AN INTRODUCTION TO DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

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An Introduction to Differential Equations and Linear Algebra

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A SHORT TABLE OF LAPLACE TRANSFORMS

Function $f(t)$	Laplace Transform $F(s)$
$f(t) = t^n$, n a nonnegative integer	$F(s) = \frac{n!}{s^{n+1}}$, $s > 0$
$f(t) = e^{at}$, a constant	$F(s) = \frac{1}{s-a}$, $s > a$
$f(t) = \sin bt$, b constant	$F(s) = \frac{b}{s^2+b^2}$, $s > 0$
$f(t) = \cos bt$, b constant	$F(s) = \frac{s}{s^2+b^2}$, $s > 0$
$f(t) = t^{-1/2}$	$F(s) = \sqrt{\frac{\pi}{s}}$, $s > 0$
$f(t) = u_a(t)$	$F(s) = \frac{e^{-as}}{s}$
$f(t) = \delta(t-a)$	$F(s) = e^{-as}$
Transform of Derivatives	
f'	$L[f'] = sL[f] - f(0)$
f''	$L[f''] = s^2 L[f] - sf(0) - f'(0)$
Shifting Theorems	
$e^{at}f(t)$	$F(s-a)$
$u_a(t)f(t-a)$	$e^{-as}F(s)$

SOME SOLUTION TECHNIQUES FOR $y' = f(x, y)$

Type	Standard Form	Technique
Separable	$p(y)\frac{dy}{dx} = q(x)$.	Separate the variables and integrate directly.
First-Order Homogeneous	$\frac{dy}{dx} = f(x, y)$, with f homogeneous of degree zero $[f(tx, ty) = f(x, y)]$.	Change variables: $y = xV(x)$ and reduce to a separable equation.
First-Order Linear	$\frac{dy}{dx} + p(x)y = q(x)$.	Rewrite as $\frac{d}{dx}(ye^{\int p(x)\,dx}) = q(x)\,e^{\int p(x)\,dx}$, and integrate with respect to x.
Bernoulli Equation	$\frac{dy}{dx} + p(x)y = q(x)y^n$.	Divide by y^n and make the change of variables $u = y^{1-n}$. This reduces the equation to a linear equation.
Exact	$M(x, y)\,dx + N(x, y)\,dy = 0$, with $M_y = N_x$.	The solution is $\phi(x, y) = c$, where ϕ is determined by integrating $\phi_x = M$, $\phi_y = N$.

THE METHOD OF UNDETERMINED COEFFICIENTS

The following table lists trial solutions for the differential equation $P(D)y = F(x)$, where $P(D)$ is a polynomial differential operator.

$F(x)$	Usual Trial Solution	Modified Trial Solution
$cx^k e^{ax}$	If $P(a) \neq 0$, choose: $y_p = e^{ax}(A_0 + A_1 x + \cdots + A_k x^k)$	If a is a root of $P(r) = 0$ of multiplicity m, choose: $y_p = x^m e^{ax}(A_0 + A_1 x + \cdots + A_k x^k)$
$x^k e^{ax}(\alpha \cos bx + \beta \sin bx)$	If $P(a + ib) \neq 0$, choose: $y_p = e^{ax}[(A_0 \cos bx + B_0 \sin bx) + x(A_1 \cos bx + B_1 \sin bx) \cdots + x^k(A_k \cos bx + B_k \sin bx)]$	If $a + ib$ is a root of $P(r) = 0$ of multiplicity m, choose: $y_p = x^m e^{ax}[(A_0 \cos bx + B_0 \sin bx) + x(A_1 \cos bx + B_1 \sin bx) + \cdots + x^k(A_k \cos bx + B_k \sin bx)]$

If $F(x)$ is the sum of functions of the above form then the appropriate trial solution is the corresponding sum.

