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Hankel and Toeplitz Matrices and Forms

Algebraic Theory

Translated by
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EDITORIAL INTRODUCTION

This book is a valuable introduction to the theory of finite Hankel and Toeplitz matrices. These matrices are characterized by the property that in one of them the entries of the matrices depend only on the sum of indices, and in the other only on the differences of the indices.

The book is dedicated in general to the algebraic aspect of the theory and the main attention is given to problems of: extensions, computations of ranks, signature, and inversion. The author has succeeded in presenting these problems in a unified way, combining basic material with new results.

Hankel and Toeplitz matrices have a long history and have given rise to important recent applications (numerical analysis, system theory, and others).

The book is self-contained and only a knowledge of a standard course in linear algebra is required of the reader. The book is nicely written and contains a system of well chosen exercises. The book can be used as a text book for graduate and senior undergraduate students.

I would like to thank Dr. Philip Thijsse for his dedicated work in translating this book and Professor I.S. Iohvidov for his assistance and cooperation.

I. Gohberg

NOTE OF THE TRANSLATOR

The text of this edition is, but for some minor corrections, identical with that of the 1974 Russian edition. In order to inform the readers of new developments a list of additional literature was added, and at the end of the Chapters II, III and IV a Remark leads the reader to this list. For technical reasons all footnotes were replaced by notes at the end of the sections. For the convenience of the readers these notes have been listed separately in the table of contents.

The production of this translation would have been impossible without the invaluable help of Mrs. Bärbel Schulte, who typed the manuscript with much skill, and showed much patience during the process, and of Professor I.S. Iohvidov, who corrected the typescript with extreme diligence, and had to endure the critical remarks of the translator, from which he would have been spared if the job had been done by a non-mathematician.

PREFACE

The theory of Hankel and Toeplitz matrices, as well as the theory of the corresponding quadratic and Hermitian forms, is related to that part of mathematics which can in no way be termed non-prolific in the mathematical literature. On the contrary, many journal papers and entire monographs have been dedicated to these theories, and interest in them has not diminished since the beginning of the present century, and in the case of Hankel matrices and forms, even since the end of the previous century. Such a continuous interest can be explained in the first place from the wide range of applications of the mentioned theories - in algebra, function theory, harmonic analysis, the moment problem, functional analysis, probability theory and many applied problems.

Besides the mentioned regions of direct application, there is still one more section of mathematics in which Toeplitz and Hankel matrices play the role of distinctive models. The point is, that the continual analogues of systems of linear algebraic equations, in which the matrices of coefficients are Toeplitz matrices (i.e., the entries of these matrices depend only on the difference of the indices of the rows and the columns), are integral equations with kernels, which depend only on the difference of the arguments, including, in particular, the Wiener-Hopf equations, a class which is of such importance for theoretical physics. Not infrequently facts, discovered on the algebraic level for the mentioned linear systems, lead instantly to analoguous new results for integral equations (a quite recent example is the paper [25]; in § 18 of this book the reader will become partially acquainted with its contents). The analoguous situation holds for Hankel matrices (i.e. matrices in which the entries depend only on the sum of the indices) and kernels which depend on the sum of the arguments.

This makes it all the more paradoxical, that at least in the Russian language no monograph has been dedicated to Toeplitz and Hankel matrices and forms in a purely algebraic setting. Moreover, although some information on Hankel matrices and forms can be obtained from the monograph "Theory of Matrices" of F.R. Gantmaher ([3],Ch.X,§ 1o and Ch.XVI,§ 1o), practically all known Russian or translated courses on linear algebra and matrix theory make no mention of Toeplitz matrices and forms, except for the literally few lines devoted to them in the book of R. Bellmann [2]. As to the well-known monograph of U. Grenander and G. Szegö "Toeplitz forms and their applications" [7], that is on the whole devoted to analytic problems. The term "Toeplitz form" itself is, in spite of the general definition given to it by the authors in the preface, used in this book almost exclusively in the sense in which it entered in the literature following the works of C. Carathédory, O. Toeplitz, E. Fischer, G. Herglotz and F. Riesz (in the years 19o7-1915). Namely, they deal basically with forms with coefficients which are connected with certain power series, Laurent series or Fourier series, and not at all with forms of general shape and their purely algebraic properties.

To date, a large number of results relating to the algebra of Hankel and Toeplitz matrices and the corresponding forms has been accumulated in the journal literature, and these results combine already to form a sufficiently well-structured theory. It originated in the memoires of G. Frobenius [19,2o] (from the years 1984 and 1912), but further results, which enter into the present book, were only found in our days.

