Lecture Notes in Mathematics
A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

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N. P. Bhatia · G. P. Szegö

Dynamical Systems: Stability Theory and Applications

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PREFACE

This book began as a series of lecture notes of the course given by N.P. Bhatia at the Western Reserve University during the Spring of 1965 and the lecture notes of the courses given by G.P. Szegö at the University of Milan during the year 1964 - 65 and at Case Institute of Technology during the summer of 1965. These courses were meant for different audiences, on one side graduate students in mathematics, and on the other graduate students in systems theory and physics.

However in the process of developing these notes we have found a number of other results of interest which we decided to include (See 1.9, 2.7, 2.8, 2.11, 2.14, 3.3, 3.4, 3.5, 3.7, 3.8, 3.9). Therefore, this monograph is of a dual nature involving both a systematic compilation of known results in dynamical systems and differential equations and a presentation of new Theorems and points of view. As a result, a certain lack of organizational unity and overlapping are evident.

The reader should consider this monograph not as a polished, finished product, but rather as a complete survey of the present state of the art including many new open areas and new problems. Thus, we feel that these notes fit the special aims of this Springer-Verlag series. We do hope that this monograph will be appropriate for a one year graduate course in Dynamical Systems.

This monograph is still devoted to a mixed audience so we have tried to make the presentation of Chapter I (Dynamical Systems in Euclidean Space) as simple as possible, using the most simple mathematical techniques and proving in detail all statements, even those which may be obvious to more mature readers. Chapter 2 (Dynamical Systems in Metric Spaces) is more advanced. Chapter 3 has a mixed composition : Sections 3.1, 3.2, 3.6, 3.7 and 3.8 are quite elementary, while the remaining part of the chapter is advanced. In this latter part we mention many problems which are still in an early developmental stage. A sizeable number of the results contained in this monograph have never been published in book form before.

We would like to thak Prof. Walter Leighton of Western Reserve University, Prof. Mihailo Mesarović of Case Institute of Technology, and Prof. Monroe Martin, Director of Institute for Fluid Dynamics and Applied Mathematics of the University of Maryland, under whose sponsorship the authors had the chance of writing this monograph. We wish to thank several students at our universities, in particular, A. Cellina, P. Fallone, C. Sutti and G. Kramerich for checking parts of the manuscript. We are also indebted to Prof. A. Strauss and Prof. O. Hajek for many helpful suggestions and inspiring discussions and to Prof. J. Yorke for allowing to present his new results in Sec. 3.4. We wish also to express our appreciation to Mrs. Carol Smith of TECH - TYPE Corp., who typed most of the manuscript.

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The authors

March 1967

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CHAPTER 0

Notation , Terminology and Preliminary Lemmas

0.1 Notation

T: topological space

X: metric space with metric ρ

E: real euclidean space of n-dimensions

E^2: the real euclidean plane

G: group

R: set of real numbers.

R^+: non-negative real numbers

R^-: non-positive real numbers

I: set of integers

I^+: set of non-negative integers

I^-: set of non-positive integers

In the sequel, when not otherwise stated, capital letters will denote matrices and sets, small latin letters vector (notable exceptions t,s,k,v and w which have been used to denote real numbers), small greek letters real numbers (notable exception π, which denotes a mapping).

If $x = (x_1, \ldots, x_n) \in E$, $||x||$ will denote the euclidean norm of x i.e.,

0.1.1 $$||x|| = (\Sigma_1^n \, x_i^2)^{1/2}$$

while

0.1.2 $$|x| = \max (|x_i|, \ i = 1,\ldots,n)$$

Given two points $x,y \in E$ $\rho(x,y)$ will denote the euclidean distance between x and y, i.e., 0.1.3 $$\rho(x,y) = [\sum_{1}^{n}{}_{i} (x_i - y_i)^2]^{1/2}$$

If M is a non-empty subset of X, $x \in X$, and $\alpha > 0$, then we write

0.1.4 $$\rho(x,M) = \inf\{\rho(x,y) : y \in M\},$$

0.1.5 $$S(M,\alpha) = \{x \in X: \rho(x,M) < \alpha\},$$

0.1.6 $$S[M,\alpha] = \{x \in X: \rho(x,M) \leq \alpha\},$$

0.1.7 $$H(M,\alpha) = \{x \in X: \rho(x,M) = \alpha\}.$$

$S(M,\alpha)$, $S[M,\alpha]$, and $H(M,\alpha)$ will sometimes be referred to as the open sphere, the closed sphere, and the spherical hypersurface (of radius α about M).

The closure, boundary, complement, and interior of any set $M \subset X$ is denoted respectively by $\bar{M}$, ∂M, $\mathcal{C}(M)$, and $\mathcal{I}(M)$.

