If a group G is factorable as a product of p-subgroups (that is, $G = AB$, where A and B are p-subgroups), does it follow that G is itself a p-group?

Sh. S. Kemkhadze

*1.37. Is every subgroup of a locally nilpotent group quasi-invariant?

Sh. S. Kemkhadze

*1.38. By an N^0-*group* we mean a group in which every cyclic subgroup is a term of some normal system of the group. Is every N^0-group an $\tilde{N}$-group?

Sh. S. Kemkhadze

1.39. Is the product of two normal weakly nilpotent subgroups a weakly nilpotent group?

Sh. S. Kemkhadze

1.40. Is a group which is the product of two normal Engel subgroups necessarily an Engel group?

Sh. S. Kemkhadze

1.46. Under what conditions is the normalizer of a relatively convex subgroup again relatively convex?[4]

A. I. Kokorin

*1.47. A subgroup H of a group G is called *strictly isolated* if, whenever $xg_1^{-1}xg_1 \cdots g_n^{-1}xg_n$ belongs to H, so do x and each $g_i^{-1}xg_i$. A group in which the identity subgroup is strictly isolated is called an *S-group*. Do there exist S-groups which are not orderable groups? Aliter: Is a strictly isolated normal subgroup of an orderable group necessarily a convex subgroup?

A. I. Kokorin

1.51. Under what conditions will a group of matrices over a field (say the complex numbers) be orderable?

A. I. Kokorin

*1.52. Describe those groups which can be ordered linearly in a unique way up to reversal of orderings.

A. I. Kokorin

1.53. Describe all possible linear orderings of a free nilpotent group with a finite number of generators. It is not difficult to see that this problem reduces to that of describing the central systems of isolated subgroups.

A. I. Kokorin

1.54. Describe all possible linear orderings of a finitely generated free metabelian group.

A. I. Kokorin

1.55. Obtain an elementary classification of the linearly ordered free groups with a fixed number of generators.

A. I. Kokorin

1.58. A subgroup of an orderable group, all of whose linear orderings can be extended to a linear ordering of the whole group, is called a *v-subgroup*. Under what conditions is a subgroup of an orderable group a *v*-subgroup?

A. I. Kokorin

1.59. Investigate the properties of the maximal partial orderings in an orderable group. We note, for example, that, under any maximal partial ordering of an orderable group, its hypercenters are linearly ordered.

A. I. Kokorin

*1.60. Can an orderable metabelian group be embedded in a complete orderable group? This is a weaker version of B. H. Neumann's well-known problem of embedding an orderable group in a complete orderable group.

A. I. Kokorin

1.61. Can an orderable group be embedded in an orderable group with a complete maximal locally nilpotent normal subgroup (or with a complete maximal abelian normal subgroup)?

A. I. Kokorin

1.62. Investigate the class of soluble orderable groups.

A. I. Kokorin

1.63. A group G is called *dense* if it has no proper isolated subgroup other than its identity.

*a) Apart from the locally cyclic groups, are there any dense torsionfree groups?

*b) Given that any two nontrivial elements x and y of a torsionfree group G satisfy the relation $x^k = y^l$, where k and l are nonzero integers depending on x and y, does it follow that G is abelian?

c) Give a description of the dense finite groups.

P. G. Kontorovich

*1.64. A torsionfree group G is called *separable* if it can be represented as the set-theoretic union of two of its proper subsemigroups. Are all R-groups separable? There is an example (due to J. Wiegold) of an inseparable group which is not an R-group.

P. G. Kontorovich

1.65. Is the class of abelian extensions of abelian groups closed under the formation of direct sums?

L. Ya. Kulikov

1.66. Given a periodic abelian group T and a nondenumerable cardinal number $\mathfrak{m}$, does there exist a torsionfree abelian group $U = U(T, \mathfrak{m})$ with the following property: For torsionfree abelian groups A of cardinality $\leqslant \mathfrak{m}$, the equation $\mathrm{Ext}(A, T) = 0$ holds if and only if A is embeddable in U?