Highly remarkable in our view, are the deep analogies and also direct relations, which were discovered only in the later years between the two classes of matrices (and forms) to which this book is dedicated. These analogies and connections, namely, were the orientation which enabled us to clear up many questions which remained, until now, in the shadow, in spite of the venerable age of the considered theory.

The reasons delineated above constitute, in all probability, sufficient justification for the purpose adopted in the writing of this book: *to restrict it in particular to the algebraic aspect of the theory, but to reflect this, if possible, completely.* We note, that the first part of this formula, to set aside all kinds of applications, as long as these are presented in other monographs in a sufficiently complete way, is (just as the second part) not wholly sustained with due consequence - we could not resist the temptation to adduce if only the simplest application of Hankel and Toeplitz forms in the theory of the separation of the roots of algebraic equations, which, besides, does not really violate the algebraic character of the book, mentioned in its subtitle. The special Appendix I is dedicated to this matter, whereas Appendix II touches, albeit also only in the same elementary way, the deep connection between our subject and the classical moment problem.

As to the basic text of the book, it is, with the exception of Chapter I, entirely devoted to the algebraic properties of Hankel (Ch. II) and Toeplitz (Ch. III) matrices and forms, and also to the various transformations of these subjects, among them the mutual transformations of matrices and forms of each of these two classes to matrices and forms of the other class (Ch. IV).

Let us linger in some detail on the contents of Chapters II - IV. The core of the whole theory is the so-called method of singular extension of Hankel and Toeplitz matrices (§§ 9 and 12 respectively) and the notions of characteristics which are developed on this basis. These notions which allow, respectively in §§ 11 and 15, to establish comparatively rapidly fundamental theorems on the rank of Hankel and Toeplitz matrices separately, are then combined in § 17 to one single systems of characteristics, covering both considered classes of matrices. In §§ 12 and 16, respectively, signature rules are established - the well-known rule of Frobenius for Hankel forms and a new rule for Toeplitz forms; this and the other are obtained by the same method of singular extensions and characteristics. Section 18 is entirely devoted to the problem of inversion of Toeplitz and Hankel matrices, and § 19 to transformations which transfer into each other the forms of the two classes which interest us.

Chapter I plays an auxiliary role. In it information from the general theory of matrices and forms, which is necessary for the subsequent chapters, is gathered. Some of this material is presented in traditional form but another part, however, had to be presented in a new way in order to make the reading of the book, if possible, independent of the direct availability of other texts. This relates in particular to §§ 6 and 8, which deal with truncated forms and the signature rule of Jacobi (and its generalizations), respectively. Somewhat distinct is § 3, which contains purely technical but, for the construction of the entire theory, very important material - a lemma on the evaluation of one special determinant and its consequences.

Of the reader is required the knowledge of the elements of mathematical analysis and algebra, and also the knowledge of a basic course in linear algebra and matrix theory to the extent of, for example, the first ten chapters of the treatise of F.R. Gantmaher [3], to which this book is, actually, presented as a supplement. Such minimal general preparatory requirements of the reader has forced us to exclude from the book the theory of infinite extensions of Hankel and Toeplitz forms with a fixed number of squares with a certain sign. This theory, developed in the papers [3o, 42, 33, 43, 31] and others, is in this book represented only in two exercises in §§ 12 and 16 respectively, since it requires the application of tools from functional analysis (operators on Hilbert spaces with an indefinite metric). In addition the study of the asymptotics of the coefficients of the mentioned infinite extensions necessitates the engagement of the appropriate analytical apparatus.

The original text of the book was completed as a manuscript of special courses which the author held in the years 1968 - 197o at the Mathematics Department of the Voronež State University and at the Department of Physics and Mathematics of the Voronež Pedagogical Institute. Subsequently this text was significantly extended by the inclusion of new results, both published and unpublished, and also in favour of examples and exercises, which conclude each section of the basic text and both appendices. The range of these exercises is sufficiently wide - from elementary numerical examples, provided either with detailed calculations or with answers, to little propositions, and sometimes also important theorems, not occurring in the basic text. The most difficult among the exercises are accompanied by hints.

In the book continuous numeration of the sections is adopted; the propositions, and also the examples and exercises are numerated anew in each section; the items, lemmata and theorems, and also the individual formulae have double numbers (of which the first number denotes the section). The references to the literature in brackets [] lead the reader to the list of cited literature at the end of the book.