BASIC INTEGRALS

Function $F(x)$	Integral $\int F(x)\,dx$
$x^n,\ n \neq -1$	$\frac{1}{n+1} x^{n+1} + c$
$\frac{1}{x}$	$\ln \lvert x \rvert + c$
$e^{ax},\ a \neq 0$	$\frac{1}{a} e^{ax} + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\tan x$	$\ln \lvert \sec x \rvert + c$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert + c$
$\csc x$	$\ln \lvert \csc x - \cot x \rvert + c$
$e^{ax} \sin bx$	$\frac{1}{a^2 + b^2} e^{ax}(a \sin bx - b \cos bx) + c$
$e^{ax} \cos bx$	$\frac{1}{a^2 + b^2} e^{ax}(a \cos bx + b \sin bx) + c$
$\ln x$	$x \ln x - x + c$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{\sqrt{a^2 - x^2}},\ a > 0$	$\sin^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\ln(x + \sqrt{a^2 + x^2}) + c$
$\frac{f'(x)}{f(x)}$	$\ln \lvert f(x) \rvert + c$
$e^{u(x)} \frac{du}{dx}$	$e^{u(x)} + c$

An Introduction to Differential Equations and Linear Algebra

To Christina,
for lost time

Preface

In *An Introduction to Differential Equations and Linear Algebra* the standard material on linear algebra and linear differential equations required in many sophomore courses for mathematics, science, and engineering majors is introduced. I have endeavored to develop an appreciation for the power of the general vector space framework in formulating and solving linear problems. In particular, the theory underlying the solution of linear differential equations is derived very simply as an application of the vector space results. My aim has been to present the material in a manner that is accessible to the student who has successfully completed three semesters of calculus. It is definitely the intention that the student read the text, not just the examples. Almost all the results are proved in detail, and therefore the level of rigor is reasonably high.

The text begins with two chapters on the classical techniques for solving first-order differential equations and some of their applications. It is here that the student gains familiarity with the terminology and notation used in differential equation theory and an appreciation for the types of problems whose mathematical formulation gives rise to differential equations. The advantage of beginning with these chapters is that differential equations can then be used to motivate and to illustrate the more abstract theorems and definitions that form the basis of the linear algebra developed in Chapters 3 to 8. For example, the problem of finding the set of all solutions to the differential equation $y'' + y = 0$ is used to motivate the more abstract idea of the kernel of a linear transformation, T. Then, having proved that the kernel of T is a vector space, we return to differential equations to conclude that the set of all solutions to $y'' + ay' + by = 0$ is a vector space. The question then arises as to the dimension of the solution space. This question is answered in Chapter 9, and the remainder of the text is concerned with introducing techniques for solving linear differential equations and linear systems of differential equations.

For many students linear algebra is their first exposure to abstract mathematics, and they invariably have a very hard time of it. However, the use of differential equations in developing the fundamental ideas can give the applied-oriented reader both the motivation and direction for persevering with the abstractness of the vector space framework.

There is certainly too much material to finish the whole text in one semester, and so the chapters have been structured for maximum flexibility. At Fullerton our fourth-semester linear algebra and differential equations course is structured around Chapters 1 to 9 and 11. This material can be completed fairly easily in one semester. In teaching this course I often cover Chapter 9 (Linear Ordinary Differential Equations) directly after Chapter 7 (Linear Transformations). This enables the student to see a concrete application of the vector space framework to differential equations and also provides a short break in the abstract development. Having completed Chapter 9, I then return to Chapter 8 (Eigenvalues and Eigenvectors) before finishing the course with a discussion of linear systems of differential equations.

There are several chapters that can be covered rather rapidly if the instructor feels there is too much detail. Particular instances are Chapters 1 and 2 (really all that is needed for the remainder of the text are the introductory ideas in Section 1.1 and familiarity with linear differential equations). Also, Chapter 5 contains a fairly detailed account of determinants for this level text. This reflects my own personal feeling that the student obtains a firmer understanding of the idea of a determinant by mastering the classical definition as opposed to the simpler inductive definition. In Chapter 7 the inverse transformation is not needed unless the Laplace transform is to be studied later, and so Section 7.3 may be omitted. Section 8.4 is not required in the remainder of the text, and Section 8.3 is required only if the matrix exponential function is going to be discussed in Chapter 11.

