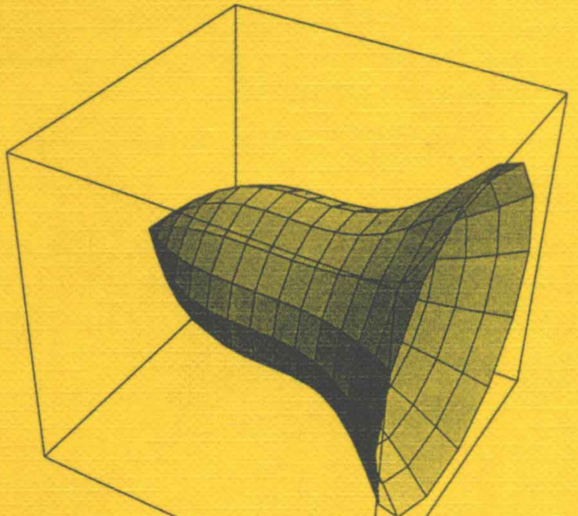
Lecture Notes in Mathematics 1517

Klaus Keimel Walter Roth

Ordered Cones and Approximation



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Introduction

Korovkin type approximation theorems typically deal with certain restricted classes of continuous linear operators on locally convex vector spaces. These may be positive operators on ordered vector spaces or contractions on normed spaces as in the seminal work by Korovkin [29], [30], Shashkin [50], [51], Wulbert [63], Bauer [9], and Behrens and Lorentz [12]. Less known situations include operators on Banach algebras [4], [39], [40] or on spaces of stochastic processes [60], [61] with certain restricting properties. More recently, there are results about order preserving linear operators on set-valued functions [59], [28], [13], [14] as well. In this case the domain of the operators under consideration is no longer a vector space but only a cone, i.e. subtraction of elements is not always defined (see also [44], [45]). Generally, if an approximation process is modelled by such a restricted class of operators, those very restrictions guarantee convergence towards the identity on a large subset of their domain if this property may be checked for a relatively small test set. Korovkin's classical theorem (see [29], [30]) states that a sequence of positive linear operators on $C[0,1]$ converges towards the identity for all functions in $C[0,1]$ if this holds for the three "test functions" $1, x$ and x^2. Unfortunately, the different situations mentioned above and the different restrictions on the classes of operators so far required different approaches and techniques.

Looking for a unified presentation of Korovkin type approximation theorems we had to leave the setting of vector spaces and turn to more general structures which we call locally convex cones. For our purposes, it is essential to include cones which are not embeddable in vector spaces. As we need to apply functional analytic concepts, in particular an appropriate duality theory and Hahn-Banach type extension and separation theorems, we have to stay reasonably close to the classical theory of locally convex vector spaces, yet allow sufficient generality in order to serve our main purpose: Various restrictions on classes of operators in Korovkin type approximation may be taken care of by the proper choice of domains and their topologies alone. Thus, we just have to investigate continuous operators between locally convex cones. The following is an outline of the main concepts and some (simplified) results of this work.

Preordered cones. A *cone* is a set $\boldsymbol{P}$ endowed with an addition $(a,b) \to a+b$ and a scalar multiplication $(\alpha, a) \to \alpha a$ for real numbers $\alpha \geq 0$. The addition is only supposed to be associative and commutative, and a neutral element 0 is required to exist (see Ch.I.1).

Of course, cones in real vector spaces are cones in the above sense. They have the *cancellation property*

$$a+c = b+c \ \text{ implies } \ a = b$$

which we do not require in general.

In addition we shall assume that $\boldsymbol{P}$ carries a *preorder*, i.e. a reflexive transitive relation "$\leq$" such that

$a \leq b$ implies $a+c \leq b+c$ and $\alpha a \leq \alpha b$ for all $a,b,c \in \boldsymbol{P}$ and all $\alpha \geq 0$.