For his initial interest in Toeplitz forms, and also in other problems of algebra and functional analysis, the author is obliged to his dear teacher Mark Grigorevič Kreĭn. In this book (especially in the Appendices I and II) the reader will repeatedly encounter some of his ideas and results, relating to our subject.

The text of this book reflects the valuable suggestions of

V.P. Potapov, expressed by him at the earlier stages of its preparation, when the idea of the book was barely thought out. At the final stage of the work the interest shown in this project by the collaborators of the chair of algebra of the Moscow State University, O.N. Golovin, E.B. Vinberg, E.S. Golod and V.N. Latyšev was a great stimulus for the author.

T.Ya. Azizov and E.I. Iohvidov, students in the special courses in which the book "originated", did indeed extend invaluable help to the author in the realization of the manuscript. In particular T.Ya. Azizov undertook the unenviable task of reading the complete text and verifying all exercises and calculations, which resulted in the insertion of numerous corrections and improvements. Useful remarks during the presentation and the reworking of the lecture courses were made by F.I. Lander.

To all those mentioned here the author wishes to express his sincere gratitude.

CONTENTS

Chapter I

SOME INFORMATION FROM THE GENERAL THEORY OF MATRICES AND FORMS.

§ 1 THE RECIPROCAL MATRIX AND ITS MINORS

1.1 We shall consider arbitrary square matrices $A = \| a_{ij} \|_{i,j=1}^{m}$ of complex numbers. If

$$i_1 < i_2 < \cdots < i_p;\ j_1 < j_2 < \cdots < j_p$$

are two sets of p indices ($1 \leqq p \leqq n$) from the indices $1,2,\cdots,n$, then we denote, as usually, through

$$A \begin{pmatrix} i_1 & i_2 \cdots i_p \\ j_1 & j_2 \cdots j_p \end{pmatrix}$$

the minor, consisting of the elements on the intersection of the rows with numbers $i_1, i_2, \cdots, i_p$ and the columns with numbers $j_1, j_2, \cdots, j_p$ of A, i.e.,

$$A \begin{pmatrix} i_1 & i_2 \cdots i_p \\ j_1 & j_2 \cdots j_p \end{pmatrix} = \det \| a_{i_\mu j_\nu} \|_{\mu,\nu=1}^{p}.$$

Evidently,

$$A \begin{pmatrix} 1 & 2 \cdots n \\ 1 & 2 \cdots n \end{pmatrix} \equiv |A|,$$

the determinant of the matrix A.

We agree to denote the n-p indices remaining after taking from the set $\{1,2,\cdots,n\}$ the indices $i_1, i_2, \cdots, i_p$ $(j_1, j_2, \cdots, j_p)$ through $k_1, k_2, \cdots, k_{n-p}$ $(\ell_1, \ell_2, \cdots, \ell_{n-p})$ (here the indices are always written in increasing order). Then

$$A \begin{pmatrix} k_1 & k_2 \cdots k_{n-p} \\ \ell_1 & \ell_2 \cdots \ell_{n-p} \end{pmatrix}$$

is, by definition, the *complementary* minor to the minor

$$A \begin{pmatrix} i_1 & i_2 \cdots i_p \\ j_1 & j_2 \cdots j_p \end{pmatrix}.$$

Evidently, complementary to the minors $a_{ij} = A\binom{i}{j}$ of the first order are the determinants

$$A\begin{pmatrix}1 & 2\cdots i-1 & i+1\cdots n\\ 1 & 2\cdots j-1 & j+1\cdots n\end{pmatrix} \equiv \tilde{a}_{ij} \quad (i,j=1,2,\cdots,n),$$

and the numbers $A_{ij} = (-1)^{i+j}\tilde{a}_{ij}$ represent the cofactors to the elements a_{ij} in the matrix A, respectively $(i,j=1,2,\cdots,n)$.

1.2 Diverging somewhat from more extended terminology, we shall, following [3], call the matrix

$$\tilde{A} = \|\tilde{a}_{ij}\|_{i,j=1}^{n}$$

consisting of the minors of order n-1 of A the *reciprocal* matrix with respect to the matrix A. We establish a rule for the computation of minors of the reciprocal matrix.

