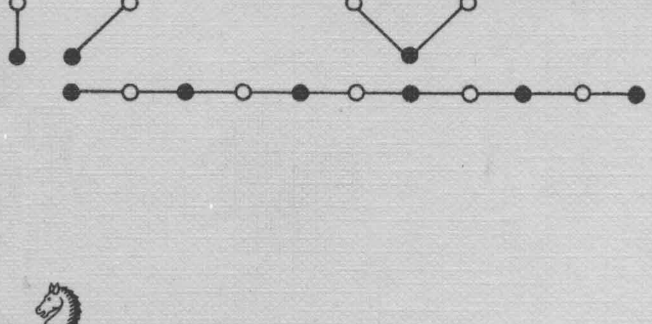
Lecture Notes in Mathematics 1542

Jürg Fröhlich Thomas Kerler

Quantum Groups, Quantum Categories and Quantum Field Theory



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Authors

Jürg Fröhlich
Theoretische Physik
ETH - Hönggerberg
CH-8093 Zürich, Switzerland

Thomas Kerler
Department of Mathematics
Harvard University
Cambridge, MA 02138, USA

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Contents

Chapter 1

Introduction and Survey of Results

Our original motivation for undertaking the work presented in this book* has been to clarify the connections between the braid (group) statistics discovered in low-dimensional quantum field theories and the associated unitary representations of the braid groups with representations of the braid groups obtained from the representation theory of quantum groups – such as $U_q(g)$, with deformation parameter $q = q_N := \exp(i\pi/N)$, for some $N = 3, 4, \ldots$. Among quantum field theories with braid statistics there are two-dimensional, chiral conformal field theories and three-dimensional gauge theories with a Chern-Simons term in their action functional. These field theories play an important role in string theory, in the theory of critical phenomena in statistical mechanics, and in a variety of systems of condensed matter physics, such as quantum Hall systems.

An example of a field theory with braid statistics is a chiral sector of the two-dimensional Wess-Zumino-Novikov-Witten model with group $SU(2)$ at level k which is closely related to the representation theory of $\widehat{su}(2)_k$-Kac-Moody algebra, with $k = 1, 2, 3, \ldots$. The braid statistics of chiral vertex operators in this theory can be understood by analyzing the solutions of the Knizhnik-Zamolodchikov equations. Work of Drinfel'd [4] has shown that, in the example of the $SU(2)$-WZNW model, there is a close connection between solutions of the Knizhnik-Zamolodchikov equations and the representation theory

*This book is based on the Ph.D. thesis of T.K. and on results in [6, 11, 24, 28, 42, 61] of $U_q(sl_2)$ if the level k is related to the deformation parameter q by the equation $q = \exp(i\pi/(k+2))$, and k is not a rational number. For an extension of these results to the negative rationals see [62]. Unfortunately, the $SU(2)$-WZNW model is a unitary quantum field theory only for the values $k = 1, 2, 3, \cdots$, not covered by the results of Drinfel'd. Our goal was to understand the connections between the field theory and the quantum group for the physically interesting case of positive integer levels. (This motivates much of our analysis in Chapters 2 through 7.)

The notion of symmetry adequate to describe the structure of superselection sectors in quantum field theories with braid statistics turns out to be quite radically different from the notion of symmetry that is used to describe the structure of superselection sectors in higher dimensional quantum field theories with permutation (group) statistics, (i.e., Fermi-Dirac or Bose-Einstein statistics). While in the latter case compact groups and their representation theory provide the correct notion of symmetry, the situation is less clear for quantum field theories with braid statistics. One conjecture has been that quantum groups, i.e., quasi-triangular (quasi-)Hopf algebras, might provide a useful notion of symmetry (or of "quantized symmetry") describing the main structural features of quantum field theories with braid statistics. It became clear, fairly soon, that the quantum groups which might appear in unitary quantum field theories have a deformation parameter q equal to a root of unity and are therefore not semi-simple. This circumstance is the source of a variety of mathematical difficulties which had to be overcome. Work on these aspects started in 1989, and useful results, eventually leading to the material in Chapters 4, 5 and 6, devoted to the representation theory of $U_q(g)$, q a root of unity, and to the so-called vertex-SOS transformation, were obtained in the diploma thesis of T.K.; see [6]. Our idea was to combine such results with the general theory of braid statistics in low-dimensional quantum field theories, in order to develop an adequate concept of "quantized symmetries" in such theories; see Chapter 7, Sects. 7.1 and 7.2.

