Lecture Notes in Mathematics

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Stochastic Methods in Finance

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Stochastic Methods in Finance

Lectures given at the
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Preface

A considerable part of the vast development in Mathematical Finance over the last two decades was determined by the application of stochastic methods. These were therefore chosen as the focus of the 2003 School on "Stochastic Methods in Finance". The growing interest of the mathematical community in this field was also reflected by the extraordinarily high number of applications for the CIME-EMS School. It was attended by 115 scientists and researchers, selected from among over 200 applicants. The attendees came from all continents: 85 were Europeans, among them 35 Italians.

The aim of the School was to provide a broad and accurate knowledge of some of the most up-to-date and relevant topics in Mathematical Finance. Particular attention was devoted to the investigation of innovative methods from stochastic analysis that play a fundamental role in mathematical modeling in finance or insurance: the theory of stochastic processes, optimal and stochastic control, stochastic differential equations, convex analysis and duality theory.

The outstanding and internationally renowned lecturers have themselves contributed in an essential way to the development of the theory and techniques that constituted the subjects of the lectures. The financial origin and motivation of the mathematical analysis were presented in a rigorous manner and this facilitated the understanding of the interface between mathematics and finance. Great emphasis was also placed on the importance and efficiency of mathematical instruments for the formalization and resolution of financial problems. Moreover, the direct financial origin of the development of some theories now of remarkable importance in mathematics emerged with clarity.

The selection of the five topics of the CIME Course was not an easy task because of the wide spectrum of recent developments in Mathematical Finance. Although other topics could have been proposed, we are confident that the choice made covers some of the areas of greatest current interest.

We now propose a brief guided tour through the topics chosen and through the methodologies that modern financial mathematics has elaborated to unveil *Risk* beneath its different masks.

We begin the tour with expected utility maximization in continuous-time stochastic markets: this classical problem, which can be traced back to the seminal works by Merton, received a renewed impulse in the middle of the 1980's, when the so-called duality approach to the problem was first developed. Over the past twenty years, the theory constantly improved, until the general case of semimartingale stochastic models was finally tackled with great success. This prompted us to dedicate one series of lectures to this traditional as well as very innovative topic:

"Utility Maximization in Incomplete Markets", Prof. Walter Schachermayer, Technical University of Vienna.
This course was mainly focused on the maximization of the expected utility from terminal wealth in incomplete markets. A part of the course was dedicated to the presentation of the stochastic model of the market, with particular attention to the formulation of the condition of No Arbitrage. Some results of convex analysis and duality theory were also introduced and explained, as they are needed for the formulation of the dual problem with respect to the set of equivalent martingale measures. Then some recent results of this classical problem were presented in the general context of semi-martingale financial models.

The importance of the above-mentioned analysis of the utility maximization problem is also revealed in the theory of asset pricing in incomplete markets, where the agent's preferences have again to be given serious consideration, since *Risk* cannot be completely hedged. Different notions of "utility-based" prices have been introduced in the literature since the middle of the 1990's. These concepts determine pricing rules which are often non-linear outside the set of marketed claims. Depending on the utility function selected, these pricing kernels share many properties with non-linear valuations: this bordered on the realm of risk measures and capital requirements. Coherent or convex risk measures have been studied intensively in the last eight years but only very recently have risk measures been considered in a dynamic context. The theory of non-linear expectations is very appropriate for dealing with the genuinely dynamic aspects of the measures of *Risk*. This leads to the next topic:

"Nonlinear expectations, nonlinear evaluations and risk measures", Prof. Shige Peng, Shandong University.
In this course the theory of the so-called "g-expectations" was developed, with particular attention to the following topics: backward stochastic differential equations, F-expectation, g-martingales and theorems of decomposition of E-supermartingales. Applications to the theory of risk measures in a dynamic context were suggested, with particular emphasis on the issues of time consistency of the dynamic risk measures.