THEOREM 1.1. *For an arbitrary natural number* p $(1 \leq p \leq n)$ *one has*

$$\tilde{A}\begin{pmatrix}i_1 & i_2\cdots i_p\\ j_1 & j_2\cdots j_p\end{pmatrix} = |A|^{p-1} \; A\begin{pmatrix}k_1 & k_2\cdots k_{n-p}\\ \ell_1 & \ell_2\cdots \ell_{n-p}\end{pmatrix}. \qquad (1.1)$$

Here, in the case where $p = n$, *formula* (1.1) *should be understood as*

$$|\tilde{A}| = A\begin{pmatrix}1 & 2\cdots n\\ 1 & 2\cdots n\end{pmatrix} = |A|^{n-1}, \qquad (1.2)$$

and for $p = 1$ *and* $|A| = 0$ *one should assume* $|A|^{p-1} = 1$.

PROOF. Without loss of generality one can restrict oneself to considering minors of the shape

$$\tilde{A}\begin{pmatrix}1 & 2\cdots p\\ 1 & 2\cdots p\end{pmatrix} \qquad (1 \leq p \leq n),$$

as the general case is obtained from this easily by appropriate permutation of rows and columns. Now formula (1.1) takes the form

$$\tilde{A}\begin{pmatrix}1 & 2\cdots p\\ 1 & 2\cdots p\end{pmatrix} = |A|^{p-1} \; A\begin{pmatrix}p+1\cdots n\\ p+1\cdots n\end{pmatrix}. \qquad (1.3)$$

In order to establish this, we multiply in the determinant

$$\tilde{A}\begin{pmatrix}1 & 2\cdots p\\ 1 & 2\cdots p\end{pmatrix} = \begin{vmatrix}\tilde{a}_{11} & \tilde{a}_{12}\cdots\tilde{a}_{1p}\\ \tilde{a}_{21} & \tilde{a}_{22}\cdots\tilde{a}_{2p}\\ \cdots & \cdots\\ \tilde{a}_{p1} & \tilde{a}_{p2}\cdots\tilde{a}_{pp}\end{vmatrix}$$

the i-th row with $(-1)^i (i=1,2,\ldots p)$ and the j-th column with $(-1)^j$ $(j=1,2,\ldots,p)$ It is easy to understand that by such a transformation the value of the determinant doesn't change, and the determinant itself takes the form

$$\tilde{A}\begin{pmatrix}1 & 2\ldots p\\ 1 & 2\ldots p\end{pmatrix} = \det \|(-1)^{i+j}\tilde{a}_{ij}\|_{i,j=1}^{p} = \begin{vmatrix} A_{11} & A_{12}\ldots A_{1p}\\ A_{21} & A_{22}\ldots A_{2p}\\ . & . \quad \ldots\\ A_{p1} & A_{p2}\ldots A_{pp}\end{vmatrix} .$$

It is convenient to assume that this minor has the shape of a determinant of order n (in the case p = n the next step is, clearly, not necessary):

$$\tilde{A}\begin{pmatrix}1 & 2\ldots p\\ 1 & 2\ldots p\end{pmatrix} = \left|\begin{array}{ccc|ccc} A_{11} & \ldots & A_{1p} & A_{1,p+1} & \ldots & A_{1n}\\ A_{21} & \ldots & A_{2p} & A_{2,p+1} & \ldots & A_{2n}\\ . & \ldots & . & . & \ldots & .\\ A_{p1} & \ldots & A_{pp} & A_{p,p+1} & \ldots & A_{pn}\\ \hline 0 & \ldots & 0 & 1 & \ldots & 0\\ . & \ldots & . & . & \ldots & .\\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{array}\right| .$$

We multiply both sides of this equality with the determinant

$$|A| = \left|\begin{array}{ccc|ccc} a_{11} & \ldots & a_{1p} & a_{1,p+1} & \ldots & a_{1n}\\ a_{21} & \ldots & a_{2p} & a_{2,p+1} & \ldots & a_{2n}\\ . & \ldots & . & . & \ldots & .\\ a_{p1} & \ldots & a_{pp} & a_{p,p+1} & \ldots & a_{pn}\\ \hline a_{p+1,1} & \ldots & a_{p+1,p} & a_{p+1,p+1} & \ldots & a_{p+1,n}\\ . & \ldots & . & . & \ldots & .\\ a_{n1} & \ldots & a_{np} & a_{n,p+1} & \ldots & a_{nn} \end{array}\right| ,$$

where we shall develop the product of the determinants on the right-hand side through the "row on row" method. Then, taking into account the well-known properties of the cofactors A_{ij}, we obtain

$$\tilde{A}\begin{pmatrix}1 & 2\ldots p\\ 1 & 2\ldots p\end{pmatrix} \cdot |A| =$$

$$= \left|\begin{array}{cccc|ccc} |A| & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & |A| & \ldots & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot & \cdot & \ldots & \cdot \\ 0 & & \ldots & |A| & 0 & \ldots & 0 \\ \hline a_{1,p+1} & a_{2,p+1} & \ldots & a_{p,p+1} & a_{p+1,p+1} & \ldots & a_{n,p+1} \\ \cdot & \cdot & \ldots & \cdot & \cdot & \ldots & \cdot \\ a_{1n} & a_{2n} & \ldots & a_{pn} & a_{p+1,n} & \ldots & a_{nn} \end{array}\right| =$$

$$= \quad |A|^p \cdot A\begin{pmatrix} p+1 & \ldots & n \\ p+1 & \ldots & n \end{pmatrix}. \tag{1.4}$$