In the course of our work, we encountered a variety of mathematical subtleties and difficulties which led us to study certain abstract algebraic structures – a class of (not necessarily Tannakian) tensor categories – which we call quantum categories. Work of

Doplicher and Roberts [29] and of Deligne [56] and lectures at the 1991 Borel seminar in Bern played an important role in guiding us towards the right concepts.

These concepts and the results on quantum categories presented in this volume, see also [61], are of some intrinsic mathematical interest, independent of their origin in problems of quantum field theory. Although problems in theoretical physics triggered our investigations, and in spite of the fact that in Chapters 2, 3 and 7, Sects. 7.1 through 7.4 we often use a language coming from local quantum theory (in the algebraic formulation of Haag and collaborators [17, 18, 19, 20]), all results and proofs in this volume (after Chapter 2) can be understood in a sense of pure mathematics: They can be read without knowledge of local quantum theory going beyond some expressions introduced in Chapters 2 and 3, and they are mathematically rigorous.

In order to dispel possible hesitations and worries among readers, who are pure mathematicians, we now sketch some of the physical background underlying our work, thereby introducing some elements of the language of algebraic quantum theory in a non-technical way. For additional details the reader may glance through Chapter 2.

For quantum field theories on a space-time of dimension four (or higher) the concept of a global gauge group, or symmetry G is, roughly speaking, the following one: The Hilbert space $\mathcal{H}$ of physical states of such a theory carries a (highly reducible) unitary representation of the group G. Among the densely defined operators on $\mathcal{H}$ there are the so-called local field operators which transform covariantly under the adjoint action of the group G. The fixed point algebra, with respect to this group action in the total field algebra, is the algebra of observables. This algebra, denoted by $\mathcal{A}$, is a C^*-algebra obtained as an inductive limit of a net of von Neumann algebras $\mathcal{A}(\mathcal{O})$ of observables localized in bounded open regions $\mathcal{O}$ of space-time. The von Neumann algebras $\mathcal{A}(\mathcal{O})$ are isomorphic to the unique hyperfinite factor of type III_1, in all examples of algebraic field theories that one understands reasonably well. The Hilbert space $\mathcal{H}$ decomposes into a direct sum of orthogonal subspaces, called superselection sectors, carrying inequivalent representations of the observable algebra $\mathcal{A}$. All these representations of $\mathcal{A}$ can be generated by composing a standard representation, the so-called vacuum representation, with *endomorphisms of

$\mathcal{A}$. Each superselection sector also carries a representation of the global gauge group G which is equivalent to a mulitple of a distinct irreducible representation of G. As shown by Doplicher, Haag and Roberts (DHR) [19], one can introduce a notion of tensor product, or "composition", of superselection sectors with properties analogous to those of the tensor product of representations of a compact group. The composition of superselection sectors can be defined even if one does not know the global gauge group G of the theory, yet. From the properties of the composition of superselection sectors, in particular from the fusion rules of this composition and from the statistics of superselection sectors, i.e., from certain representations of the permutation groups canonically associated with superselection sectors, one can reconstruct important data of the global gauge group G. In particular, one can find its character table and its 6-j symbols. As proven by Doplicher and Roberts [29], those data are sufficient to reconstruct G. The representation category of G turns out to reproduce all properties of the composition of superselection sectors, and one is able to reconstruct the algebra of local field operators from these data. One says that the group G is dual to the quantum theory described by $\mathcal{A}$ and $\mathcal{H}$.