Most of the exercises have been checked using the symbolic computer algebra system Maple that is being developed at the University of Waterloo, Canada. In fact I have used Maple quite extensively in constructing many of the exercise sets. Some of the graphics in Chapter 10 and Section 13.6 were generated by Mathematica. All the other figures were drawn using the graphics software Cricket Draw, and the whole manuscript was produced on an Apple Macintosh computer.

Acknowledgments

The text has been extensively class tested over the passed three years. I would like to thank my colleagues Harriet Edwards, Ted Hromadka, Vyron Klassen, John Mathews, Ron Miller, and Edsel Stiel, who used various versions of the manuscript at Fullerton, and especially Ernie Solheid, who checked the galley proofs thoroughly for mathematical accuracy. Their comments, criticisms, and suggestions have contributed significantly to the final product. Indeed, to a large extent it was the encouragement of Dr. Mathews and Dr. Stiel that provided the initial motivation for the development of this project from a set of supplementary class notes to a full-blown textbook.

I would also like to acknowledge the thoughtful comments of the many people involved in reviewing the several drafts of this text, in particular William L. Briggs, University of Colorado, Denver; Paul W. Britt, Louisiana State University, Baton Rouge; John E. Brown, Purdue University; David Lesley, San Diego State University; and David B. Surowski, Kansas State University. All these comments were considered carefully in the final preparation of the text, and they have been of invaluable help in reinforcing my own feelings as to how the material in this text should be presented.

The person who has made the largest contribution to the accuracy of this text is Walfred Lester. As Visiting Lecturer in the mathematics department at Fullerton during the academic year 1988–1989, Walfred taught from the manuscript and worked all the exercises. While doing so, he uncovered and corrected many of the errors that were present at the time and made several suggestions regarding the presentation of the material. I wish to express my thanks and appreciation for a helpful, lively, and enjoyable interaction.

Thanks are also due to the production editor, Kathleen Lafferty, who has done a superb job in overseeing all aspects of the production of this text.

I owe the greatest debt of gratitude to my wife, Christina Goode. Her continued support and encouragement throughout the development of this project has been a constant source of inspiration, particularly during those times when it seemed as though the manuscript would never be completed. In addition, her careful proofreading has helped to minimize errors and to clarify the explanations in several places. I dedicate this book to her.

Finally I would like to acknowledge the indirect influence of my mentor and friend Professor John Wainwright.

Stephen W. Goode

MATRICES

Matrix Multiplication: If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then

$$AB = \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right].$$

Zero Matrix: The $m \times n$ matrix whose elements are all zero.

Transpose, A^T: Interchange row and column vectors in A.

Identity Matrix: $I_n = [\delta_{ij}]$, where $\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$

Symmetric Matrix: $A^T = A$.

Skew-symmetric Matrix: $A^T = -A$.

Upper Triangular Matrix: $a_{ij} = 0$ whenever $i > j$.

Lower Triangular Matrix: $a_{ij} = 0$ whenever $j > i$.

Rank(A) = number of nonzero rows in any row echelon form of A.

DETERMINANTS

If A is an $n \times n$ matrix then

$$\det(A) = \Sigma\, \sigma(p_1, p_2, \ldots, p_n)\, a_{1p_1} a_{2p_2} a_{3p_3} \cdots a_{np_n},$$

where the summation is over the $n!$ permutations $(p_1, p_2, \ldots, p_n)$ of the integers $1, 2, 3, \ldots, n$.

The *Cofactor Expansion Theorem* states that $\det(A)$ can be evaluated using either of the following formulas:

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij},$$

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij},$$

where A_{ij} denotes the cofactor of the element a_{ij}.