If $\{x_n\}$ is any sequence such that $\lim_{n \to \infty} x_n = x$, then this fact is simply denoted by $x_n \to x$.

We shall frequently be concerned with transformations Q from X to 2^X (the set of all subsets of X). Given $Q:X \to 2^X$, and $M \subset X$, we write

0.1.8 $$Q(M) = \bigcup\{Q(x) : x \in M\}.$$

where

0.1.9 $$\bigcup\{Q(x) : x \in M\} = \bigcup_{x \in M} \{Q(x)\}$$

If $\{Q_i\}$, $i \in I$, is a family of transformations from X to 2^X with I as an index set, then

0.1.10 $$Q = \bigcup\{Q_i : i \in I\}$$

denotes the transformation from X to 2^X defined by

0.1.11 $$Q(x) = \bigcup\{Q_i(x) : i \in I\}.$$

Given two sets $M, N \subset X$, their difference is denoted by $M \setminus N$. Given two maps π_1 and π_2 with $\pi_1 \circ \pi_2$ we will denote the composition map.
Sometimes we will use the logic symbols $\exists$, $\in$, $\ni$, $\forall$ and $\Rightarrow$ meaning "there exists", "belonging to", "such that", "for all" and "implies".
Sometimes the following simplified symbols will be used :

$$\sum_1^n {}_i \triangleq \sum_{i=1}^{n}$$

and

$$\bigcup(\gamma(x) : x \in M) \triangleq \bigcup_{x \in M} \gamma(x) .$$

0.2 *Terminology*

0.2.1 *DEFINITION*

Given a compact set $M \subset E$, *a continuous scalar function* $v = \phi(x)$, *defined in an open neighborhood* $N(M)$ *of* M *is said to be positive (negative) semidefinite for the set* M *in the open neighborhood* $N(M)$ *if*

$$\phi(x) = 0 \qquad \text{for all } x \in M$$

$$\phi(x) \geq 0 \quad (\phi(x) \leq 0) \qquad \text{for all } x \in N(M) \setminus M$$

If $N(M) = E$, *then the scalar function* $v = \phi(x)$ *is said to be positive (negative) semidefinite for the set* M. *If* $M = \{0\}$ *and* $N(M) = E$, *then the scalar function* $v = \phi(x)$ *is called positive (negative) semidefinite. If for the set* M, *a function* $v = \phi(x)$ *defined in a neighborhood* $N(M)$ *with* $\phi(x) = 0$ *for* $x \in M$ *is not semidefinite, we shall call it indefinite.*

0.2.2 *Remark*

The definition (0.2.1) as well as the following definitions (0.2.4) applies to a slightly larger class of sets than the compact sets, namely for the class of closed sets with a compact vicinity; viz closed sets M, such that for some $\beta > 0$ the set $\overline{S(M,\beta) \setminus M}$ is compact.

0.2.3 *Example*

If X is locally compact, then for sufficiently small $\delta > 0$, the set $C(S[x,\delta])$ is a set with a compact vicinity.

0.2.4 *DEFINITION*

Given a compact set $M \subset E$, *a continuous scalar function* $v = \phi(x)$, *defined in an open neighborhood* $N(M)$ *of* M *is said to be positive (negative) definite for the set* M *in the neighborhood* $N(M)$ *if it is*

$$\phi(x) = 0 \qquad x \in M$$

$$\phi(x) > 0 \quad (\phi(x) < 0) \quad \textit{for all} \quad x \in N(M) \setminus M.$$

If $N(M) = E$, *then the real-valued function* $v = \phi(x)$ *is said to be* *__positive__* *(negative)* *__definite for the set__* M. *If* $M = \{0\}$ *and* $N(M) = E$, *then the scalar function* $\phi(x)$ *is called* *__positive__* *(negative)* *__definite__*.

0.2.5 *DEFINITION*

A scalar function $\alpha = \alpha(\mu)$ *is called* *__strictly increasing__* *if* $\alpha(\mu_1) > \alpha(\mu_2)$ *whenever* $\mu_1 > \mu_2$, *and it is called* *__increasing__* *if* $\alpha(\mu_1) \geq \alpha(\mu_2)$ *whenever* $\mu_1 > \mu_2$.

0.2.6 *DEFINITION*

Given a scalar function $v = \phi(x)$, *if there exists an increasing function* $\alpha = \alpha(\mu)$ *such that*

0.2.7 $$\alpha(\mu) \to +\infty \quad \textit{as} \quad \mu \to +\infty$$

and such that $\phi(x)$ *satisfies in* E *the inequality*

0.2.8 $$\alpha(\rho(M,x)) \leq |\phi(x)| \quad , \qquad M \textit{ a compact set}$$

then the real-valued function $v = \phi(x)$ *is called* *__radially unbounded for the set__* M.

