INSTITUTE OF ACTUARIES'

TEXT-BOOK

OF THE

PRINCIPLES OF INTEREST, LIFE ANNUITIES, AND ASSURANCES,

AND THEIR PRACTICAL APPLICATION.

PART I.

INTEREST (INCLUDING ANNUITIES-CERTAIN).

NEW EDITION.

[REVISED.]

BY

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PREFACE.

The Council of the Institute of Actuaries, while recognizing the skill with which the first Text-Book, Part I., on Interest, had been written, felt that in some ways it might be made more suitable for the students for whom it was intended. When, therefore, a new edition was needed, they laid the matter before Mr. Todhunter, requesting him to consider it from this point of view, giving him full liberty to act as he might think best. He found it desirable to re-write the volume, and has accordingly done so.

It is hoped that the following pages, including in due proportion theoretical explanation and practical example, will prove increasingly useful to all whose duty or pleasure it may be to apply themselves to this important subject, and that Mr. Todhunter's ability and care will earn the gratitude which they surely merit.

C. D. H.

24 *June* 1901.

INTRODUCTION BY THE AUTHOR.

In the preparation of a New Edition of the Text-Book, Part I., it has been found necessary to re-write the work. The general Theory of Compound Interest has been presented in a form which will, it is hoped, afford a comprehensive view of the subject, and special attention has been given to the applications of the Theory to practical financial problems. For the convenience of those students who have no previous knowledge of the methods of the Infinitesimal Calculus, a chapter on the elements of this subject has been included.

In the compilation of the volume assistance has been derived from numerous papers and notes in the *Journal*, and from various treatises on Compound Interest—more especially from Mr. George King's *Theory of Finance*, to which no subsequent writer could fail to be greatly indebted—but, in accordance with precedent, references to authorities have not been given.

The author takes this opportunity of acknowledging his indebtedness to the Council of the Institute for the critical examination which they have given to the work during its progress, while according him entire liberty in the treatment of the subject. He also offers his best thanks to Mr. J. E. FAULKS, B.A., for many valuable suggestions, and to Mr. A. LEVINE, M.A., for assistance in the revision of the earlier proof-sheets of the two concluding chapters and other parts of the work.

R. T.

London, 12 *June* 1901.

THE necessity for another edition having arisen, the book has been revised by the author in consultation with Mr. W. PALIN ELDERTON and Mr. H. M. TROUNCER, M.A., to whom the author is much indebted.

R. T.

London, *July* 1915.

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INSTITUTE OF ACTUARIES' TEXT-BOOK.

PART I.

THE THEORY OF COMPOUND INTEREST AND ANNUITIES-CERTAIN.

CHAPTER I.

DEFINITIONS AND ELEMENTARY PROPOSITIONS.

1. INTEREST may be defined as the consideration for the use of capital, or as that which is earned by the productive investment of capital. In theory, it is not necessary that the invested capital and the consideration for the use of it should be expressed in terms of one or the same commodity, but in practice it is usual and convenient to express both in terms of some one unit; in the investigations that follow, it will be assumed that both are expressed in terms of a unit of money, without specification of the particular currency to which that unit belongs.

2. The invested capital is called the PRINCIPAL. The consideration for the use of capital usually becomes due at stated intervals, and, being itself of the same nature as capital, may be employed, when received, as capital. In the Theory of Compound Interest, it is assumed that the consideration will not be allowed to remain idle, but will immediately be productively invested.

B

3. The total interest earned on a given principal in a given time will obviously depend on (1) the given principal, (2) the interest contracted to be paid for each stated interval in respect of each unit of principal, (3) the given time. The second of these quantities is, in the strictest sense, the RATE OF INTEREST, and in some investigations it will be found convenient to take this quantity—the interest contracted to be paid in respect of each unit of principal for each stated interval—and the number of such intervals in the given time as data. The expression "the rate of interest" is, however, more generally used with reference to a *year*, the accepted unit of time in the Theory of Finance, and it is so used to denote:—

(1) the rate per unit *per annum* at which interest is calculated for each stated interval for which interest is contracted to be paid, or, in other words, the interest that would be earned on each unit of principal in a year if the interest received at the end of each stated interval were not itself productively invested;

(2) the total interest earned on each unit of principal in a year on the assumption that the actual interest as received at the end of each stated interval is invested on the same terms as the original principal.

4. It is obvious that, except in the case when the stated interval for which interest is to be paid is a year, these two senses in which the expression "the rate of interest" is employed represent two different things. To take a simple example, let it be supposed that a principal of 100 is invested in consideration of the payment of $2\frac{1}{2}$ at the end of each half-year. In this case the rate per unit *per annum* at which interest is calculated, or the interest that would be earned in a year on each unit of principal if the interest received at the end of the first half-year were not productively invested, is ·05, whereas the total interest earned on each unit of principal in a year on the assumption that the $2\frac{1}{2}$ received at the end of the first half-year is invested on the same terms as the original principal (*i.e.*, in consideration of the payment of $2\frac{1}{2}$ per-cent on the $2\frac{1}{2}$, or ·0625, at the end of each half-year) is ·050625. It is convenient, therefore, to distinguish between the two senses in which the expression "rate of interest" is employed by the use of distinct expressions and distinct symbols.

