ELEMENTS OF SET THEORY

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PREFACE

This is an introductory undergraduate textbook in set theory. In mathematics these days, essentially everything is a set. Some knowledge of set theory is a necessary part of the background everyone needs for further study of mathematics. It is also possible to study set theory for its own interest—it is a subject with intriguing results about simple objects. This book starts with material that nobody can do without. There is no end to what can be learned of set theory, but here is a beginning.

The author of a book always has a preferred manner for using the book: A reader should simply study it from beginning to end. But in practice, the users of a book have their own goals. I have tried to build into the present book enough flexibility to accommodate a variety of goals.

The axiomatic material in the text is marked by a stripe in the margin. The purpose of the stripe is to allow a user to deemphasize the axiomatic material, or even to omit it entirely.

A course in axiomatic set theory might reasonably cover the first six or seven chapters, omitting Chapter 5. This is the amount of set theory that everyone with an interest in matters mathematical should know. Those with a special interest in set theory itself are encouraged to continue to the end of the book (and beyond). A very different sort of course might emphasize the set-theoretic construction of the number systems. This course might cover the first five chapters, devoting only as much attention to the axiomatic material as desired. The book presupposes no specific background. It does give real proofs. The first difficult proof is not met until part way through Chapter 4.

The hierarchical view of sets, constructed by transfinite iteration of the power set operation, is adopted from the start. The axiom of regularity is not added until it can be proved to be equivalent to the assertion that every set has a rank.

The exercises are placed at the end of each (or nearly each) section. In addition, Chapters 2, 3, and 4 have "Review Exercises" at the ends of the chapters. These are comparatively straightforward exercises for the reader wishing additional review of the material. There are, in all, close to 300 exercises.

There is a brief appendix dealing with some topics from logic, such as truth tables and quantifiers. This appendix also contains an example of how one might discover a proof.

At the end of this text there is an annotated list of books recommended for further study. In fact it includes diverse books for several further studies in a variety of directions. Those wishing to track down the source of particular results or historical points are referred to the books on the list that provide specific citations.

There are two stylistic matters that require mention. The end of a proof is marked by a reversed turnstile (⊣). This device is due to C. C. Chang and H. J. Keisler. In definitions, I generally pass up the traditionally correct "if" in favor of the logically correct "iff" (meaning "if and only if").

Two preliminary editions of the text have been used in my courses at UCLA. I would be pleased to receive comments and corrections from further users of the book.

LIST OF SYMBOLS

The number indicates the page on which the symbol first occurs in the text or the page on which it is defined.

Symbol	Page	Symbol	Page	Symbol	Page
$\in$	1	or	13	$\bigcap_{X \in \mathscr{A}}$	32
$\notin$	1	$\Rightarrow$	13	$\langle x, y \rangle$	35
$\{,\}$	1	$\Leftrightarrow$	13	$A \times B$	37
$=$	2	ZF	15	xRy	40
iff	2	VNB	15	I_S	40
$\varnothing$	2	GB	15	dom	40
$\cup$	3	$\{x \in c \mid \_\}$	21	ran	40
$\cap$	3	$A - B$	21	fld	40
$\subseteq$	3	ω	21	$\langle x, y, z \rangle$	41
$\mathscr{P}$	4	$\dashv$	22	$\langle x \rangle$	42
$\{x \mid \_\}$	4	$\bigcup$	23	$F(x)$	43
V_α	7	$\bigcap$	25	$F: A \to B$	43
$\forall$	13	$A \setminus B$	27	F^{-1}	44
$\exists$	13	$\mathbb{R}$	27	$F \circ G$	44
$\neg$	13	$-A$	28	$F \restriction A$	44
&	13	$\bigcup_{X \in \mathscr{A}}$	32	$F[\![A]\!]$	44

