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Geometric Properties of Banach Spaces and Nonlinear Iterations

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To my beloved family:
Ifeoma (wife),
and children:
Chu Chu; Ada; KK and Okey (Okido).

Preface

The contents of this monograph fall within the general area of nonlinear functional analysis and applications. We focus on an important topic within this area: *geometric properties of Banach spaces and nonlinear iterations,* a topic of intensive research efforts, especially within the past 30 years, or so.

In this theory, some geometric properties of Banach spaces play a crucial role. In the first part of the monograph, we expose these geometric properties most of which are well known. As is well known, among all infinite dimensional Banach spaces, Hilbert spaces have the *nicest* geometric properties. The availability of the inner product, the fact that the *proximity map or nearest point map* of a real Hilbert space H onto a closed convex subset K of H is Lipschitzian with constant 1, and the following two identities

$$||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \qquad (*)$$

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2, \qquad (**)$$

which hold for all $x, y \in H$, are some of the geometric properties that characterize inner product spaces and also make certain problems posed in Hilbert spaces more manageable than those in general Banach spaces. However, as has been rightly observed by M. Hazewinkel, "... many, and probably most, mathematical objects and models do not naturally live in Hilbert spaces". Consequently, to extend some of the Hilbert space techniques to more general Banach spaces, analogues of the identities $(*)$ and $(**)$ have to be developed. For this development, the duality map which has become a most important tool in nonlinear functional analysis plays a central role. In 1976, Bynum [61] obtained the following analogue of $(*)$ for l_p spaces, $1 < p < \infty$:

$$||x+y||^2 \leq (p-1)||x||^2 + ||y||^2 + 2\langle x, j(y)\rangle, \ 2 \leq p < \infty,$$

$$(p-1)||x+y||^2 \leq ||x||^2 + ||y||^2 + 2\langle x, j(y)\rangle, \ 1 < p \leq 2.$$

Analogues of (**) were also obtained by Bynum. In 1979, Reich [408] obtained an analogue of (*) in uniformly smooth Banach spaces. Other analogues of (*) and (**) obtained in 1991 and later can be found, for example, in Xu [509] and in Xu and Roach [525].

In Chapters 1 and 2, basic well-known facts on geometric properties of Banach spaces which are used in the monograph are presented. The materials here (and much more) can be found in any of the excellent books on this topic (e.g., Diestel [206]; Lindenstrauss and Tzafriri [312]). The *duality map* which is central in our work is presented in Chapter 3. Here, we have also computed explicitly the duality maps in some concrete Banach spaces. In Chapters 4 and 5, we sketch the proofs of the analogues of the identities (*) and (**) obtained in 1991 and later. In the last section of Chapter 5, we present characterizations of real uniformly smooth Banach spaces and Banach spaces with uniformly Gâteaux differentiable norms by means of continuity properties of the normalized duality maps. Applications of the geometric properties of Banach spaces presented in Chapters 1 to 5 to iterative algorithms for solutions of nonlinear equations, an intensive and extensive area of research work (Berinde [28] contains 1575 entries in the reference list on this topic) begin in Chapter 6. To motivate some of the reasons for our choices of the classes of nonlinear operators studied in this monograph, we begin with the following.

Let K be a nonempty subset of a real normed linear space E and let $T : K \to K$ be a map. A point $x \in K$ is said to be *a fixed point* of T if $Tx = x$. Now, consider the differential equation $\frac{du}{dt} + Au(t) = 0$ which describes an evolution system where A is an *accretive map* from a Banach space E to itself. In Hilbert spaces, accretive operators are called *monotone*. At equilibrium state, $\frac{du}{dt} = 0$, and so a solution of $Au = 0$ describes the equilibrium or stable state of the system. This is very desirable in many applications in, for example, ecology, economics, physics, to name a few. Consequently, considerable research efforts have been devoted to methods of solving the equation $Au = 0$ when A is accretive. Since generally A is nonlinear, there is no closed form solution of this equation. The standard technique is to introduce an operator T defined by $T := I - A$ where I is the identity map on E. Such a T is called *a pseudo-contraction* (or is called *pseudo-contractive*). It is then clear that any zero of A is a fixed point of T. As a result of this, the study of fixed point theory for pseudo-contractive maps has attracted the interest of numerous scientists and has become a flourishing area of research, especially within the past 30 years or so, for numerous mathematicians. A very important subclass of the class of pseudo-contractive mappings is that of *nonexpansive* mappings, where $T : K \to K$ is called nonexpansive if $||Tx - Ty|| \leq ||x - y||$ holds for arbitrary $x, y \in K$.