Among the many forms of *Risk* considered in finance, credit risk has received major attention in recent years. This is due to its theoretical relevance but certainly also to its practical implications among the multitude of investors. Credit risk is the risk faced by one party as a result of the possible decline in the creditworthiness of the counterpart or of a third party. An overview of the current state of the art was given in the following series of lectures:

"Stochastic methods in credit risk modeling: valuation and hedging", Prof. Tomasz Bielecki, Illinois Institute of Technology.
A broad review of the recent methodologies for the management of credit risk was presented in this course: structural models, intensity-based models, modeling of dependent defaults and migrations, defaultable term structures, copula based models. For each model the main mathematical tools have been described in detail, with particular emphasis on the theory of martingales, stochastic control, Markov chains. The written contribution to this volume involves, in addition to the lecturer, two co-authors, they too are among the most prominent current experts in the field.

The notion of *Risk* is not limited to finance, but has a traditional and dominating place also in insurance. For some time the two fields have evolved independently of one another, but recently they are increasingly interacting and this is reflected also in the financial reality, where insurance companies are entering the financial market and viceversa. It was therefore natural to have a series of lectures also on insurance risk and on the techniques to control it.

"Financial control methods applied in insurance", Prof. Christian Hipp, University of Karlsruhe.
The methodologies developed in modern mathematical finance have also met with wide use in the applications to the control and the management of the specific risk of insurance companies. In particular, the course showed how the theory of stochastic control and stochastic optimization can be used effectively and how it can be integrated with the classical insurance and risk theory.

Last but not least we come to the topic of partial and asymmetric information that doubtlessly is a possible source of *Risk*, but has considerable importance in itself since evidently the information is neither complete nor equally shared among the agents. Frequently debated also by economists, this topic was analyzed in the lectures:

"Partial and asymmetric information", Prof. Kerry Back, University of St. Louis.
In the context of economic equilibrium, a survey of incomplete and asymmetric information (or insider trading) models was presented. First, a review of filtering theory and stochastic control was introduced. In the second part of the course some work on incomplete information models was analyzed, focusing on Markov chain models. The last part was concerned with asymmetric information models, with particular emphasis on the Kyle model and extensions thereof.

As editors of these Lecture Notes we would like to thank the many persons and Institutions that contributed to the success of the school. It is our pleasure to thank the members of the CIME (Centro Internazionale Matematico Estivo) Scientific Committee for their invitation to organize the School; the Director, Prof. Pietro Zecca, and the Secretary, Prof. Elvira Mascolo, for their efficient support during the organization. We were particularly pleased by the fact that the European Mathematical Society (EMS) chose to co-sponsor this CIME-School as one of its two Summer Schools for 2003 and that it provided additional financial support through UNESCO-Roste.
Our special thanks go to the lecturers for their early preparation of the material to be distributed to the participants, for their excellent performance in teaching the courses and their stimulating scientific contributions. All the participants contributed to the creation of an exceptionally friendly atmosphere which also characterized the various social events organized in the beautiful environment around the School. We would like to thank the Town Council of Bressanone/Brixen for additional financial and organizational support; the Director and the staff of the Cusanus Academy in Bressanone/Brixen for their kind hospitality and efficiency as well as all those who helped us in the realization of this event.
This volume collects the texts of the five series of lectures presented at the Summer School. They are arranged in alphabetic order according to the name of the lecturer.

Firenze and Padova, March 2004

Marco Frittelli and Wolfgang J. Runggaldier

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Incomplete and Asymmetric Information in Asset Pricing Theory

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These notes could equally well be entitled "Applications of Filtering in Financial Theory." They constitute a selective survey of incomplete and asymmetric information models. The study of asymmetric information, which emphasizes differences in information, means that we will be concerned with equilibrium theory and how the less informed agents learn in equilibrium from the more informed agents. The study of incomplete information is also most interesting in the context of economic equilibrium.

Excellent surveys of incomplete information models in finance [48] and of asymmetric information models [10] have recently been published. In these notes, I will not attempt to repeat these comprehensive surveys but instead will give a more selective review.