Locally convex cone topologies. Our model for introducing a locally convex topology on $\boldsymbol{P}$ is the set $Conv(E)$ of all non-empty convex subsets of a locally convex topological vector space E. It has a natural addition and a scalar multiplication by non-negative reals; it is ordered by inclusion. An arbitrary base $\boldsymbol{V} \subset Conv(E)$ of convex neighborhoods of 0 in E induces three hyperspace topologies on $Conv(E)$ given by the respective neighborhood bases for $A \in Conv(E)$:

in the *upper topology* $V(A) = \{B \in Conv(E) \mid B \subset A+V\}$, $V \in \boldsymbol{V}$,

in the *lower topology* $(A)V = \{B \in Conv(E) \mid A \subset B+V\}$, $V \in \boldsymbol{V}$,

in the *symmetric topology* $V(A) \cap A(V)$, $V \in \boldsymbol{V}$.

Identifying elements of E with singleton sets, E is naturally embedded in $Conv(E)$, and all the three topologies on $Conv(E)$ coincide with the given locally convex topology on E.

For an abstract formulation we use order theoretical concepts to introduce locally convex topologies on cones (see I.2.2): A subset $\boldsymbol{V}$ of the cone $\boldsymbol{P}$ is called an *(abstract) 0-neighborhood system*, if the following properties hold:

$0 < v$ for all $v \in \boldsymbol{V}$;

for all $u,v \in \boldsymbol{V}$ there is $w \in \boldsymbol{V}$ with $w \leq u$ and $w \leq v$;

$u+v \in \boldsymbol{V}$ and $\alpha v \in \boldsymbol{V}$ whenever $u,v \in \boldsymbol{V}$ and $\alpha > 0$.

For every $a \in \boldsymbol{P}$ we define

$$v(a) = \{b \in \boldsymbol{P} \mid b \leq a+v\}$$

to be a neighborhood of a in the *upper topology*, and

$$(a)v = \{b \in \boldsymbol{P} \mid a \leq b+v\}$$

to be a neighborhood of a in the *lower topology*. The common refinement of these two topologies is called the *symmetric topology* on $\boldsymbol{P}$. We call $(\boldsymbol{P},\boldsymbol{V})$ a *full locally convex cone*. We also consider subcones $\boldsymbol{Q}$ of $\boldsymbol{P}$ not necessarily containing $\boldsymbol{V}$. They will be endowed with the topologies induced from $\boldsymbol{Q}$ and denoted as *locally convex cones* $(\boldsymbol{Q},\boldsymbol{V})$. They are the general subject of our study.

Cones are asymmetric structures, so asymmetric conditions come as no surprise: For technical reasons we require the elements of a locally convex cone to be *bounded below*, i.e. for every $a \in \boldsymbol{Q}$ and $v \in \boldsymbol{V}$ we have $0 \leq a+\rho v$ for some $\rho > 0$.

Note that the upper neighborhoods $v(a)$ are decreasing convex sets and the $(a)v$ are increasing convex sets. The neighborhoods in the symmetric topology are both convex and order convex. Thus, all of these three topologies merit to be called locally convex. Of course, the upper and the lower topology are far from being Hausdorff. Since all of the three topologies are defined in terms of the preorder on Q, continuity properties etc. will be expressible by means of the ordering and the 0-neighborhoods alone.

The global preorder (see I.3.9). On a locally convex cone (Q,V) we define the *global preorder* "$\preceq$" for $a,b \in Q$ by

$$a \preceq b \quad \text{if and only if} \quad a \leq b+v \quad \text{for all} \quad v \in V.$$

It is easy to check that $a \leq b$ for our original preorder always implies $a \preceq b$. Furthermore, mappings between locally convex cones which are continuous either with respect to their upper or their lower topologies need to be monotone with respect to their global preorders. This is the main reason for the usefulness of locally convex cones in Korovkin type approximation theory.

The following standard examples will be used throughout our text:

Examples. (a) Clearly every locally convex topological vector space E with 0-neighborhood base V is a locally convex cone (E,V) in this sense. (E is a subcone of the full locally convex cone $(Conv(E),V)$.)

(b) The cone $\overline{R} = R \cup \{+\infty\}$ may be endowed with the abstract neighborhood system $V = \{\varepsilon \in R \mid \varepsilon > 0\}$. For $a \in R$ the intervals $(-\infty, a+\varepsilon)$ are the upper and the intervals $(a-\varepsilon, +\infty)$ the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{R}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology on $\overline{R}$ is the usual topology on R with $\{+\infty\}$ as an isolated point.