Apart from being an obvious generalization of the contraction mappings, nonexpansive maps are important, as has been observed by Bruck [59], mainly for the following two reasons:

- Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.
- Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form $0 \in \frac{du}{dt} + T(t)u$, where the operators $\{T(t)\}$ are, in general, set-valued and are *accretive* or *dissipative* and *minimally continuous.*

If K is a closed nonempty subset of a Banach space and $T : K \to K$ is nonexpansive, it is known that T may not have a fixed point (unlike the case if T is a strict contraction), and even when it has, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n, n \geq 1$ (the so-called *Picard sequence*) may fail to converge to such a fixed point. This can be seen by considering an anti-clockwise rotation of the unit disc of $\mathbb{R}^2$ about the origin through an angle of say, $\frac{\pi}{4}$. This map is nonexpansive with the origin as the unique fixed point, but the Picard sequence fails to converge with any starting point $x_0 \neq 0$. Krasnosel'skii [291], however, showed that in this example, if the Picard iteration formula is replaced by the following formula,

$$x_0 \in K, \quad x_{n+1} = \frac{1}{2}\Big(x_n + Tx_n\Big), n \geq 0, \tag{0.1}$$

then the iterative sequence converges to the unique fixed point. In general, if E is a normed linear space and T is a nonexpansive mapping, the following generalization of (0.1) which has proved successful in the approximation of a fixed point of T (when it exists) was given by Schaefer [431]:

$$x_0 \in K, \quad x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, \quad n \geq 0, \quad \lambda \in (0,1). \tag{0.2}$$

However, the most general iterative formula for approximation of fixed points of nonexpansive mappings, which is called the *Mann iteration formula* (in the light of Mann [319]), is the following:

$$x_0 \in K, \; x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{0.3}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0,1)$ satisfying the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$ and $(ii) \sum_{n=1}^{\infty} \alpha_n = \infty$. The recursion formula (0.2) is consequently called the *Krasnoselskii-Mann (KM)* formula for finding fixed points of *nonexpansive (ne)* mappings. The following quotation indicates part of the interest in iterative approximation of fixed points of *nonexpansive mappings.*

- "Many well-known algorithms in signal processing and image reconstruction are iterative in nature .... A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the KM iteration procedure, for particular choices of the ne operator...." (Charles Byrne, [63]).

For the past 30 years or so, the study of the Krasnoselskii-Mann iterative procedures for the approximation of fixed points of nonexpansive mappings and fixed points of some of their generalizations, and approximation of zeros of accretive-type operators have been flourishing areas of research for many mathematicians. Numerous applications of analogues of (*) and (**) to nonlinear iterations involving various classes of nonlinear operators have since then been topics of intensive research. Today, substantial definitive results have been proved, some of the methods have reached their boundaries while others are still subjects of intensive research activity. However, it is apparent that the theory has now reached a level of maturity appropriate for an examination of its central themes.

The aim of this monograph is to present an in-depth and up-to-date coverage of the main ideas, concepts and most important results on iterative algorithms for approximation of fixed points of nonlinear nonexpansive mappings and some of their important generalizations; iterative approximation of zeros of accretive-type operators; iterative approximation of solutions of variational inequality problems involving these operators; iterative algorithms for solutions of Hammerstein integral equations; and iterative approximation of common fixed points (and common zeros) of families of these mappings. Furthermore, some important open questions related to these selected topics are included.

