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ABSTRACT

We present a modern approach to the classical problem of Plateau based purely on differential geometric concepts. We not only reprove the classical results of Douglas but also develop a new geometric criterion on a given finite system of disjoint Jordan curves in three-dimensional Euclidean space which guarantees the existence of a minimal surface of a prescribed genus having these curves as boundary.

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INTRODUCTION

In 1931 Jesse Douglas and, simultaneously, Tibor Rado solved the famous problem of Plateau, namely that every Jordan wire in $\mathbb{R}^n$ bounds at least one disc type surface of least area [4,28]. For his work Douglas received the first Fields medal in 1936. By this time he had shown that his methods would allow one to prove that there exist minimal surfaces of genus zero and connectivity k spanning k Jordan curves $\Gamma_1,\ldots,\Gamma_k$ in $\mathbb{R}^n$ provided that one such surface exists having strictly less area than the infimum of the areas of all disconnected genus zero surfaces spanning $\Gamma_1,\ldots,\Gamma_k$ [6]. Somewhat later he announced and published proofs of theorems giving similar sufficient conditions which guarantee the existence of a minimal surface of arbitrary genus spanning one or more wires in Euclidean space [7,8,9]. The method of Douglas, being of great historical significance deserves some description and we shall begin with an analytical formulation of the problem.

Let Γ be a Jordan curve in $\mathbb{R}^n$ and $B \subset \mathbb{R}^2$ be the closed unit disc. The classical problem of Plateau asks that we minimize the area integral

$$A(u) = \int \sqrt{EG - F^2}\, dxdy$$

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among all differentiable mappings $u : \mathcal{B} \to \mathbb{R}^n$ such that

(1) $u : \partial\mathcal{B} \to \Gamma$ is a homeomorphism.

Here we have used the traditional abbreviations

$$E = \sum_{k=1}^{n} \left(\frac{\partial u^k}{\partial x}\right)^2, \quad G = \sum_{k=1}^{n} \left(\frac{\partial u^k}{\partial y}\right)^2, \quad F = \sum_{k=1}^{n} \frac{\partial u^k}{\partial x}\frac{\partial u^k}{\partial y} .$$

The Euler equations of this variational problem form a system of non-linear partial differential equations expressing the condition that the surface u have mean curvature zero, i.e. it is a minimal surface. One may, however, try to take advantage of the fact that the area integral is invariant under the diffeomorphism group of the disc and to transform these equations into a particularly simple form by using coordinate representations. Following Riemann, Weierstraß, H.A.Schwarz, and Darboux one introduces isothermal coordinates

(2) $E = G \ , \ F = 0$

which in fact linearize the Euler equations of least area, namely they reduce to Laplace's equation

(3) $\Delta u = 0.$

One is thus led to the definition of a classical disc type minimal surface as a map $u : \mathcal{B} \to \mathbb{R}^n$ which fulfills conditions (2) and (3).

For unknotted curves Garnier [15] was able to prove the existence of solutions of (2) and (3) subject to the boundary condition (1) by function theoretic methods. The general case evaded researches until the work of Douglas and

Rado. They both use the direct method of the calculus of the variations and thus obtain an area minimizing solution while Garnier's solution might be unstable. In applying the direct method one now replaces the complicated area functional by the simpler Dirichlet integral D where

$$D(u) = \frac{1}{2}\int (E+G)\,dxdy \ .$$

It is important to note that

$$A(u) \leqq \int \sqrt{EG}\,dxdy \leqq \frac{1}{2}\int (E+G)\,dxdy = D(u) \ .$$

and equality holds if and only if $E = G$, $F = 0$. This and the analogy with the length and energy functionals of geodesics [24] make it plausible that minima of D should be minima of A. This is, in fact, the case. In his prize winning paper Douglas, however, did not explicitly attempt to find a minimum for Dirichlet's integral but for another functional H which is now called the Douglas functional. Using Poisson's integral formula for harmonic functions Douglas obtained the expression

$$H(u) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{(u(\cos\alpha,\sin\alpha) - u(\cos\beta,\sin\beta))^2}{4\sin^2\frac{1}{2}(\alpha-\beta)}\,d\alpha d\beta$$

which equals $D(u)$ if u is harmonic. The replacement of $D(u)$ by $H(u)$ transforms a variational problem involving derivatives to one that does not, an important feature of Douglas' existence proof. In the case of two contours Γ_1 and Γ_2 in $\mathbb{R}^n$, where the domain of our mappings u is an annulus, the functional $H(u)$ is similar. However, in the general case of surfaces of connectivity k and genus