(c) Let $(E,\leq)$ be a locally convex ordered topological vector space with 0-neighborhood base V. For $a,b \in E$, and $V \in V$ we define

$$a \leq b + V \quad \text{if there is some} \quad v \in V \quad \text{such that} \quad a \leq b+v.$$

Thus (E,V) is a locally convex cone and the symmetric topology on E coincides with the original one if the neighborhoods $V \in V$ are order convex.

(d) If (Q,V) is a locally convex cone then there is a canonical way to define a locally convex topology on the cone $Conv(Q)$ of non-empty convex subsets of Q:
For convex sets $A,B \in Conv(Q)$ and a neighborhood $v \in V$ we set

$$A \leq B + \overline{v} \quad \text{if and only if for all} \quad a \in A \quad \text{there is some} \quad b \in B \quad \text{such that} \quad a \leq b + v.$$

Thus $(Conv(Q),\overline{V})$ is a locally convex cone (we set $\overline{V} = \{\overline{v} \mid v \in V\}$). It does not satisfy the cancellation property.

Moreover, it may be shown that every locally convex cone satisfying a minor additional assumption (see Theorem II.2.20) admits a representation as a subcone of $(Conv(E),\overline{V})$, where (E,V) denotes a suitable locally convex topological vector space.

(e) If (Q,V) is a locally convex cone and X is a compact space by $C(X,Q)$ we denote the cone of Q-valued functions on X which are continuous with respect to the symmetric topology on Q. For $f,g \in C(X,Q)$ and $v \in V$ we set

$$f \leq g + \overline{v} \quad \text{if and only if} \quad f(x) \leq g(x) + v \quad \text{for all} \quad x \in X.$$

Thus, endowed with this topology of uniform convergence $(C(X,Q),\overline{V})$ is a locally convex cone as well.

(f) In the context of Korovkin type approximation theorems for linear contractions on normed spaces the following locally convex cone is of interest: Let $(E,\|\ \|)$ be a normed vector space with unit ball B. Let $Q = \{a+\rho B \mid a \in E,\ \rho \geq 0\}$ be provided with the canonical neighborhood basis $V = \{\rho B \mid \rho > 0\}$ and the set inclusion as preorder.

Uniformly continuous linear operators (see II.1.1 and II.1.2). For cones Q and P, a map $T : Q \to P$ is called a *linear operator*, if

$$T(a+b) = T(a)+T(b) \qquad \text{for all } a,b \in Q \text{ and}$$
$$T(\alpha a) = \alpha T(a) \qquad \text{for all } a \in Q \text{ and } \alpha \geq 0.$$

If (Q,V) and (P,W) are locally convex cones then the linear operator $T : Q \to P$ is called *uniformly continuous* or *u-continuous* for short, if for every $w \in W$ one can find a $v \in V$ such that

$$a \leq b+v \quad \text{implies} \quad T(a) \leq T(b)+w.$$

Uniform continuity is not just continuity. It is immediate from the definition that it implies and combines continuity with respect to the upper, lower and symmetric topologies on Q and P. Every u-continuous linear operator is monotone with respect to the global preorder. In Example (f), for example, if we extend a given linear operator T on the normed space E to a linear operator $\overline{T}$ on $Q = \{a+\rho B \mid a \in E,\ \rho \geq 0\}$ by setting $\overline{T}(B) = B$, then u-continuity is equivalent for T to be contractive.

The dual cone (see II.2). By the above a linear functional $\mu : Q \to \overline{R}$ is u-continuous if there is a neighborhood $v \in V$ such that

$$a \leq b+v \quad \text{implies} \quad \mu(a) \leq \mu(b)+1.$$

The u-continuous linear functionals on Q form again a cone, denoted by Q^* and called the *dual cone* of Q. We endow Q^* with the topology $w(Q^*,Q)$ of pointwise convergence of the elements of Q, considered as functions on Q^* with values in $\overline{R}$ with its usual topology. The *polar* v_Q° of a neighborhood $v \in V$ consists of all linear functionals fulfilling the above condition. It is seen (II.2.4) to be $w(Q^*,Q)$-compact and convex.