We assume familiarity with basic concepts of analysis and topology. The monograph is addressed to graduate students of mathematics, computer science, statistics, informatics, engineering, to mathematicians interested in learning about the subject, and to numerous specialists in the area.

I have great pleasure in thanking Professor Giovanni Vidossich of Institute for Advanced Studies, SISSA, Trieste, Italy for his constant encouragement, and Professor Billy Rhoades of the Department of Mathematics, Indiana University, Bloomington, Indiana, USA, who read a version of the first draft and whose comments spurred me on. Professors Vasile Berinde and Naseer Shahzad helped with putting my Latex files into the Spinger LNM format. I am very grateful to them for this. Very special thanks go to the staff of the Publications Department of The Abdus Salam ICTP, for their patience and ever-ready assistance in typesetting the original version of the monograph. Finally, I have great pleasure in expressing my sincere gratitude to my wife, Ify, and to our children for their encouragement and understanding.

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Trieste,
June 2008

Charles Ejike Chidume

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Chapter 1
Some Geometric Properties of Banach Spaces

1.1 Introduction

In the first part of this monograph (Chapters 1 to 5), we explore selected geometric properties of Banach spaces that will play crucial roles in our study of iterative algorithms for nonlinear operators in various Banach spaces.

In this chapter, we introduce the classes of *uniformly convex* and *strictly convex* spaces, and in Chapter 2, we shall introduce the class of *smooth spaces.* All the results presented in these two chapters are well-known and standard and can be found in several books on geometry of Banach spaces, for example, in Diestel [206], or in Lindenstrauss and Tzafriri [312]. Consequently, we shall skip some details and long proofs.

It is well known that if E is a real normed space, the following identities hold

$$||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \tag{1.1}$$

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2, \tag{1.2}$$

for all $x, y \in E, \lambda \in (0,1)$ if and only if E is a real inner product space.

These geometric identities which characterize inner product spaces make numerous problems posed in real Hilbert spaces more manageable than those posed in arbitrary real Banach spaces. Consequently, to extend some of the Hilbert space techniques to more general Banach spaces, analogues of these identities have to be developed in such Banach spaces.

In Chapter 3, we introduce the *duality map* which has become a most important tool in nonlinear functional analysis. We compute the duality map explicitly for some specific Banach spaces. In arbitrary normed spaces, the duality map will serve as the analogue of the inner product in Hilbert spaces.

C. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, Lecture Notes in Mathematics 1965,

In Chapters 4 and 5, we present the analogues of the identities (1.1) and (1.2) in uniformly convex and uniformly smooth Banach spaces, respectively. Most of the results presented in these chapters were developed in 1991 or later.

At the end of Chapter 5, we characterize uniformly smooth spaces and spaces with uniformly Gâteaux differentiable norm in terms of *uniform continuity of the normalized duality map on bounded sets.* These characterizations will be used extensively in the monograph. We begin with some basic notions.

In 1936, A.J. Clarkson [191] published his famous paper on *uniform convexity* (defined below). This work signalled the beginning of extensive research efforts on the geometry of Banach spaces and its applications in functional analysis.

1.2 Uniformly Convex Spaces

Let X be an arbitrary normed space and for fixed $x_0 \in X$, let $S_r(x_0)$ denote the *sphere* centred at x_0 with radius $r > 0$, that is,

$$S_r(x_0) := \{x \in X : \|x - x_0\| = r\}.$$

Definition 1.1. A normed space X is called *uniformly convex* if for any $\varepsilon \in (0, 2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ with $\|x\| = 1, \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\left\|\frac{1}{2}\,(x + y)\right\| \leq 1 - \delta$.

Thus, a normed space is uniformly convex if for any two distinct points x and y on the unit sphere centred at the origin the midpoint of the line segment joining x and y is never on the sphere but is close to the sphere only if x and y are sufficiently close to each other.