SYSTEMS OF LINEAR EQUATIONS

1. Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let r denote the rank of A, and let $r^\#$ denote the rank of the augmented matrix of the system. Then:
   (a) If $r < r^\#$ the system is inconsistent.
   (b) If $r = r^\#$ the system is consistent and:
      (i) There exists a unique solution if and only if $r^\# = n$.
      (ii) There exists an infinite number of solutions if and only if $r^\# < n$.
2. An $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det(A) \neq 0$.
3. An $n \times n$ *homogeneous* linear system $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions if and only if $\det(A) = 0$.
4. *Cramer's rule:* If $\det(A) \neq 0$, then the unique solution to $A\mathbf{x} = \mathbf{b}$ is $(x_1, x_2, \ldots, x_n)$, where $x_k = \dfrac{\det(B_k)}{\det(A)}$, $k = 1, 2, \ldots, n$, and B_k denotes the matrix obtained by replacing the kth column vector of A with $\mathbf{b}$.

VECTOR SPACES

1. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is said to:
   (a) be *linearly dependent,* if there exist scalars $c_1, c_2, \ldots, c_k$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$.
   (b) be *linearly independent,* if the *only* values of the scalars $c_1, c_2, \ldots, c_k$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ are $c_1 = c_2 = \cdots = c_k = 0$.
   (c) *span* V, if *every* vector in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, that is, if for any $\mathbf{v} \in V$ there exist scalars $c_1, c_2, \ldots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.
2. A set of linearly independent vectors that spans a vector space V is called a *basis* for V.
   (a) All bases in a finite dimensional vector space contain the same number of vectors, and this number is called the *dimension* of V, denoted $\dim[V]$.
   (b) If $\dim[V] = n$, then *any* set of n linearly independent vectors in V forms a basis for V.

LINEAR TRANSFORMATIONS

A mapping $T : V \dashrightarrow W$ from the vector space V into the vector space W is called a linear transformation if it satisfies

$$T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}), \text{ for all } \mathbf{x} \text{ and } \mathbf{y} \text{ in } V,$$
$$T(c\mathbf{x}) = cT(\mathbf{x}), \text{ for all } \mathbf{x} \text{ in } V \text{ and all scalars } c.$$

1. The *kernel* of T, denoted $\mathrm{Ker}(T)$, is the set of all vectors in V that are mapped to the zero vector in W. Thus $\mathrm{Ker}(T) = \{\mathbf{x} \in V : T(\mathbf{x}) = \mathbf{0}\}$. $\mathrm{Ker}(T)$ is a subspace of V.
2. The *range* of T, denoted $\mathrm{Rng}(T)$, is the set of vectors in W that we obtain when we allow T to act on every vector in V. Thus $\mathrm{Rng}(T) = \{\mathbf{y} \in W : T(\mathbf{x}) = \mathbf{y} \text{ for at least one } \mathbf{x} \in V\}$. $\mathrm{Rng}(T)$ is a subspace of W.

EIGENVALUES AND EIGENVECTORS

1. For a given $n \times n$ matrix A, the eigenvalue-eigenvector problem consists of determining all scalars λ and all *nonzero* vectors $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
2. The eigenvalues of A are the roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = 0, \quad (1)$$

   and the eigenvectors are obtained by solving the homogeneous linear systems $(A - \lambda I)\mathbf{v} = \mathbf{0}$ when λ assumes the values obtained in (1).
3. If A is nondefective and $S = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent eigenvectors of A, then $S^{-1}AS = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$, are the eigenvalues of A corresponding to the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
4. If A is a *real symmetric* matrix, then it has a complete set of (real) orthonormal eigenvectors, say $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$. If $S = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n]$, then S is an orthogonal matrix ($S^{-1} = S^T$), and $S^TAS = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

THE INVERSE OF AN $n \times n$ MATRIX A

An $n \times n$ matrix is said to be nonsingular if there exists a matrix A^{-1} satisfying $AA^{-1} = A^{-1}A = I_n$. A^{-1} is called the *inverse* of A and is unique if it exists.

1. A^{-1} exists if and only if $\det(A) \neq 0$.
2. A^{-1} exists if and only if $\mathrm{rank}(A) = n$.
3. If A is nonsingular then $A^{-1} = \dfrac{1}{\det(A)}\,\mathrm{adj}(A)$, where $\mathrm{adj}(A)$ denotes the adjoint of A.

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