The following is derived using Hahn-Banach type theorems as in [22]:

Extension Theorem (II.2.9). *Let* P *be a subcone of the locally convex cone* (Q,V). *Then every u-continuous linear functional on* P *can be extended to a u-continuous linear functional on* Q; *more precisely: For every* $\mu \in v_P^\circ$ *there is a* $\tilde{\mu} \in v_Q^\circ$ *such that* $\mu = \tilde{\mu}|_P$.

Superharmonicity with respect to a subcone. Introducing the notation for an element of a locally convex cone to be superharmonic with respect to a given subcone, in Chapter III we turn to applications in approximation theory. This notion is well-known and useful in classical Korovkin theory. It serves the same purpose in our more general setting:

Let Q_0 be a subcone of the locally convex cone (Q,V). Let $\mu \in Q^*$ and $a \in Q$. We shall say that the element a is *Q_0-superharmonic in* μ (see III.1.1) if firstly $\mu(a)$ is finite and if secondly, for all $\nu \in Q^*$,

$$\nu(b) \leq \mu(b) \text{ for all } b \in Q_0 \text{ implies } \nu(a) \leq \mu(a).$$

In order to give an example how classical results may be transferred to our more general concept, recall the following well-known statement in the $C_0(X)$-case which is due to Bauer and Donner [11]:

Theorem. *Let X be a locally compact space and G a linear subspace of $C_0(X)$, the space of continuous real-valued functions on X vanishing at infinity. For a function $f \in C_0(X)$ the following conditions are equivalent:*

(i) For every net $(T_\alpha)_{\alpha \in A}$ of equicontinuous positive linear operators on $C_0(X)$,

$$T_\alpha(g) \to g \quad \text{for all } g \in G \quad \text{implies } T_\alpha(f) \to f.$$

(ii) For every $x \in X$ we have

$$f(x) = \sup_{\varepsilon > 0} \inf \{g(x) \mid g \in G,\ f \leq g + \varepsilon\} = \inf_{\varepsilon > 0} \sup \{g(x) \mid g \in G,\ g \leq f + \varepsilon\}.$$

(iii) For every $x \in X$ and every bounded positive Radon measure μ on X,

$$\mu(g) = g(x) \quad \text{for all } g \in G \quad \text{implies } \mu(f) = f(x).$$

Convergence in (i) is meant with respect to the topology of uniform convergence on $C_0(X)$.

Consider the locally convex cone $(C_0(X),V)$ where V consists of the strictly positive constant functions on X. The dual cone then is formed by the bounded positive Radon measures on X. Clearly, condition (iii) in the preceding theorem means that both f and $-f$ are G-superharmonic in all point evaluations of X. Thus, the equivalence of (ii) and (iii) is a consequence of our

Sup-Inf-Theorem (III.1.3). *Let Q_0 be a subcone of the locally convex cone (Q,V). Let $a \in Q$ and $\mu \in Q^*$ such that $\mu(a)$ is finite. Then a is Q_0-superharmonic in μ if and only if*

$$\mu(a) = \sup_{v \in V} \inf\{\mu(b) \mid b \in Q_0,\ a \leq b + v\}.$$

Like its classical counterpart this theorem may be used to derive Stone-Weierstraß type theorems for locally convex cones (see III.3.6 and III.3.7).

In Chapter IV we turn to Korovkin type approximation theory in locally convex cones and introduce techniques involving adjoint operators (see II.2.15) on the dual cones. We use the above notation of superharmonicity in order to derive our General Convergence Theorem IV.1.13. For a net $(a_\alpha)_{\alpha \in A}$ in a locally convex cone (Q,V) and an element $a \in Q$ we shall denote by $a_\alpha \uparrow a$ the convergence of (a_α) towards a with respect to the upper topology (IV.1.7). We state a simplified version of this theorem which is however sufficient to derive the classical results including the above by Bauer and Donner (c.f. [4], [11], [19], [20], [49], [61], etc.):