The results of Doplicher and Roberts can be viewed as the answer to a purely mathematical duality problem (see also [56]): The fusion rules and the 6-j symbols obtained from the composition of superselection sectors are nothing but the structure constants of a symmetric tensor category with C^* structure. The problem is how to reconstruct from such an abstract category a compact group whose representation category is isomorphic to the given tensor category. It is an old result of Tannaka and Kreĭn that it is always possible to reconstruct a compact group from a symmetric tensor category if the category is Tannakian, i.e., if we know the dimensions of the representation spaces and the Clebsch-Gordan matrices, or 3-j symbols, which form the basic morphism spaces. The results of Doplicher and Roberts represent a vast generalization of the Tannaka-Kreĭn results, since the dimensions and Clebsch-Gordan matrices are not known a priori.

Another duality theorem related to the one of Doplicher and Roberts is due to Deligne [56] which requires integrality of certain dimensions but no C^* structure on the symmetric tensor category. (It enables one to reconstruct algebraic groups from certain symmetric tensor categories.) Disregarding some subtleties in the hypotheses of these duality theorems, they teach us that it is equivalent to talk about compact groups or certain symmetric tensor categories.

Quantum field theories in two and three space-time dimensions can also be formulated within the formalism of algebraic quantum theory of DHR, involving an algebra $\mathcal{A}$ of observables and superselection sectors carrying representations of $\mathcal{A}$ which are compositions of a standard representation with *endomorphisms of $\mathcal{A}$. This structure enables us to extract an abstract tensor category described in terms of an algebra of fusion rules and 6-j symbols. Contrary to the categories obtained from quantum field theories in four or more space-time dimensions, the tensor categories associated with quantum field theories in two and three space-time dimensions are, in general, not symmetric but only braided. Therefore, they cannot be representation categories of cocommutative algebras, like group algebras. In many physically interesting examples of field theories, these categories are not even Tannakian and, therefore, cannot be identified, naïvely, with the representation category of a Hopf algebra or a quantum group; see [61]. The complications coming from these features motivate many of our results in Chapters 6 through 8.

The following models of two- and three-dimensional quantum field theories yield non-Tannakian categories:

(1) Minimal conformal models [7] and Wess-Zumino-Novikov-Witten models [8] in two space-time dimensions .

The basic feature of these models is that they exhibit infinite-dimensional symmetries. The example of the $SU(n)$-WZW model can be understood as a Lagrangian field theory with action functional given by

$$\begin{aligned} S(g) \;=\; & \tfrac{k}{16\pi} \int_{S^2} tr\,((g^{-1}\partial_\mu g)(g^{-1}\partial^\mu g))\, d^2x \\ & + \tfrac{k}{24\pi} \int_{B^3} tr\,((\tilde{g}^{-1} d\tilde{g})^{\wedge 3}), \end{aligned}$$

where, classically, a field configuration g is a map from the two-sphere S^2 to the group $G = SU(n)$, and $\tilde{g}$ is an arbitrary extension of g from $S^2 = \partial B^3$ to the ball B^3; (such an extension always exists, since π_2 of a group is trivial). The second term in $S(g)$ is the so-called Wess-Zumino term which is defined only $mod\, k\mathbb{Z}$. Classically, the theory exhibits a symmetry which is the product of two loop groups, for right- and left movers, respectively. For $k = 1, 2, 3, \ldots$, the quantum theory associated with $S(g)$ has conserved currents generating two commuting $\widehat{su}(n)$-Kac-Moody algebras at level k, whose universal enveloping algebras contain Virasoro algebras; (Sugawara construction). From the representation theory of the infinite-dimensional Lie algebras of symmetry generators in these models, i.e., the representation theory of Virasoro- or Kac-Moody algebras, one can construct algebras of so-called chiral vertex operators which play the role of Clebsch-Gordan operators of (a semi-simple quotient of) the representation category of the Virasoro- or Kac-Moody algebra. Local conformally covariant field operators are then constructed by taking linear combinations of products of two such chiral vertex operators, a holomorphic one (left movers) and an anti-holomorphic one (right movers).