The first part of this article provides a review of filtering theory, in particular establishing the notation to be used in the later parts. The second part reviews some work on incomplete information models, focusing on recent work using simple Markov chain models to model the behavior of the market portfolio. The last part reviews asymmetric information models, focusing on the Kyle model and extensions thereof.

1 Filtering Theory

Let us start with a brief review of filtering theory, as exposited in [33]. Note first that engineers and economists tend to use the term "signal" differently. Engineers take the viewpoint of the transmitter, who sends a "signal," which is then to be estimated (or "filtered") from a noisy observation. Economists tend to take the viewpoint of the receiver, who observes a "signal" and then uses it to estimate some other variable. To avoid confusion, I will try to avoid the term, but when I use it (in the last part of the chapter), it will be in the sense of economists.

We work on a finite time horizon $[0, T]$ and a complete probability space $(\Omega, \mathcal{A}, P)$. The problem is to estimate a process X from the observations of another process Y. In general, one considers estimating the conditional expectation $E[f(X_t)|\mathcal{F}_t^Y]$, where $\{\mathcal{F}_t^Y\}$ is the the filtration generated by Y augmented by the P–null sets in $\mathcal{A}$, and f is a real-valued function satisfying some minimal regularity conditions but otherwise arbitrary. By estimating $E[f(X_t)|\mathcal{F}_t^Y]$ for arbitrary f, one can obtain the distribution of X_t conditional on $\mathcal{F}_t^Y$.

For any process θ, we will use the conventional notation $\hat{\theta}_t$ to denote $E[\theta_t|\mathcal{F}_t^Y]$. More precisely, $\hat{\theta}_t$ denotes for each t a version of $E[\theta_t|\mathcal{F}_t^Y]$ chosen so that the resulting process $(t, \omega) \mapsto \hat{\theta}_t(\omega)$ is jointly measurable.

Let W be an n–dimensional Wiener process on its own filtration and define $\mathcal{F}_t$ to be the σ–field generated by $(X_s, W_s; s \leq t)$ augmented by the P–null sets in $\mathcal{A}$. We assume for each t that $\mathcal{F}_t$ is independent of the σ–field generated by $(W_v - W_u; t \leq u \leq v \leq T)$, which simply means that the future changes in the Wiener process cannot be foretold by X. Henceforth, we will assume that all processes are $\{\mathcal{F}_t\}$–adapted.

The Wiener process W creates the noise that must be filtered from the observation process. Specifically, assume the observation process Y satisfies

$$dY_t = h_t\, dt + dW_t; \quad Y_0 = 0 \tag{1}$$

where h is a jointly measurable $\Re^n$–valued process satisfying $E\int_0^T \|h_t\|^2\, dt < \infty$.

Assume X takes values in some complete separable metric space, define $f_t = f(X_t)$,and assume

$$df_t = g_t\, dt + dM_t, \tag{2}$$

for some jointly measurable process g and right-continuous martingale M such that $E\int_0^T |g_t|^2 dt < \infty$. If X is given as the solution of a stochastic differential equation and f is smooth, the processes g and M can of course be computed from Itô's formula. We assume further that $E[f_t^2] < \infty$ for each t and $E\int_0^T \|f_t h_t\|^2\, dt < \infty$.

The "innovation process" is defined as

$$\begin{aligned} dZ_t &= dY_t - \hat{h}_t\, dt \\ &= (h_t - \hat{h}_t)\, dt + dW_t \end{aligned} \tag{3}$$

with $Z_0 = 0$. The differential dZ is interpreted as the innovation or "surprise" in the variable Y, which consists of two parts, one being the error in the estimation of the drift h_t and the other being the random change dW.

The main results of filtering theory, due to Fujisaka, Kallianpur, and Kunita [22], are the following.

1) The innovation process Z is an $\{\mathcal{F}_t^Y\}$–Brownian Motion.