$p > 0$, one is forced to take as parameter domain a Riemann surface of genus p bounded by k circles; the construction of $H(u)$ becomes not only less elementary, but from the point of view of these authors incredibly complicated. Douglas was able to accomplish this generalization by making essential use of the theory of Abelian functions on Riemann surfaces, the theory of theta functions defined on their Jacobi varieties and their dependence on the moduli of the underlying Riemann surfaces. Namely, in order to obtain minimal surfaces through the minimization of D or H it is necessary to minimize over all conformal classes of Riemann surfaces. This was carried out by Douglas at a point in mathematical history[1)] when the structure of such conformal classes was not understood. That Douglas's work was a tour de force of classical function theory is an understatement.

According to C.Reid's book "Courant" [29] Douglas gave a lecture at N.Y.U. in 1936 which stimulated Courant's interest in Plateau's problem and its generalizations to higher topological structure. In the same year he gave a new proof of Douglas's original 1931 result using Dirichlet's integral instead of the Douglas functional.Courant felt that his approach was simpler and more general than that of Douglas. There is no question that for the genus zero case he was correct in this assertion. He was also correct in pointing out that the use of Dirichlet's inte-

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[1)] It is interesting to note that Teichmüller's pioneering work [32] was appearing at about the same time.

gral permitted one to attack other boundary value problems for minimal surfaces, a subject which has received a great deal of current interest, cf. [19,27]. However, Courant, in these authors' opinion remains vague on the higher genus case. A typical comment can be found in his paper [2] where he states (p.78) "Higher topological structure does not affect our reasoning". Courant's method of attack for higher genus was later worked out by Shiffman [31]. In both the case of Shiffman and Courant, their approach uses the theory of conformal mappings in order to represent conformal classes of Riemann surfaces as slit domains in the upper half plane. Variation of the conformal structure could be achieved by varying the position of the slits. An approach similar in spirit though different in detail was recently undertaken by Jost [20]. He uses fundamental polygons in the Poincaré upper half plane as normal domains. Douglas, on the other hand did not need to refer to any theorems in conformal mapping, a point which Courant often, justifiably, mentions in his papers. The crux of all approaches mentioned so far, in the opinion of the authors, lies in the fact that the dependence of the variational functional from the underlying conformal structure is not at all transparent, but only very implicit. This will not be the case in the approach suggested in the present paper.

The purpose of our paper is twofold. First of all we want to present a modern approach to the classical Plateau problem in which all basic concepts and methods are of purely differential geometric nature. The whole theory is embedded in the framework of global analysis; minimal surfaces appear as the critical points of a differentiable functional on a differentiable manifold, derivatives can be computed explicitly. This will allow us to apply directly methods and results from nonlinear functional analysis like Sard's theorem, bifurcation theory, degree theory, etc. This will be carried out in forthcoming papers.

As a second goal of our paper we want to give a sufficient geometric-topological criterion on a system of Jordan curves in $\mathbb{R}^3$ guaranteeing the existence of a minimal surface of prescribed topological type spanning these curves. Part of this result was announced in [34]. This criterion (Theorems 5.1 , 5.2) is of a completely different nature than Douglas's.

Finally we wish to make some remarks on the relation of the classical theory of Plateau's problem to the geometric measure theory. This theory was mainly designed to attack the higher dimensional form of Plateau's problem, a realm inaccessible to the classical theory. But, admittedly also in the classical case of two-dimensional furfaces in $\mathbb{R}^3$ the geometric measure theory approach yields beautiful results which could not easily - if at all - be obtained within the classical theory, like the following one: any sufficiently smooth Jordan curve in $\mathbb{R}^3$ spans a differentiably embedded (up to the boundary) minimal surface of some (unknown) topological type [16]. Geometric measure theory in the opinion of the authors is, however, not so well suited for questions where one is interested in surfaces of a prescribed topological type. We are therefore convinced that the classical theory continues to hold its place within the general theories of minimal surfaces.