5. The rate per unit per annum at which the actual interest for each stated interval is calculated when that interval differs from a year, or, in other words, the interest that would be earned on each unit of principal in a year if the interest as received were not productively invested, is called the NOMINAL RATE OF INTEREST, and will be distinguished by the symbol j. The frequency with which interest is actually payable, or the stated interval of payment, is defined by the expressions "payable half-yearly, quarterly, or m times a year" (as the case may be), "convertible half-yearly, quarterly, monthly, &c.", or "with half-yearly, quarterly, monthly, &c., rests." Thus, when interest is said to be at the nominal rate of 5 per-cent per annum payable (or convertible) half-yearly, or at the nominal rate of 5 per-cent per annum with half-yearly rests, it is meant that $2\frac{1}{2}$ is to be paid at the end of each half year for each 100 owing at the beginning of the half year. The frequency of conversion of a given nominal rate may be denoted by means of a suffix placed in brackets at the lower right-hand corner of the symbol representing the rate. Thus $j_{(m)}$ denotes a nominal rate j convertible m times a year.

6. The total interest earned on 1 in a year, on the assumption that the actual interest (if receivable otherwise than yearly) is immediately invested as it becomes due, on the same terms as the original principal, is called the EFFECTIVE RATE OF INTEREST, and will be distinguished by the symbol i.

7. To every nominal rate of interest, convertible with a given frequency, there is a *corresponding* effective rate, for the total interest earned on each unit of principal in a year—in other words, the effective rate of interest—may be found by accumulating a unit, on the assumption of compound interest, at the given nominal rate. Thus, if the nominal rate be j, convertible m times a year, an original unit of principal, together with the interest upon it at the end of the first $\frac{1}{m}$th of a year, will amount to $1+\frac{j}{m}$. By assumption the $\frac{j}{m}$ is immediately invested on the same terms as the original unit of principal, so that the interest due at the end of the second $\frac{1}{m}$th of a year will be $\frac{j}{m}\left(1+\frac{j}{m}\right)$; hence the original unit with interest will amount to $1+\frac{j}{m}+\frac{j}{m}\left(1+\frac{j}{m}\right)$ or $\left(1+\frac{j}{m}\right)^2$. Similarly, in each $\frac{1}{m}$th of a year the amount of the original unit with interest at the beginning of the interval will be increased in the ratio of $1+\frac{j}{m}$ to 1. Consequently, at the end of a year, the unit of principal with interest will amount to $\left(1+\frac{j}{m}\right)^m$. The total interest earned on each unit of principal in the year is, therefore, $\left(1+\frac{j}{m}\right)^m-1$. In symbols,

$$i=\left(1+\frac{j}{m}\right)^m-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence

$$j=m\left\{(1+i)^{\frac{1}{m}}-1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$m \log\left(1+\frac{j}{m}\right)=\log(1+i) \quad . \quad . \quad . \quad . \quad (3)$$

8. From these equations the effective rate of interest corresponding to a given nominal rate convertible with a given frequency, or, conversely, the nominal rate convertible with any required frequency corresponding to a given effective rate, may be calculated.

9. It will be observed that if *two* of the three quantities j, m and i are given, the equation gives a single value for the third quantity; that is to say, to a given nominal rate convertible with a given frequency there is one, and only one, corresponding effective rate; to a given effective rate, there is one, and only one, corresponding nominal rate convertible with a given frequency; and, finally, there is one, and only one, frequency for which a given nominal rate and a given effective rate will correspond.

10. But if only *one* of the three quantities is given, any number of corresponding values may be found for either of the remaining two by assigning successive values to the other.

Thus, if j be given, any number of corresponding values of i may be found by giving successive values to m. As m increases from 1 to ∞, the value of $\left(1+\frac{j}{m}\right)^m-1$ increases from j to e^j-1. Hence the effective rate corresponding to a given nominal rate increases as the frequency of conversion of the latter is increased. For example, a nominal rate of 5 per-cent per annum convertible quarterly gives a higher effective rate than the same nominal rate convertible half-yearly. Again, if i be given, any number of corresponding values of j may be found by giving successive values to m. As m increases from 1 to ∞, the value of $m\left\{(1+i)^{\frac{1}{m}}-1\right\}$ decreases from i to $\log_e(1+i)$. Hence the nominal rate corresponding to a given effective rate *decreases* as the frequency of conversion is increased. For example, the nominal rate convertible *half-yearly* corresponding to an effective rate of 5 per-cent exceeds the nominal rate convertible *quarterly* corresponding to the same effective rate. It will be noticed, however, that the nominal rate corresponding to the effective rate i does not decrease indefinitely as m is increased, but gradually approaches the value $\log_e(1+i)$, this being the limiting value of $m\left\{(1+i)^{\frac{1}{m}}-1\right\}$, when m is made infinitely large. This limiting value is called the FORCE OF INTEREST corresponding to the effective rate i, and is distinguished by the special symbol δ.