We observe at once that, for p = n, it is simple to obtain that

$$\tilde{A}\begin{pmatrix} 1 & 2 \ldots n \\ 1 & 2 \ldots n \end{pmatrix} \cdot |A| = |A|^n. \tag{1.5}$$

If the matrix A is nonsingular ($|A| \neq 0$) then formula (1.3) (resp.(1.2)) follows immediately from (1.4) (resp.(1.5)). In the case where $|A| = 0$ the identity (1.3) (resp.(1.2)) is obtained by a standard limit transition. Namely, the matrix $A_\varepsilon = A + \varepsilon E$ - where E is the identity matrix on order n - is considered. The determinant $|A_\varepsilon|$ is a polynomial in ε. Therefore, in an arbitrarily small neighbourhood of zero there can be found values ε for which $|A_\varepsilon| \neq 0$. Having noted that for such ε the identity in formula (1.3)(resp.(1.2)) is valid for A_ε (strictly speaking for the minors of the reciprocal $\tilde{A}_\varepsilon$) we take the limit for $\varepsilon \to 0$ over those values of ε for which $|A_\varepsilon| \neq 0$. Hereby the minors of A_ε and $\tilde{A}_\varepsilon$ go to the respective minors of the matrices A and $\tilde{A}$ we obtain the identity (1.3)(resp.(1.2)).

EXAMPLES AND EXERCISES

1. Let

$$A = \left\| \begin{matrix} 3 & 0 & -1 \\ 2 & 7 & -2 \\ -3 & 4 & 0 \end{matrix} \right\|.$$

Whithout constructing $\tilde{A}$ we evaluate $\tilde{A}\binom{1\ 2}{2\ 3}$. Here p = 2,
$|A| = 3.8+(-1)(29) = -5$,

$$|A|^{p-1} = -5, \quad A\binom{3}{1} = -3.$$

With formula (1.1)

$$\tilde{A}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = |A|^{p-1} A \begin{pmatrix} 3 \\ 1 \end{pmatrix} = (-5)(-3) = 15. \qquad (1.6)$$

2. For the matrix A of example 1 we evaluate $|\tilde{A}|$. With formula (1.2)

$$|\tilde{A}| = |\tilde{A}|^{n-1} = (-5)^2 = 25.$$

We note, that the matrix $\tilde{A}$ itself has, in the given case, the form

$$\tilde{A} = \begin{Vmatrix} 8 & -6 & 29 \\ 4 & -3 & 12 \\ 7 & -4 & 21 \end{Vmatrix}.$$

Returning to example 1 we verify the result (1.6):

$$\tilde{A}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{vmatrix} -6 & 29 \\ -3 & 12 \end{vmatrix} = -72 + 87 = 15.$$

3. If the matrix A is "lower triangular",

$$A = \begin{Vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{Vmatrix},$$

i.e., $a_{ij} = 0$ for $j > i$, then the reciprocal matrix $\tilde{A}$ is "upper triangular",

$$\tilde{A} = \begin{Vmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \tilde{a}_{nn} \end{Vmatrix}$$

i.e., $\tilde{a}_{ij} = 0$ for $i > j$.

4. Is the converse of the statement, for lated in example 3, correct?

§ 2 THE SYLVESTER IDENTITIES FOR BORDERED MINORS

2.1 We consider in the matrix $A = \|a_{ij}\|_{i,j=1}^{n}$ $(n \geqq 2)$ the minor

$$A\begin{pmatrix} 1 & 2 & \cdots & p \\ 1 & 2 & \cdots & p \end{pmatrix} \qquad (1 \leqq p < n)$$

and we shall border it, adding any row and any column from the remaining n-p rows and n-p columns, i.e., forming the minors

$$b_{rs} \equiv A\begin{pmatrix} 1 & 2 & \cdots & p & r \\ 1 & 2 & \cdots & p & s \end{pmatrix}.$$