Convergence Theorem (IV.1.11). *Let Q_0 be a subcone of the locally convex cone (Q,V). Suppose that for all $v \in V$ the element $a \in Q$ is Q_0-superharmonic in all elements of the $w(Q^*,Q)$-closure of the extreme points of v_Q°. Then for every net $(T_\alpha)_{\alpha \in A}$ of equicontinuous linear operators on Q*

$$T_\alpha(b) \uparrow b \quad \text{for all } b \in Q_0 \text{ implies } T_\alpha(a) \uparrow a.$$

Our text is not meant to be a complete survey of Korovkin type approximation theory. We do not repeat the classical cases in detail. They have been dealt with in many places and we globally refer to Donner's book [19] on the subject and to the very useful bibliographies and summaries by Altomare/Campiti [7] and Pannenberg [41]. We shall, however, in Chapter IV.2 give a few examples that may indicate how to apply our general results to various classical situations. Chapters V and VI are devoted to further applications of our General Convergence Theorem, most of which are new. We consider cone-valued functions which generate locally convex cones in various ways. In particular, we generalize the concept of Nachbin spaces (see V.1) which is described in [43] and which formalizes the concept of weighted approximation. We thus extend Korovkin type theorems to problems of weighted approximation far beyond those by Gadzhiev in [23], [24]. To give an example of our new results we formulate a Korovkin type theorem for set-valued functions (V.3.2 and V.3.4). For finite dimensional vector spaces, $X = [0,1]$ and $M = \{1, x, x^2\}$ it is due to Vitale [59]. A more general version for the finite dimensional case is contained in [28]:

For a locally convex vector space E with neighborhood base V we denote by $CConv(E)$ the locally convex cone of all non empty compact convex subsets of E (c.f. Example (d) from above). Finally, for a compact space X, $C(X,CConv(E))$ denotes the locally convex cone of all continuous (with respect to the symmetric topology) $CConv(E)$-valued functions. It is ordered by (pointwise) inclusion. We consider convergence with respect to the symmetric topology on $C(X,CConv(E))$ (c.f. 4(e)); i.e. the net $(f_\alpha)_{\alpha \in A}$ of functions in $C(X,CConv(E))$ converges to $f \in C(X,CConv(E))$ if and only if for all $V \in V$ there is some $\alpha_0 \in A$ such that

$$f_\alpha(x) \subset f(x)+V \text{ and } f(x) \subset f_\alpha(x)+V \text{ for all } x \in X \text{ and } \alpha \geq \alpha_0.$$

Recall that a *Korovkin system for* $C(X)$ is a subset M of $C(X)$ such that, for the linear subspace G spanned by M, every function in $C(X)$ fulfills the equivalent conditions of the Theorem quoted on page 5.

In a similar way we say that a subset $\overline{M}$ of $C(X,CConv(E))$ is a *Korovkin system for* $C(X,CConv(E))$ if for every $f \in C(X,CConv(E))$ and every net $(T_\alpha)_{\alpha \in A}$ of equicontinuous monotone linear operators on $C(X,CConv(E))$

$$T_\alpha(g) \to g \quad \text{for all } g \in \overline{M}, \quad \text{implies } T_\alpha(f) \to f.$$

Theorem. *Let E be a locally convex vector space, X a compact space and M a Korovkin system for $C(X)$ consisting of positive functions. Let $\boldsymbol{U}$ be a subset of* $\boldsymbol{CConv}(E)$ *such that $0 \in U$ for all $U \in \boldsymbol{U}$ and $\bigcup\{\lambda U \mid U \in \boldsymbol{U},\ \lambda \geq 0\} = E$. Then the set-valued functions*

$$x \to g(x)U \ :\ X \to \boldsymbol{Q}, \quad g \in M,\ U \in \boldsymbol{U}$$

together with the constant functions

$$x \to C \ :\ X \to \boldsymbol{Q}, \quad C \in \boldsymbol{CConv}(E)$$

form a Korovkin system for $C(X, \boldsymbol{CConv}(E))$.

Chapter VI investigates quantitative approximation theory using the full strength of our General Convergence Theorem. We derive results on the order of convergence for Korovkin type approximation processes on cone-valued functions. These situations include real- and set-valued functions (see VI.4.3, VI.4.5 and VI.4.6) and stochastic processes (VI.4.10). Most of our results are either new or considerable generalizations of known ones.