2) For any separable L^2–bounded $\{\mathcal{F}_t^Y\}$–martingale H, there exists a jointly measurable $\{\mathcal{F}_t^Y\}$–adapted $\Re^n$–valued process ϕ such that $E\int_0^T \|\phi_t\|^2 \, dt < \infty$, and

$$dH_t = \sum_{i=1}^{n} \phi_t^i \, dZ_t^i.$$

3) There exist jointly measurable adapted processes α^i such that $d[M, W^i]_t = \alpha_t^i \, dt$, for $i = 1, \ldots, N$.
4) $\hat{f}$ evolves as

$$d\hat{f}_t = \hat{g}_t \, dt + \left(\widehat{fh}_t - \hat{f}_t \hat{h}_t + \hat{\alpha}_t \right)' dZ_t, \tag{4}$$

where $\widehat{fh}_t$ denotes $E[f_t h_t | \mathcal{F}_t^Y]$.

Part (1) means in particular that Z is a martingale; thus the innovations dZ are indeed "unpredictable." Given that it is a martingale, the fact that it is a Brownian motion follows from Levy's theorem and the fact, which follows immediately from (3), that the covariations are $d\langle Z^i, Z^j \rangle = dt$ if $i = j$ and 0 otherwise. Part (2) means that the process Z "spans" the $\{\mathcal{F}_t^Y\}$–martingales (which would follow from $\{\mathcal{F}_t^Y\} = \{\mathcal{F}_t^Z\}$, though this condition does not hold in general). Part (3) means that the square-bracket processes are absolutely continuous, though in our applications we will assume M and the W^i are independent, implying $\alpha^i = 0$ for all i.

Part (4) is the filtering formula. The estimate $\hat{f}$ is updated because f is expected to change (which is obviously captured by the term $\hat{g}_t \, dt$) and because new information from dZ is available to estimate f. The observation process Y (or equivalently the innovation process Z) is useful for estimating f due to two factors. One is the possibility of correlation between the martingales W and M. This is reflected in the term $\hat{\alpha}_t \, dZ_t$. The other factor is the correlation between f and the drift h_t of Y. This is reflected in the term $(\widehat{fh}_t - \hat{f}_t \hat{h}_t) \, dZ_t$. Note that $\widehat{fh}_t - \hat{f}_t \hat{h}_t$ is the covariance of f_t and h_t, conditional on $\mathcal{F}_t^Y$. The formula (4) generalizes the linear prediction formula

$$\hat{x} = \bar{x} + \frac{\text{cov}(x, y)}{\text{var}(y)} (y - \bar{y}),$$

which yields $\hat{x} = E[x|y]$ when x and y are joint normal.

We consider two applications.

1.1 Kalman-Bucy Filter

Assume X_0 is distributed normally with variance σ^2 and

$$\begin{aligned} dX_t &= aX_t \, dt + dB_t, \\ dY_t &= cX_t \, dt + dW_t, \end{aligned}$$

where B and W are independent real-valued Brownian motions that are independent of X_0. In this case, the distribution of X_t conditional on $\mathcal{F}_t^Y$ is normal with deterministic variance Σ_t. Moreover,

$$d\hat{X}_t = a\hat{X}_t\,dt + c\Sigma_t\,dZ_t, \tag{5}$$

where the innovation process Z is given by

$$dZ_t = dY_t - c\hat{X}_t\,dt. \tag{6}$$

Furthermore,

$$\Sigma_t = \frac{\gamma\alpha e^{\lambda t} - \beta}{\gamma e^{\lambda t} + 1}, \tag{7}$$

where α and $-\beta$ are the two roots of the quadratic equation $1+2ax-c^2x^2=0$, with both α and β positive, $\lambda = c^2(\alpha+\beta)$ and $\gamma = (\sigma^2+\beta)/(\alpha-\sigma^2)$. One can consult, e.g., [33] or [41] for the derivation of these results from the general filtering results cited above. In the multivariate case, an equation of the form (5) also holds, where Σ_t is the covariance matrix of X_t conditional on $\mathcal{F}_t^Y$. In this circumstance, the covariance matrix evolves deterministically and satisfies an ordinary differential equation of the Riccati type, but there is in general no closed-form solution of the differential equation.