L. Ya. Kulikov

*1.67. Suppose we are given a finitely presented group G, a free group F whose rank is equal to the minimal number of generators of G, and a homomorphism of F onto G with kernel N. Find a complete system of invariants of the factor group of the group N modulo the commutator subgroup $[F, N]$.

L. Ya. Kulikov

*1.68 (A. Tarski). Let $\mathfrak{K}$ be a class of groups and $Q\mathfrak{K}$ the class of all homomorphic images of members of $\mathfrak{K}$. If $\mathfrak{K}$ is axiomatizable does it follow that $Q\mathfrak{K}$ is?

Yu. I. Merzlyakov

*1.70. Let p be a prime and G the group of all matrices of the form

$$\begin{pmatrix} 1 + p\alpha & p\beta \\ p\gamma & 1 + p\delta \end{pmatrix}$$

where α, β, γ and δ are rational numbers with denominators coprime to p. Does G have the property $\overline{RN}$? If so, the property $\overline{RN}$ is not inherited by subgroups. If not, there exist $\overline{RI}$-groups which are not $\overline{RN}$-groups.

Yu. I. Merziyakov

*1.71. Let G be a connected algebraic group over an algebraically closed field. Is the number of conjugacy classes of maximal soluble subgroups of G finite? This is known to be false for fields which are not algebraically closed. At the same time, the classical Borel-Chevalley result shows that the maximal connected soluble subgroups of G form a single conjugacy class. It may be that this question is bound up with the as yet untackled problem of classifying the connected soluble algebraic groups.

V. P. Platonov

*1.72. D. Hertzig has shown that a connected algebraic group over an algebraically closed field is soluble if it has a rational regular automorphism. Is this result true for an arbitrary field? It can be shown to hold for fields of characteristic zero.

V. P. Platonov

*1.73. Are there only finitely many conjugacy classes of maximal periodic subgroups in a finitely generated integer linear group? This has been proved for soluble integral groups by Mal'tsev, and has recently been established for the integral part of an algebraic group (the group of units) by Borel and Harish-Chandra.

V. P. Platonov

1.74. Describe all minimal topological groups, that is, nondiscrete groups with discrete closed subgroups. The minimal locally bicompact groups can be described without much trouble, but the problem is probably quite complicated in the general case.

V. P. Platonov

*1.75. Classify the infinite simple periodic linear groups. Infinite series of such groups are known at present over limiting fields, analogous to the classical simple Lie groups. The following is a more special question: Are infinite simple periodic linear groups countable? The above problem is clearly connected with the problem of classifying the finite simple groups, but it is not clear a priori how close this connection is.

V. P. Platonov

*1.76. Does there exist a topologically simple, locally nilpotent, locally bicompact group?

V. P. Platonov

*1.78. If a group G is the product of two complete abelian p-groups of finite rank, does it follow that G itself is a complete abelian p-group of finite rank?

N. F. Sesekin

*1.80. Does there exist a finite simple group whose Sylow 2-subgroups are a direct product of quaternion groups?

A. I. Starostin

1.81. The *width* of a group G is, by definition, the smallest cardinal $m = m(G)$ with the property that any subgroup of G generated by a finite set $S \subset G$ is generated by a subset of S of cardinality at most m.

a) Do all groups of finite width possess the minimum condition for subgroups?

b) Do all groups with the minimum condition for subgroups have finite width?

*c) The same questions with the extra condition of local finiteness (it is clear that groups of finite width are periodic). In particular, is a locally finite group of finite width a Chernikov group? For several classes of groups, locally radical groups for example, the condition of finite width and the minimum condition for subgroups are equivalent.

L. N. Shevrin

*1.82. Two sets of identity relations are *equivalent* if they determine the same variety of groups. Construct an infinite set of identities which is not equivalent to any finite one.

A. L. Shmel'kin