Chapter I: Locally Convex Cones

In the first three sections of this chapter we present the basic definitions of ordered and locally convex cones and their associated topologies. In Section 4 we discuss the question of embeddability in locally convex vector spaces. Roughly speaking, we show that the subcone B_Q of *bounded elements* (see 2.3) of a locally convex cone Q is embeddable in a canonical way into a locally convex ordered vector space. In particular, if a locally convex cone happens to be a vector space, then, endowed with its *symmetric topology* (see 2.2), it is a locally convex vector space. Thus, our notion of local convexity reduces to the usual notion of local convexity in the case of vector spaces.

In Section 5 we present an alternative approach for locally convex cones through quasi-uniform structures. Indeed, we believe that this approach is the most appropriate and natural one; its disadvantage is that (quasi)uniform structures are less appealing to our intuition. Quasi-uniform structures had been introduced under the name of "semiuniform structures" by L. Nachbin [34] as a common generalization of order and uniform structures. In vector spaces one has a canonical uniform structure induced by any vector space topology, and we believe that this uniformity is essential for all the basic facts of functional analysis. Thus, for ordered cones, the appropriate locally convex structure should be defined in terms of quasiuniform structures which carry the information both about order and topology.

1. Cones and preordered cones.

1.1 Cones. We define a *cone* to be a set P endowed with an addition $(a,b) \to a+b$ and a scalar multiplication $(\alpha, a) \to \alpha a$ for real numbers $\alpha > 0$. The addition is only supposed to be associative and commutative and a neutral element 0_P (shortly 0) is required to exist, i.e.:

$$(a+b)+c = a+(b+c) \qquad \text{for all } a,b,c \in P,$$
$$a+b = b+a \qquad \text{for all } a,b \in P,$$
$$0+a = a \qquad \text{for all } a \in P.$$

For the scalar multiplication we require as usual:

$$\alpha(\beta a) = (\alpha\beta)a \qquad \text{for all } \alpha,\beta > 0 \text{ and } a \in P,$$
$$(\alpha+\beta)a = \alpha a+\beta a \qquad \text{for all } \alpha,\beta > 0 \text{ and } a \in P,$$
$$\alpha(a+b) = \alpha a+\alpha b \qquad \text{for all } \alpha > 0 \text{ and } a,b \in P,$$
$$1 \cdot a = a \qquad \text{for all } a \in P.$$

In this definition of a cone P, the scalar multiplication is only required to be defined for real numbers $\alpha > 0$. We may - and we shall do this in the sequel - extend the scalar multiplication to $\alpha = 0$ by defining $0 \cdot a = 0$ for all $a \in P$, and all of the above rules remain valid. On the other hand, $\alpha \cdot 0 = 0$ for all $\alpha > 0$ is a consequence of these rules. Indeed, for all $a \in P$ we have

$$a = \alpha\,(\alpha^{-1} a + 0) = a + \alpha 0,$$

whence $\alpha \cdot 0 = 0$ by the unicity of the neutral element.

1.2 Subcones and faces. A subset Q of a cone P is called a *subcone* if

$$a+b \in Q \quad \text{and} \quad \alpha a \in Q \qquad \text{for all } a,b \in Q \text{ and } \alpha \geq 0.$$

Note that every subcone of P contains 0. A *face* F is a subset of P such that

$$a+b \in F \quad \text{implies} \quad a,b \in F \qquad \text{for all } a,b \in P.$$

Of course, cones in real vector spaces are cones in the above sense. They have the *cancellation property*

(C) $\qquad a+c = b+c \quad$ implies $\quad a = b$

for arbitrary elements a,b,c. Conversely, cones which satisfy the cancellation property are embeddable in real vector spaces. It is important to note that cones in our sense are in general far from being embeddable in vector spaces, as the addition is not supposed to be cancellative. This is essential, as we want to include examples like the following:

1.3 Example. With its straightforward addition and multiplication with $\alpha \geq 0$, the set $\overline{R} = R \cup \{+\infty\}$ is a cone.

1.4 Example: Cones of convex sets. Let P be a cone. A subset A of P is called *convex*, if $\alpha a + (1-\alpha) b \in A$, whenever $a,b \in A$ and $0 \leq \alpha \leq 1$.