0.2.9 *DEFINITION*

If M *is* *__closed set__* *(not necessarily compact) and the function* $v = \phi(x)$ *satisfies the requirements of definition (0.2.1) (or 0.2.4) then* $\phi(x)$ *is called* *__weakly semidefinite__* *(or* *__weakly definite__*) *for the set* M *in the open set* $N(M)$. *If further* $\phi(x)$ *is defined in* $S(M,\delta)$ *for some* $\delta > 0$, *and if there is a strictly increasing function* $\alpha(\varepsilon), \alpha(0) = 0$, *such that*

0.2.10 $$\alpha(\rho(x,M)) \leq \phi(x), \qquad x \in S(M,\delta),$$

holds, then $\phi(x)$ *is called* *(positive) definite for the set* M *in the neighborhood* $S(M,\delta)$.

0.2.11 *DEFINITION*

If M *is a closed set and in the neighborhood* $N(M) (\supset S(M,\alpha))$ *the real-valued function* $v = \phi(x)$ *satisfies the condition*

0.2.12 $$|\phi(x)| \leq \beta(\rho(M,x))$$

where $\beta = \beta(\mu)$ *is an increasing function, then the function* $v = \phi(x)$ *is called* *uniformly bounded for the set* M *in* $N(M)$.

0.2.13 *DEFINITION*

If $M \subset E$ *is a closed set and there does not exist an* $\eta > 0$ *such that the real-valued function* $v = \phi(x)$ *is at least weakly semidefinite for the set* M *in the set* $S(M,\eta)$, *then* $\phi(x)$ *will be called indefinite for the set* M.

If $M \subset E$ *is a closed set, a continuous real-valued function* $\phi(x)$ *which is not at least weakly semidefinite for the set* M *in an open neighborhood* $N(M)$, *will be called indefinite for the set* M *in* $N(M)$.

The properties of the scalar function $v = \phi(x)$ can be investigated in two different spaces: the $(n + 1)$ dimensional Euclidean space (v,x) and the n-dimensional Euclidean space (x). In this latter case one actually considers the properties of the sets $\phi(x) = k$ $(-\infty < k < +\infty)$.

0.2.14 *DEFINITION.*

A set D *of real numbers is called relatively dense if there is a* $T > 0$ *such that*

$$D \cap (t - T, t + T) \neq \phi \quad \text{for all} \quad t \in R.$$

0.3 *Preliminary Lemmas*

We shall now state a few obvious properties of definite (or semidefinite) functions both in the space (v,x) and in the space (x). We shall define in the following corollaries properties of real-valued functions with respect to a compact set. The statements are identical in the case of sets with a compact vicinity and weaker when, instead of considering compact sets, one considers closed, non compact sets. In particular, the statements concerning definite functions become statements on weakly definite functions, as it must be obvious to the reader by comparing definitions (0.2.1) and (0.2.4) with the definition (0.2.9).

0.3.1 *LEMMA*

A continuous scalar function $v = \phi(x)$ *is positive (negative) definite for a compact set* M *if* M *is the absolute minimum (maximum) of the function.*

0.3.2 *LEMMA*

A continuous scalar function $v = \phi(x), \phi(x) = 0$ *for* $x \in M$, *is at least semidefinite for the compact set* M *if and only if there does not exist in* E *any hypersurface on which* $\phi(x)$ *changes its sign and it is definite if there does not exist any point* $y \notin M$ *such that* $\phi(y) = 0$.

0.3.3 *LEMMA*

Necessary and sufficient condition for the continuous real-valued function $v = \phi(x)$ *to be positive definite for the compact set* M *in some open neighborhood* $N(M)$ *is that there exists two strictly increasing, continuous functions* $\alpha = \alpha(\mu)$ *and* $\beta = \beta(\mu)$ *such that*

0.3.4 $$\alpha(\rho(M,x)) \leq \phi(x) \leq \beta(\rho(M,x)), \quad \alpha(0) = \beta(0) = 0$$

Proof: The condition (0.3.4) is clearly sufficient. To see the necessity, define

$$\alpha^*(\gamma) = \inf\{\phi(x) : \gamma \leq \rho(x,M) \leq \delta\},$$

and

$$\beta^*(\gamma) = \sup\{\phi(x) : \rho(x,M) \leq \gamma\},$$

where $\delta > 0$ is such that $N(M) \supset S(M,\delta)$. Then indeed

$$\alpha^*(\rho(x,M)) \leq \phi(x) \leq \beta^*(\rho(x,M)) \qquad ,$$

and $\alpha^*(\gamma)$ and $\beta^*(\gamma)$ are continuous.