The functional (Dirichlet's) whose critical points are minimal surfaces of a given genus is a function of two variables. One variable is a mapping $u : M \to \mathbb{R}^N$ from a Riemann surface M with boundary into Euclidian space and the second is an equivalence class of conformal structures on M. This class of conformal structures we take to be Teichmüller's moduli space, and we begin our first section with a discussion of its construction.

§1. ON TEICHMÜLLER THEORY FOR ORIENTED SURFACES

In this section we follow the approach to Teichmüller theory as developed in [12,13,14]. We suppose that M is a compact oriented surface without boundary of genus greater than one. It is well known that there is a collection of coordinate charts $\{U_i,\varphi_i\}_{i\in I}$ covering $M = \cup U_i$, with orientation preserving coordinate mappings $\{\varphi_i\}$, $\varphi_i : U_i \to \mathbb{C}$, so that when defined $\varphi_i \circ \varphi_j^{-1}$ is holomorphic. Such a collection of $\{U_i,\varphi_i\}$ is called a complex structure which we denote by c. When we think of M having a fixed complex structure c we shall designate this by writing the pair (M,c). However, a given M may have many complex structures. For example, let $f : M \circlearrowleft$ be a C^∞ diffeomorphism. Then we can construct a new complex structure f^*c, the pull back of c by f, considering the coordinate pairs $\{f^{-1}(U_i),\varphi_i \circ f\}$. Then $(\varphi_i \circ f)\circ(\varphi_j \circ f)^{-1}$ are also holomorphic and hence f^*c is indeed a complex structure, with $f : (M,f^*c) \to (M,c)$ a holomorphic mapping. Riemann wanted to identify (M,c) with (M,f^*c) and thus (under this relation) to consider equivalence classes of complex structures on a fixed M.

To be more precise let $\mathcal{C} = \mathcal{C}(M)$ be the space of all complex structures on M and $\mathcal{D} = \mathcal{D}(M)$ the space of all C^∞ diffeomorphisms of M to itself. Then Riemann was interested in the space of equivalence classes $\mathcal{C}/\mathcal{D} = \mathcal{R}(M)$, as described above. The space $R(M)$ is the Riemann space of moduli of M.

Riemann conjectured that $R(M)$ is a 6(genus M) - 6 dimensional space in the case genus M is greater than one. The structure of the space C/D is not well understood today. In 1939 Oswald Teichmüller in a series of brilliant papers [32] broke up the problem into two parts as follows. Let $D_0 \subset D$ be those diffeomorphisms which are isotopic to the identity (i.e. homotopic through diffeomorphisms). Define the Teichmüller space $T = T(M)$ to be the quotient space C/D_0. The modular group Γ of the surface M defined by $\Gamma = D/D_0$ is well known to be a discrete group. Then $R(M) = T/\Gamma$. We can attempt to attain an understanding of $R(M)$ by first understanding $T(M)$ and then the action of Γ on $T(M)$. It is this second question whose answer is not yet at hand. Teichmüller was able to put a topology (in fact a metric) on $T(M)$ and to show that $T(M)$ is homeomorphic to Euclidian space $\mathbb{R}^{6p-6}$, $p = \text{genus}(M)$.

In the next paragraphs we shall not employ any of the ideas of Teichmüller. We shall rather give an outline of a description of Teichmüller space based on the work [12].

The space C is a bit difficult to "get one's hand on" or much less to understand in a very concrete way. We shall come to understand this space and the action of D on it through a somewhat circuitous route.

<u>Definition 1.1</u> An almost complex structure J on M is

a C^∞ section of the (1,1) tensor bundle $T_1^1(M)$ over M, such that $J^2 = -I$, I the identity map.

More colloquially, for each $x \in M$, $J(x)$ is a linear map of the tangent space T_xM into itself such that $J^2(x) = -I(x)$, $I(x) : T_xM \circlearrowleft$ the identity and with $x \to J(x)$ C^∞ smooth. We say that J is <u>orientation preserving</u> if for each x, and for $X_x \in T_xM$ non-zero, the pair $(X_x, J(x)X_x)$ is a positively oriented basis for T_xM.