We note immediately that the following definition is also used: A normed space X is *uniformly convex* if for any $\varepsilon \in (0, 2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ with $||x|| \leq 1, ||y|| \leq 1$ and $||x - y|| \geq \varepsilon$, then $||\frac{1}{2}(x + y)|| \leq 1 - \delta$. In the sequel we shall use either of the two definitions.

Theorem 1.2. *L_p spaces, $1 < p < \infty$, are uniformly convex.*

Proof. See e.g., Diestel [206].

Theorem 1.3. *Let X be a uniformly convex space. Then, for any $d > 0$, $\varepsilon > 0$ and arbitrary vectors $x, y \in X$ with $\|x\| \leq d, \|y\| \leq d, \|x - y\| \geq \varepsilon$, there exists a $\delta > 0$ such that*

$$\left\|\frac{1}{2}\,(x + y)\right\| \leq \left[1 - \delta\left(\frac{\varepsilon}{d}\right)\right] d\,.$$

Proof. For arbitrary $x, y \in X$, let $z_1 = \frac{x}{d}, z_2 = \frac{y}{d}$, and set $\bar{\varepsilon} = \frac{\varepsilon}{d}$. Obviously $\bar{\varepsilon} > 0$. Moreover, $\|z_1\| \leq 1, \|z_2\| \leq 1$ and $\|z_1 - z_2\| = \frac{1}{d}\,\|x - y\| \geq \frac{\varepsilon}{d} = \bar{\varepsilon}$. Now, by uniform convexity, we have for some $\delta = \delta(\frac{\varepsilon}{d}) > 0$,

$$\left\|\frac{1}{2}\,(z_1+z_2)\right\| \le 1-\delta(\bar{\varepsilon}),$$

that is,

$$\left\|\frac{1}{2d}\,(x+y)\right\| \le 1-\delta\left(\frac{\varepsilon}{d}\right),$$

which implies,

$$\left\|\frac{1}{2}\,(x+y)\right\| \le \left[1-\delta\left(\frac{\varepsilon}{d}\right)\right] d.$$

The proof is complete. □

Proposition 1.4. *Let X be a uniformly convex space and let $\alpha \in (0,1)$ and $\varepsilon > 0$. Then for any $d > 0$, if $x, y \in X$ are such that $\|x\| \le d, \|y\| \le d$, $\|x-y\| \ge \varepsilon$, then there exists $\delta = \delta\left(\frac{\varepsilon}{d}\right) > 0$ such that*

$$\|\alpha x + (1-\alpha)y\| \le \left[1 - 2\delta\left(\frac{\varepsilon}{d}\right)\min\{\alpha, 1-\alpha\}\right] d.$$

Proof. See Exercises 1.1, Problem 3.

1.3 Strictly Convex Banach Spaces

Definition 1.5. A normed space E is called *strictly convex* if for all $x, y \in E, x \neq y, ||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1\ \forall\ \lambda \in (0,1)$.

Theorem 1.6. *Every uniformly convex space is strictly convex.*

Proof. See Exercises 1.1, Problem 4.

Theorem 1.6 gives a large class of strictly convex spaces. However, we shall see later that some well known Banach spaces are *not* strictly convex.

We first give two examples of Banach spaces which are *strictly convex* but not *uniformly convex.*

Example 1.7. (Goebel and Kirk, [230]). Fix $\mu > 0$ and let $C[0,1]$ be endowed with the norm $||.||_\mu$ defined as follows,

$$||x||_\mu := ||x||_0 + \mu\Big(\int_0^1 x^2(t)dt\Big)^{\frac{1}{2}},$$

where $||.||_0$ is the usual supremum norm. Then,

$$||x||_0 \le ||x||_\mu \le (1+\mu)||x||_0, \quad x \in C[0,1],$$

and the two norms are equivalent with $||.||_\mu$ near $||.||_0$ for small μ. However, $(C[0,1], ||.||_0)$ is not strictly convex while for any $\mu > 0$, $(C[0,1], ||.||_\mu)$ is. On the other hand, for any $\varepsilon \in (0,2]$ there exist functions $x, y \in C[0,1]$