11. The force of interest corresponding to a given effective rate i may therefore be defined as the nominal rate convertible at infinitely short intervals corresponding to that effective rate.

12. From the foregoing it will be seen that the basis upon which interest is to be calculated in any given case may be defined by means of an *effective rate of interest*, i; *a nominal rate of interest*, j, *convertible with a given frequency*, m; or, finally, a *force of interest*, δ; and that when any one of these three quantities is given the *corresponding values* of the other two may be determined by the equations

$$1+i=\left(1+\frac{j}{m}\right)^m=e^\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

13. To proceed to the general theory of the accumulation of principal under the operation of compound interest.

Let P be a given principal, S the sum to which it will amount if accumulated at compound interest for n years, and I the total interest earned on P in the given period. Let i be the effective rate of interest at which the given principal is to be accumulated, j the corresponding nominal rate of interest convertible m times a year, and δ the corresponding force of interest. Then by reasoning precisely similar to that by which it was shown that a unit accumulated for a year at compound interest at the nominal rate j convertible m times a year will amount to $\left(1+\frac{j}{m}\right)^m$, it follows that

$$S=P(1+i)^n=P\left(1+\frac{j}{m}\right)^{mn}=Pe^{n\delta} \quad . \quad . \quad . \quad . \quad (5)$$

This system of equations affords the means of calculating the amount, in a given number of years, of a given principal at any given rate of interest—effective or nominal—provided only that rate continues uniformly in operation throughout the entire period. The appropriate formula to employ in any given case will be

$$S=P\,(1+i)^n$$

or

$$S=P\left(1+\frac{j}{m}\right)^{mn}$$

or

$$S=Pe^{n\delta}$$

according as the given rate is an effective rate, a nominal rate, or a force of interest. It would, of course, be practicable to obtain the amount of a given principal at a given nominal rate or force of interest, by first finding the effective rate corresponding to the given rate, and then employing the formula $S=P(1+i)^n$, but in general it will be found more convenient to use the directly appropriate formula.

14. It should be noted that tables giving the amount of 1 in any number of years (within the limits of the tables), at various effective rates of interest, may often be employed for the purpose of calculating the amount of a given principal at a given nominal rate. For example, let it be required to find the amount of 100 in 20 years at 6 per-cent convertible half-yearly. By the appropriate formula the amount $=100(1{\cdot}03)^{40}$, which also represents the amount of 100 in *40* years at 3 per-cent *effective*. Hence the required result will be obtained by taking 100 times the tabulated value of the amount of 1 in *40* years at 3 per-cent per annum. In fact, a table of amounts may be regarded more generally as a table of $(1+x)^n$, and used for any purpose for which the value of this function is required.

15. In the derivation of formula (5), it has been implicitly assumed that n is integral. In order to extend the formula to cases in which n is not an integer, it is necessary to adopt some convention as to the interest to be assumed for a fractional part of a year. When the given rate is an effective rate, or when it is a nominal rate and the fractional part of a year does not contain an integral number of the intervals of conversion, it is permissible to adopt any convention that may appear suitable, for the stated conditions do not prescribe any rule. When, however, the given rate is a nominal rate—say, j convertible m times a year—and the given period of accumulation contains an exact number

of the intervals of conversion, being, say, $n+\frac{t}{m}$ years, a given principal P will amount to $P\left(1+\frac{j}{m}\right)^{nm+t}$, and this quantity, by algebraical substitution, $=P(1+i)^{n+\frac{t}{m}}$, where i is the effective rate corresponding to j. It appears, therefore, in this case, that the interest on 1 at the effective rate i for $\frac{t}{m}$ of a year is $(1+i)^{\frac{t}{m}}-1$. This result suggests the usual and convenient assumption that the interest on 1 for any fractional part, say $\frac{1}{p}$, of a year at the effective rate i, may be taken as $(1+i)^{\frac{1}{p}}-1$, and the adoption of this convention leads to the generalization that

$$S=P(1+i)^n=P\left(1+\frac{j}{m}\right)^{mn}=Pe^{n\delta}$$

for all values of n, *integral or fractional*. Similarly, the total interest earned on P will be given, in all cases, by the formula

$$I=S-P=P[(1+i)^n-1]=P\left[\left(1+\frac{j}{m}\right)^{mn}-1\right]=P[e^{n\delta}-1].$$

16. The foregoing articles deal with the *accumulation* of principal under the operation of compound interest. It is now necessary to consider the converse process of *discounting*. The general theory of compound discount may be developed on precisely the same lines as the theory of compound interest.

17. DISCOUNT may be defined as the consideration for the immediate payment of a sum due at a future date, and the total discount to be allowed for the present payment of a given sum due may be determined by reference to an *effective rate of discount* per annum, a *nominal rate of discount* per annum convertible with a given frequency, or a *force of discount*, the last-mentioned quantity being, in other words, a nominal rate of discount convertible at infinitely short intervals.

18. The sum due, less the total discount upon it, is called its PRESENT VALUE.

19. As there is an effective rate of interest corresponding to any given nominal rate of interest, so also there is an effective rate of discount corresponding to a given nominal rate of discount. For, if the nominal rate of discount be f per annum convertible m times a