Notice that $\alpha^*(\gamma) > 0$, $\beta^*(\gamma) > 0$ for $\gamma \neq 0$ and $\alpha^*(0) = \beta^*(0) = 0$, and the functions $\alpha^*(\gamma)$, $\beta^*(\gamma)$ are increasing. Now, there exists strictly increasing functions $\alpha(\gamma)$ and $\beta(\gamma)$ defined over an interval $0 \leq \gamma \leq \delta' < \delta$, such that

$$\alpha(\gamma) \leq \alpha^*(\gamma) \leq \beta^*(\gamma) \leq \beta(\gamma)$$

and $\alpha(0) = \beta(0) = 0$. For example, $\alpha(\gamma)$ may be chosen as follows. Let $\alpha^*(\delta') = \eta$, $\eta > 0$. Then there is a sequence of points $\gamma_1 > \gamma_2 > \gamma_3 > \ldots. > 0$, $\gamma_n \to 0$ as $n \to \infty$, such that $\alpha^*(\gamma_n) \geq \frac{\eta}{n}$, and $\gamma_1 = \delta'$. Now define

$$\alpha(\gamma) = \frac{\eta}{n+1} - \frac{\eta(\gamma_n - \gamma)}{(n+1)(n+2)(\gamma_n - \gamma_{n-1})}$$

$$\text{for } \gamma_n \geq \gamma \geq \gamma_{n+1}, \ n = 1,2, \quad \ldots$$

The existence of $\beta(\gamma)$ may be demonstrated in the same way and (0.3.4) holds with these $\alpha(\gamma)$ and $\beta(\gamma)$. The theorem is proved.

CHAPTER 1

DYNAMICAL SYSTEMS IN A EUCLIDEAN SPACE

1.1 *Definition of a continuous dynamical system.*

1.1.1 *DEFINITION*

A transformation $\pi : E \times R \to E$ *is said to define a dynamical system* (E,R,π) *(or continuous flow* $\mathcal{F}$*) on* E *if it has the following properties:*

i) $\pi(x,0) = x$ *for all* $x \in E$

1.1.2 ii) $\pi(\pi(x,t),s) = \pi(x,t+s)$ *for all* $x \in E$ *and all* $t,s \in R$.

iii) π *is continuous*

For every $x \in E$ the mapping π induces a continuous map $\pi_x : R \to E$ of R into E such that $\pi_x(t) = \pi(x,t)$. This mapping π_x is called the <u>motion through x</u>.

For every $t \in R$ the mapping π induces a continuous map $\pi^t : E \to E$ such that $\pi^t(x) = \pi(x,t)$. The map π^t is called <u>transition</u> (or <u>action</u>).

1.1.3 *THEOREM*

The mapping π^{-t} *defined by*

$$\pi^{-t}(x) = \pi(x,-t)$$

is the inverse of the mapping π^t.

Proof. It must be proved that $(\pi^t)^{-1} = \pi^{-t}$. This can be easily shown by applying to the point $x \in E$ the mapping π^t, then to the image point of $x : y = \pi(x,t)$ the mapping π^{-t}. The image point of y under this mapping:

$z = \pi^{-t}(y)$ must coincide with x. In fact, using axioms (i) and (ii) we have

$$z = \pi^{-t}(\pi(x,t)) = \pi(\pi(x,t),-t) = \pi(x,t-t) = \pi(x,0) = x,$$

which proves the theorem.

1.1.4 *THEOREM*

The mapping π^t *is a topological transformation of* E *onto itself.*

Proof. The map π^t is an onto mapping. In fact, all points $x \in E$ are image points of points $\pi(x,-t) \in E$. For the same reasons the map π^t is one to one. In fact the statement

$$\pi(x,t) = \pi(y,t) = z \qquad x,y,z \in E \qquad t \in R \quad \text{fixed}$$

implies, by application of the inverse map π^{-t}, that

$$x = y = \pi(z,-t)$$

which shows that π^t is one to one.

Since, by the definition 1.1.1, π^{-t} is obviously continuous the theorem is proved.

As a consequence of this fact, it follows that the dynamical system $\mathcal{F}$ is a one-parameter group of topological transformations, meaning by this that for each value of $t \in R$ a topological transformation is defined and, furthermore, the transformation π^t forms a group. We claim that the set $\{\pi^t\}$, $t \in R$ is a group with the group operation defined by

1.1.5 $$\pi^t \pi^s = \pi^{t+s} \quad .$$