1.2 Two-State Markov Chain

A very simple model that lies outside the Gaussian family is a two-state Markov chain. There is no loss of generality in taking the states to be 0 and 1, and it is convenient to do so. Consider the Markov chain X satisfying

$$dX_t = (1 - X_{t-})\,dN_t^0 - X_{t-}\,dN_t^1, \tag{8}$$

where $X_{t-} \equiv \lim_{s\uparrow t} X_s$ and the N^i are independent Poisson processes with parameters λ^i that are independent of X_0. This means that X stays in each state an exponentially distributed amount of time, with the exponential distribution determining the transition from state i to state j having parameter λ^i. This fits in our earlier framework as

$$dX_t = g_t\,dt + dM_t,$$

where

$$\begin{aligned} g_t &= (1 - X_{t-})\lambda^0 - X_{t-}\lambda^1, \text{and} \\ dM_t &= (1 - X_{t-})\,dM_t^0 - X_{t-}\,dM_t^1, \end{aligned}$$

with M^i being the martingale $M_t^i = N_t^i - \lambda^i t$.

Assume

$$dY_t = h(X_{t-})\,dt + dW_t, \tag{9}$$

where W is an n–dimensional Brownian motion independent of the N^i and X_0. Thus, the drift vector of Y is $h(0)$ or $h(1)$ depending on the state X_{t-}. In terms of our earlier notation, $h_t = h(X_{t-})$.

Write π_t for $\hat{X}_t$. This is the conditional probability that $X_t = 1$. The general filtering formula (4) implies[1]

$$d\pi_t = \big[(1-\pi_t)\lambda^0 - \pi_t\lambda^1\big]\,dt + \pi_t(1-\pi_t)\big[h(1)-h(0)\big]'\,dZ_t, \tag{10}$$

where the innovation process Z is given by

$$dZ_t = dY_t - \big[(1-\pi_t)h(0) + \pi_t h(1)\big]\,dt. \tag{11}$$

This is a special case of the results on Markov chain filtering due to Wonham [47].

Note the similarity of (10) with the Kalman-Bucy filter (5): $h(1)-h(0)$ is the vector c in the equation

$$\begin{aligned} dY_t &= h(X_{t-})\,dt + dW_t \\ &= \big[(1-X_{t-})h(0) + X_{t-}h(1)\big]\,dt + dW_t \\ &= h(0)\,dt + cX_{t-}\,dt + dW_t, \end{aligned}$$

and $\pi_t(1-\pi_t)$ is the variance of X_t conditional on $\mathcal{F}_t^Y$.

2 Incomplete Information

2.1 Seminal Work

Early work in portfolio choice and market equilibrium under incomplete information includes [16], [19], and [23]. These papers analyze models of the following sort. The instantaneous rate of return on an asset is given by

$$\begin{aligned} \frac{dS}{S} &= \mu_t\,dt + \sigma\,dW, \quad \text{where} \\ d\mu_t &= \kappa(\theta-\mu_t)\,dt + \phi\,dB \end{aligned}$$

and W and B are Brownian motions with a constant correlation coefficient ρ, and where μ_0 is normally distributed and independent of W and B. It is assumed that investors observe S but not μ; i.e., their filtration is the filtration generated by S (augmented by the P–null sets). The innovation process is

$$dZ = \frac{\mu_t - \hat{\mu}_t}{\sigma}\,dt + dW,$$

which is an $\{\mathcal{F}_t^S\}$–Brownian motion. Moreover, we can write

[1] Note that (4) implies π is continuous and then from bounded convergence we have $\pi_t = E\left[X_{t-}|\mathcal{F}_t^Y\right]$, so $\hat{g}_t = (1-\pi_t)\lambda^0 - \pi_t\lambda^1$.