Of interest in relation to the main subject of our work is that the algebras of chiral vertex operators, the holomorphic ones, say, appearing in these models provide us with categorial data corresponding to non-Tannakian braided tensor categories. (This can be understood by studying the multi-valuedness properties and operator product expansions of chiral vertex operators. A very thorough analysis of the $SU(2)$-WZW model can be found in the papers of Tsuchiya and Kanie and of Kohno quoted in [9]; see also [8, 61].)

Zamolodchikov and others have studied "non-critical perturbations" of minimal conformal models which are integrable field theories [10]. Their results suggest that there are plenty of massive quantum field theories in two space-time dimensions with fields exhibiting non-abelian braid statistics, as originally described in [11]. (A perturbation of minimal conformal models giving rise to massive integrable field theories is obtained from the $\phi_{(1,3)}$-field; a field with braid statistics is the field obtained from a chiral factor of the $\phi_{(3,1)}$-field, after the perturbation has been turned on [12].) To such non-conformal field theories one can also associate certain braided tensor categories. However, the general theory of superselection sectors in two-dimensional, massive quantum field theories leads to algebraic structures more general than braided tensor categories, including ones with non-abelian fusion rule algebras. A general understanding of these structures has not been accomplished, yet.

(2) Three-dimensional Chern-Simons gauge theory, [13, 14, 15] .

Consider a gauge theory in three space-time dimensions with a simply connected, compact gauge group $G \overset{e.g.}{=} SU(n)$. Let A denote the gauge field (vector potential) with values in $g \equiv Lie(G)$, the Lie algebra of the gauge group G, and let ψ be a matter field, e.g. a two-component spinor field in the fundamental representation of G. There may be further matter fields, such as Higgs fields. The action functional of the theory is given by

$$\begin{aligned} S[A,\bar{\psi},\psi] \overset{e.g.}{=} \; & g^{-2} \int tr\,(F^2) d\,vol. \\ & - \tfrac{il}{4\pi} \int tr\,(A \wedge dA + \tfrac{2}{3}\, A \wedge A \wedge A) \\ & + \lambda \int \bar{\psi} (\not{D}_A + m)\, \psi \, d\,vol. + \cdots , \end{aligned} \tag{1.1}$$

where g, λ and m are positive constants, and l is an integer.

This class of gauge theories has been studied in [13, 14, 15]. Although the results in these papers are not mathematically rigorous, the main properties of these theories are believed to be as follows:

The gluon is massive, and there is no confinement of colour. Interactions persisting over arbitrarily large distances are purely topological and are, asymptotically, described by a pure Chern-Simons theory. Thus the statistics of coloured particles in Chern-Simons gauge theory is believed to be the same as the statistics of static colour sources in a pure Chern-Simons theory which is known explicitly [16]. The statistics of coloured asymptotic particles can be studied by analyzing the statistics of fields creating coloured states from the vacuum sector. Such fields are the Mandelstam string operators, $\psi_\alpha(\gamma_x)$, which are defined, heuristically, by

$$\psi_\alpha(\gamma_x) \;=\; \text{``}\sum_\beta N[\psi_\beta(x) P(\exp \int_{\gamma_x} A_\mu(\xi) d\xi^\mu)_{\beta\alpha}]\text{''}, \tag{1.2}$$

where α and β are group indices; γ_x is a path contained in a space-like surface, starting at x and reaching out to infinity, N is some normal ordering prescription, and P denotes path ordering. (Similarly, conjugate Mandelstam strings $\bar{\psi}_\alpha(\gamma_x)$ are defined.)