Symbol	*Page*	*Symbol*	*Page*	*Symbol*	*Page*
$[x, y]$	44	$<_Z$	98	E	182
$\bigcup_{i \in I}$	51	$\mathbb{Q}$	102	$\cong$	184
$\bigcap_{i \in I}$	51	0_Q	102	$<_L$	185
F_i	51	1_Q	102	$<$	189
${}^A B$	52	$+_Q$	103	Ω	194
$\times_{i \in I}$	54	$\cdot_Q$	105	$\aleph_1$	199
$[x]_R$	57	r^{-1}	107	V_α	200
$[x]$	57	$s \div r$	107	rank S	204
A/R	58	$<_Q$	108	$\aleph_\alpha$	212
0	67	$\|r\|$	109	$\beth_\alpha$	214
1	67	$\mathbb{R}$	113	sup S	216
2	67	$<_R$	113	it$\langle A, R\rangle$	220
3	67	$+_R$	114	kard S	222
a^+	68	0_R	115	$\oplus$	222
ω	69	$-x$	117	$\rho + \sigma$	222
σ	71	$\|x\|$	118	$\bar{\alpha}$	223
$\mathbb{Z}$	75	$\cdot_R$	118	η	223
$m + n$	79	1_R	119	ρ^*	224
A1	79	$\approx$	129	$R * S$	224
A2	79	(x, y)	130	$\rho \cdot \sigma$	224
$m \cdot n$	80	card S	136	$\alpha + \beta$	228
M1	80	$\aleph_0$	137	$\alpha \cdot \beta$	228
M2	80	$\kappa + \lambda$	139	$<_H$	228
m^n	80	$\kappa \cdot \lambda$	139	A3	229
E1	80	κ^λ	139	M3	229
E2	80	$\kappa!$	144	α^β	232
$\underline{\in}$	83	$\preccurlyeq$	145	E3	232
$\in_S$	83	$\kappa \leq \lambda$	145	ε_0	240
$\subset$	85	$\kappa < \lambda$	146	R^ι	243
$\mathbb{Z}$	91	Sq(A)	160	σ^M	249
$+_Z$	92	$\subset_S$	167	ZFC	253
0_Z	92	$\leq$	168	HF	256
$-a$	95	seg	173	cf λ	257
$\cdot_Z$	95	${}^{<A}B$	175	ssup S	262
1_Z	97	TC S	178	$\ulcorner\varphi\urcorner$	263

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CHAPTER 1

INTRODUCTION

BABY SET THEORY

We shall begin with an informal discussion of some basic concepts of set theory. In these days of the "new math," much of this material will be already familiar to you. Indeed, the practice of beginning each mathematics course with a discussion of set theory has become widespread, extending even to the elementary schools. But we want to review here elementary-school set theory (and do it in our notation). Along the way we shall be able to point out some matters that will become important later. We shall not, in these early sections, be particularly concerned with rigor. The more serious work will start in Chapter 2.

A *set* is a collection of things (called its *members* or *elements*), the collection being regarded as a single object. We write "$t \in A$" to say that t is a member of A, and we write "$t \notin A$" to say that t is not a member of A.

For example, there is the set whose members are exactly the prime numbers less than 10. This set has four elements, the numbers 2, 3, 5, and 7. We can name the set conveniently by listing the members within braces (curly brackets):

$$\{2, 3, 5, 7\}.$$

Call this set A. And let B be the set of all solutions to the polynomial equation

$$x^4 - 17x^3 + 101x^2 - 247x + 210 = 0.$$

Now it turns out (as the industrious reader can verify) that the set B has exactly the same four members, 2, 3, 5, and 7. For this reason A and B are the same set, i.e., $A = B$. It matters not that A and B were defined in different ways. Because they have exactly the same elements, they are equal; that is, they are one and the same set. We can formulate the general principle:

Principle of Extensionality If two sets have exactly the same members, then they are equal.

Here and elsewhere, we can state things more concisely and less ambiguously by utilizing a modest amount of symbolic notation. Also we abbreviate the phrase "if and only if" as "iff." Thus we have the restatement:

Principle of Extensionality If A and B are sets such that for every object t,

$$t \in A \quad \text{iff} \quad t \in B,$$

then $A = B$.