We denote by $Conv(P)$ the set of all non-empty convex subsets of P. With the addition and scalar multiplication defined as usual by

$$A+B = \{a+b \mid a \in A \text{ and } b \in B\} \quad \text{for } A,B \in Conv(P),$$

$$\alpha A = \{\alpha a \mid a \in A\} \quad \text{for } A \in Conv(P) \text{ and } \alpha \geq 0,$$

it is easily verified that $Conv(P)$ is again a cone. Convexity is required to show that $(\alpha+\beta)A$ equals $\alpha A + \beta A$: Clearly $(\alpha+\beta)A$ is a subset of $\alpha A + \beta A$. To show the converse, consider an arbitrary element $c \in \alpha A + \beta A$; it can be written $c = \alpha a + \beta b$ with $a, b \in A$; as

$$c = (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta} a + \frac{\beta}{\alpha+\beta} b\right); \quad \text{(the case } \alpha = \beta = 0 \text{ is trivial.)}$$

and as

$$\frac{\alpha}{\alpha+\beta} a + \frac{\beta}{\alpha+\beta} b \in A \quad \text{by the convexity of } A,$$

we conclude that $c \in (\alpha+\beta)A$.

Note that every subcone Q of P is convex and satisfies $Q+Q=Q$. In particular, the non-empty convex subsets of a real vector space form a cone in our sense which is far from being cancellative.

1.5 Example: Cones of cone-valued functions. Let P be a cone, X any a set. For P-valued functions on X the addition and scalar multiplication may be defined pointwise. The set $F(X,P)$ of all such functions then is a cone in our sense. But again, the addition is in general not cancellative, as it is not in P.

1.6 Preordered cones. *A preordered cone* is a cone P with a reflexive transitive relation $\leq$ such that

$$a \leq b \text{ implies } a+c \leq b+c \text{ and } \alpha a \leq \alpha b \text{ for all } a,b,c \in P \text{ and all } \alpha \geq 0.$$

If $\leq$ is in addition antisymmetric, i.e. $\leq$ is a partial ordering, then P is called an *ordered cone.*

Examples of preordered cones are $\overline{R}$ with its usual order (Ex.1.3), the set $Conv(P)$ of non-empty convex subsets of a cone P ordered by inclusion (Ex 1.4) and if P is a preordered cone, the set of P-valued functions on a given set X endowed with the pointwise ordering (Ex. 1.5). Every cone P is preordered by its *natural preorder* defined by $a \leq_n b$ if $a+c=b$ for some $c \in P$.

Convex sets in cones may look rather peculiar. For example in $\overline{R}$ all the two element sets $\{a,+\infty\}$ are convex. This phenomenon is somehow remedied by considering only increasing or decreasing sets, or more generally convex sets that are also order convex:

1.7 Example: Cones of decreasing convex sets. A subset a of a preordered cone is called *decreasing*, if $a \in A$ and $b \leq a$ for some $b \in P$ imply $b \in A$. For a subset B of P we denote by:

$$\downarrow B = \{a \in P \mid a \leq b \text{ for some } b \in B\},$$

the decreasing subset generated by B. In a dual way one defines the notion of an *increasing* subset and $\uparrow B$, the increasing subset generated by B. It is easily verified, that $\downarrow B$ and $\uparrow B$ both are convex, whenever B is convex. We denote by $DConv(P)$ the set of all non-empty decreasing convex subsets of P.

For a decreasing convex set A and $\alpha > 0$, the set αA is also decreasing and convex. But $A+B$ need not be decreasing, if A and B are. We therefore modify the addition on $DConv(P)$ and define

$$A \oplus B = \downarrow(A+B) = \{c \in P \mid c \leq a+b \text{ for some } a \in A,\ b \in B\}.$$

With this addition and the usual scalar multiplication $DConv(P)$ becomes a cone ordered by inclusion; the set $\{0\}$ acts as the additive zero element. There is a natural map

$$a \to \downarrow\{a\} \text{ of } P \text{ into } DConv(P),$$

which is order preserving. It is an embedding, i.e. injective, if and only if the preorder on P is