We shall denote by $\mathcal{A}$ the space of all almost complex structures. It is not difficult to see that given a $c \in \mathcal{C}$ we can associate a unique $J \in \mathcal{A}$. We do this as follows. Let $V_i = \varphi_i(U_i)$, (φ_i, U_i) a coordinate chart for c. Define $J_i(x) : T_xM \circlearrowleft$ by

$$J_i(x) = d\varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\varphi_i^{-1}$$

where $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \mathbb{R}^2 \circlearrowleft$ is the canonical almost complex structure on $\mathbb{R}^2$.

$$J_i(x) = J_j(x) \qquad \text{if}$$

$$d\varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\varphi_i^{-1} = d\varphi_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\varphi_j^{-1}$$

or if

$$d\psi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\psi^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where ψ is a complex analytic mapping. Write $\psi = x + iy$, $x = x(u,v)$, $y = y(u,v)$. Then $d\psi$ is represented by the matrix $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ which by the Cauchy Riemann equations is equal to $\begin{pmatrix} \frac{\partial x}{\partial u} & -\frac{\partial y}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial x}{\partial u} \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Also $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Thus $\frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and consequently $J_i(x) = J_j(x)$. What this computation shows is simply that each $c \in \mathcal{C}$ induces a $J \in \mathcal{A}$ by defining $J(x) = J_j(x)$ for any coordinate chart $(u_j, \varphi_j) \in c$.

In two dimensions, as we shall see, the converse is also true, but is a much deeper fact. That is, given a $J \in \mathcal{A}$ there is a unique $c \in \mathcal{C}$ which induces J in the above described manner.

The diffeomorphism group $\mathcal{D}$ acts on $\mathcal{A}$ in a natural way. For $f \in \mathcal{D}$ define $(f^*J)(x) = df^{-1}J_{f(x)}df(x)$. It is clear that $f^*J \in \mathcal{A}$ if $J \in \mathcal{A}$ and one easily sees that in the correspondence $c \leadsto J$ that $f^*c \leadsto f^*J$. Thus the map which sends $c \leadsto J$ is $\mathcal{D}$-equivariant, a very important fact.

We shall now introduce another space of objects into the picture, the space of all C^∞ Riemannian metrics $\mathcal{M}$ on M.

Definition 1.2 The space $\mathcal{M}$ of C^∞ Riemannian metrics on M is the space of positive C^∞ sections of the (0,2) tensor bundle $T^2_0(M)$.

Again, more colloquially if $g \in \mathcal{M}$, then for each $x \in M$ $g(x)$ is a positive definite symmetric bilinear form on T_xM, $g(x) : T_xM \times T_xM \to \mathbb{R}$ so that $x \to g(x)$ is C^∞. It is easy to see that $\mathcal{M}$ is a Frechêt manifold since $\mathcal{M}$ is open in the linear space S_2 of all symmetric two tensors on M (S_2 is defined in the same way as $\mathcal{M}$ except that positive definiteness is not required).

Let $\mathcal{P}$ be the space of all positive C^∞ functions on M and let $\mathcal{M}/\mathcal{P}$ denote the quotient space; that is we identify two metrics g_1 and g_2 if $g_1(x) = p(x)g_2(x)$ for all x, where p is a strictly positive C^∞ function on M. The following theorem is taken from [12].

<u>Theorem 1.1</u> There is a bijective equivalence (in fact, diffeomorphism) between $\mathcal{A}$ and $\mathcal{M}/\mathcal{P}$.

<u>Proof.</u> Given $g \in \mathcal{M}$ there is a standard way to construct a new, unique, non-degenerate alternating anti symmetric bilinear form $\mu_g(x) : T_xM \times T_xM \to \mathbb{R}$ such that if X_x, Y_x is an oriented basis for T_xM $\mu_g(x)(X_x, Y_x) > 0$. μ_g is called the volume element determined by g and the orientation of M.

Since g is also non-degenerate we can, for each $x \in M$, transform $\mu_g(x)$ into a linear map $\Phi(g)(x) : T_xM \circlearrowleft$ via the rule

$$g(x)(\Phi(g)(x)X_x, Y_x) = -\mu_g(x)(X_x, Y_x).$$

Let $J(x) = \Phi(g)(x)$. One then checks that $J^2(x) = -I(x)$ and that J is an almost complex structure on M. Define the map $\Phi : \mathcal{M} \to \mathcal{A}$ by $g \mapsto \Phi(g)$. Φ is not bijective, since one can easily see that $\Phi(p \cdot g) = \Phi(g)$ for $p \in \mathcal{P}$. However, one can check that Φ passes to a bijective map from $\mathcal{M}/\mathcal{P}$ to $\mathcal{A}$. □