For example, the set of primes less than 10 is the same as the set of solutions to the equation $x^4 - 17x^3 + 101x^2 - 247x + 210 = 0$. And the set $\{2\}$ whose only member is the number 2 is the same as the set of even primes.

Incidentally, we write "$A = B$" to mean that A and B are the same object. That is, the expression "A" on the left of the equality symbol names the same object as does the expression "B" on the right. If $A = B$, then automatically (i.e., by logic) anything that is true of the object A is also true of the object B (it being the same object). For example, if $A = B$, then it is automatically true that for any object t, $t \in A$ iff $t \in B$. (This is the converse to the principle of extensionality.) As usual, we write "$A \neq B$" to mean that it is not true that $A = B$.

A small set would be a set $\{0\}$ having only one member, the number 0. An even smaller set is the empty set $\varnothing$. The set $\varnothing$ has no members at all. Furthermore it is the only set with no members, since extensionality tells us that any two such sets must coincide. It might be thought at first that the empty set would be a rather useless or even frivolous set to mention, but, in fact, from the empty set by various set-theoretic operations a surprising array of sets will be constructed.

For any objects x and y, we can form the pair set $\{x, y\}$ having just the members x and y. Observe that $\{x, y\} = \{y, x\}$, as both sets have exactly the same members. As a special case we have (when $x = y$) the set $\{x, x\} = \{x\}$.

For example, we can form the set $\{\varnothing\}$ whose only member is $\varnothing$. Note that $\{\varnothing\} \neq \varnothing$, because $\varnothing \in \{\varnothing\}$ but $\varnothing \notin \varnothing$. The fact that $\{\varnothing\} \neq \varnothing$ is reflected in the fact that a man with an empty container is better off than a man with nothing—at least he has the container. Also we can form $\{\{\varnothing\}\}$, $\{\{\{\varnothing\}\}\}$, and so forth, all of which are distinct (Exercise 2).

Similarly for any objects x, y, and z we can form the set $\{x, y, z\}$. More generally, we have the set $\{x_1, \ldots, x_n\}$ whose members are exactly the objects $x_1, \ldots, x_n$. For example,

$$\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$$

is a three-element set.