For the field theories described in (1) and (2), one observes that when the group G is $SU(2)$ the combinatorial data of a braided tensor category, an algebra of fusion rules and 6-j symbols (braid- and fusion matrices), can be reconstructed from these field theories which is isomorphic to a braided tensor category that is obtained from the representation theory of the quantum group $U_q(sl_2)$, where

$$q = e^{\frac{i\pi}{k+2}}, \quad k = 1, 2, 3, \cdots,$$

(with $k = l + const.$). These categories are manifestly non-Tannakian. This is the reason why it is not possible to reconstruct field operators transforming covariantly under some representation of $U_q(sl_2)$ on the Hilbert space of physical states of those theories. However, passing to a quotient of the representation category of $U_q(sl_2)$, $q = \exp(i\pi/(k+2))$, described in Chapters 6 and 7, we can construct a semi-simple, non-Tannakian, braided tensor category describing the composition and braid statistics of superselection sectors in these quantum field theories. In this sense, $U_q(sl_2)$ is the "quantized symmetry" dual to the quantum field theories described above. (For precise details see Chapter 7.)

The strategy used to prove this duality is to compare the fusion rules and the 6-j symbols of $U_q(sl_2)$ with the corresponding data of the field theories found, e.g., in [9], and to show that they coincide. More precisely, it is quite easy to show that the representations of the braid groups associated with tensor products of the fundamental representation of $U_q(sl_2)$ coincide with those associated with arbitrary compositions of the "fundamental superselection sector" of the corresponding field theories. One implication of our work is that, in fact, the entire braided tensor categories coincide. This result follows from a much more general uniqueness theorem stating that whenever a braided tensor category with C^* structure is generated by arbitrary tensor products of a selfconjugate object, ρ, whose tensor square decomposes into two irreducible objects, i.e.,

$$\rho \otimes \rho = 1 \oplus \psi, \tag{1.3}$$

(where 1 is the neutral object, corresponding to the trivial representation of $U_q(sl_2)$, to the vacuum sector of the field theory, respectively), and a certain invariant associated with ρ, the so-called monodromy of ρ with itself, is non-scalar, then the category is isomorphic to the semi-simple subquotient of the representation category of $U_q(sl_2)$, for $q = \pm\, e^{\pm \frac{i\pi}{k+2}}$, $k = 1, 2, 3, \ldots$.

The abstract nature of eq. (1.3) suggests that this result applies to a class of local quantum field theories more general than the models described above. This observation and the fact that those models are not rigorously understood in every respect led us to work within the general framework of algebraic field theory. In this framework, ρ and ψ can be interpreted as irreducible *endomorphisms of the observable algebra $\mathcal{A}$, with 1 the identity endomorphisms of $\mathcal{A}$, and eq. (1.3) for a selfconjugate object ρ of a braided tensor category with C^* structure is equivalent to some bounds on a scalar invariant associated with ρ, its statistical dimension, $d(\rho)$; namely (1.3) is equivalent to

$$1 < d(\rho) < 2 \,. \tag{1.4}$$

The main result of this book is a complete classification of braided tensor categories with C^*-structure that are generated by a not necessarily selfconjugate, irreducible object ρ whose statistical dimension, $d(\rho)$, satisfies (1.4). This is the solution to a very limited generalization of the duality problem for groups. Our method of classification is unlikely to be efficient for much larger values of $d(\rho)$ than those specified in eq. (1.4) – except, perhaps, for certain families of examples connected with more general quantum groups. However, our solution to the problem corresponding to the bounds on $d(\rho)$ in eq. (1.4) might serve as a guide for more general attempts. In particular, our notions of product category and induced category might be useful in a general context.

The constructive part of our classification consists in the description of two families of categories: First, we need to understand the representation theory and tensor-product decompositions of $U_q(sl_2)$, with q a root of unity; (Chapters 4 and 5, and [6]). This will permit us to construct a non-Tannakian, braided tensor category by passing to the semi-simple quotient of the representation category of $U_q(sl_2)$; (vertex-SOS transformation; see