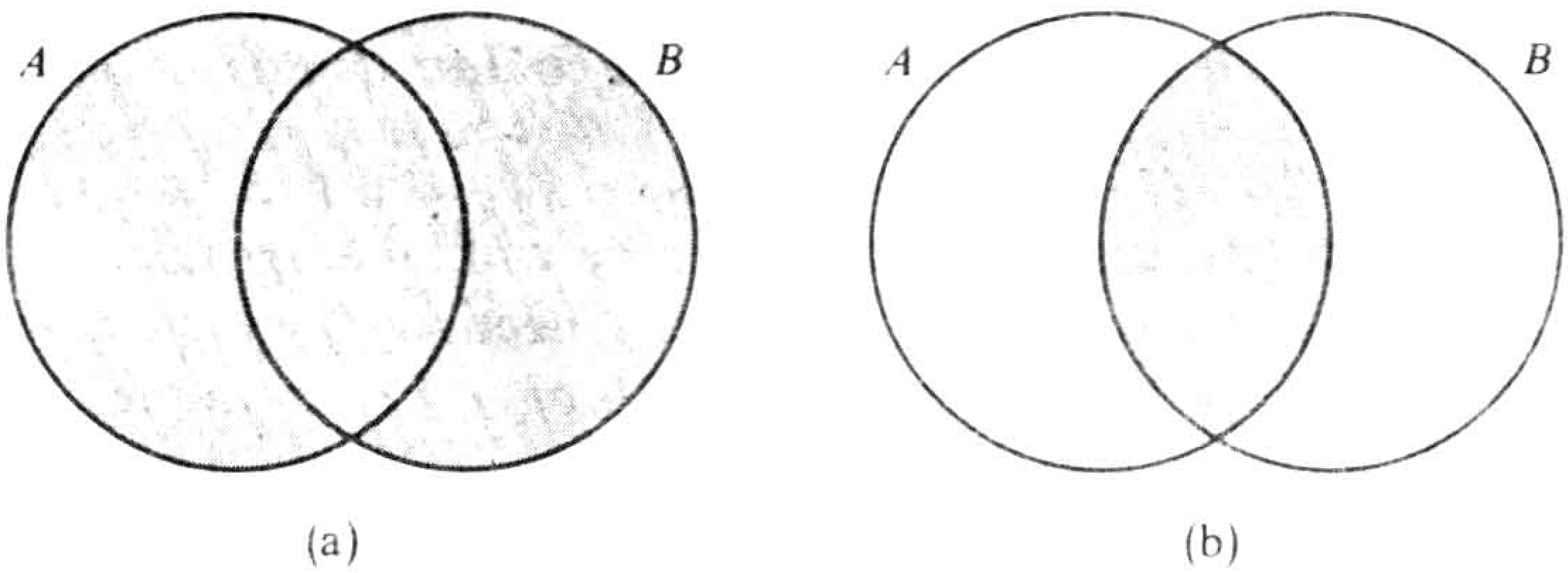

Fig. 1. The shaded areas represent (a) $A \cup B$ and (b) $A \cap B$.

Two other familiar operations on sets are union and intersection. The *union* of sets A and B is the set $A \cup B$ of all things that are members of A or B (or both). Similarly the *intersection* of A and B is the set $A \cap B$ of all things that are members of both A and B. For example,

$$\{x, y\} \cup \{z\} = \{x, y, z\}$$

and

$$\{2, 3, 5, 7\} \cap \{1, 2, 3, 4\} = \{2, 3\}.$$

Figure 1 gives the usual pictures illustrating these operations. Sets A and B are said to be *disjoint* when they have no common members, i.e., when $A \cap B = \varnothing$.

A set A is said to be a *subset* of a set B (written $A \subseteq B$) iff all the members of A are also members of B. Note that any set is a subset of itself. At the other extreme, $\varnothing$ is a subset of every set. This fact (that $\varnothing \subseteq A$ for any A) is "vacuously true," since the task of verifying, for every member of $\varnothing$, that it also belongs to A, requires doing nothing at all.

If $A \subseteq B$, then we also say that A is *included* in B or that B *includes* A. The inclusion relation ($\subseteq$) is not to be confused with the membership relation ($\in$). If we want to know whether $A \in B$, we look at the set A as a single object, and we check to see if this single object is among the members of B. By contrast, if we want to know whether $A \subseteq B$, then we must open up the set A, examine its various members, and check whether its various members can be found among the members of B.

Examples 1. $\varnothing \subseteq \varnothing$, but $\varnothing \notin \varnothing$.

2. $\{\varnothing\} \in \{\{\varnothing\}\}$ but $\{\varnothing\} \nsubseteq \{\{\varnothing\}\}$. $\{\varnothing\}$ is not a subset of $\{\{\varnothing\}\}$ because there is a member of $\{\varnothing\}$, namely $\varnothing$, that is not a member of $\{\{\varnothing\}\}$.

3. Let Us be the set of all people in the United States, and let Un be the set of all countries belonging to the United Nations. Then

$$\text{John Jones} \in Us \in Un.$$

But John Jones $\notin Un$ (since he is not even a country), and hence $Us \nsubseteq Un$.

Any set A will have one or more subsets. (In fact, if A has n elements, then A has 2^n subsets. But this is a matter we will take up much later.) We can gather all of the subsets of A into one collection. We then have the set of all subsets of A, called the *power*[1] *set* $\mathscr{P}A$ of A. For example,

$$\mathscr{P}\varnothing = \{\varnothing\},$$
$$\mathscr{P}\{\varnothing\} = \{\varnothing, \{\varnothing\}\},$$
$$\mathscr{P}\{0, 1\} = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}.$$

A very flexible way of naming a set is the method of *abstraction*. In this method we specify a set by giving the condition—the entrance requirement—that an object must satisfy in order to belong to the set. In this way we obtain the set of all objects x such that x meets the entrance requirement. The notation used for the set of all objects x such that the condition $\_\_ x \_\_$ holds is

$$\{x \mid \_\_ x \_\_\}.$$

For example:

1. $\mathscr{P}A$ is the set of all objects x such that x is a subset of A. Here "x is a subset of A" is the entrance requirement that x must satisfy in order to belong to $\mathscr{P}A$. We can write

$$\mathscr{P}A = \{x \mid x \text{ is a subset of } A\}$$
$$= \{x \mid x \subseteq A\}.$$

[1] The reasons for using the word "power" in this context are not very convincing, but the usage is now well established.

2. $A \cap B$ is the set of all objects y such that $y \in A$ and $y \in B$. We can write

$$A \cap B = \{y \mid y \in A \text{ and } y \in B\}.$$

It is unimportant whether we use "x" or "y" or another letter as the symbol (which is used as a pronoun) here.

3. The set $\{z \mid z \neq z\}$ equals $\varnothing$, because the entrance requirement "$z \neq z$" is not satisfied by any object z.

4. The set $\{n \mid n \text{ is an even prime number}\}$ is the same as the set $\{2\}$.

There are, however, some dangers inherent in the abstraction method. For certain bizarre choices of the entrance requirement, it may happen that there is no set containing exactly those objects meeting the entrance requirement. There are two ways in which disaster can strike.

One of the potential disasters is illustrated by

$$\{x \mid x \text{ is a positive integer definable in one line of type}\}.$$

The tricky word here is "definable." Some numbers are easy to define in one line. For example, the following lines each serve to define a positive integer:

12,317,
the millionth prime number,
the least number of the form $2^{2^n} + 1$ that is not prime,
the 23rd perfect[2] number.

Observe that there are only finitely many possible lines of type (because there are only finitely many symbols available to the printer, and there is a limit to how many symbols will fit on a line). Consequently

$$\{x \mid x \text{ is a positive integer definable in one line of type}\}$$

is only a finite set of integers. Consider the least positive integer not in this set; that is, consider

the least positive integer not definable in one line of type.

The preceding line defines a positive integer in one line, but that number is, by its construction, not definable in one line! So we are in trouble, and the trouble can be blamed on the entrance requirement of the set, i.e., on the phrase "is a positive integer definable in one line of type." While it may have

[2] A positive integer is *perfect* if it equals the sum of its smaller divisors, e.g., $6 = 1 + 2 + 3$. It is *deficient* (or *abundant*) if the sum of its smaller divisors is less than (or greater than, respectively) the number itself. This terminology is a vestigial trace of numerology, the study of the mystical significance of numbers. The first four perfect numbers are 6, 28, 496, and 8128.

appeared originally to be a meaningful entrance requirement, it now appears to be gravely defective. (This example was given by G. G. Berry in 1906. A related example was published in 1905 by Jules Richard.)

There is a second disaster that can result from an overly free-swinging use of the abstraction method. It is exemplified by

$$\{x \mid x \notin x\},$$

this is, by the set of all objects that are not members of themselves. Call this set A, and ask "is A a member of itself?" If $A \notin A$, then A meets the entrance requirement for A, whereupon $A \in A$. But on the other hand, if $A \in A$, then A fails to meet the entrance requirement and so $A \notin A$. Thus both "$A \in A$" and "$A \notin A$" are untenable. Again, we are in trouble. The phrase "is not a member of itself" appears to be an illegal entrance requirement for the abstraction method. (This example is known as Russell's paradox. It was communicated by Bertrand Russell in 1902 to Gottlob Frege, and was published in 1903. The example was independently discovered by Ernst Zermelo.)

These two sorts of disaster will be blocked in precise ways in our axiomatic treatment, and less formally in our nonaxiomatic treatment. The first sort of disaster (the Berry example) will be avoided by adherence to entrance requirements that can be stated in a totally unambiguous form, to be specified in the next chapter. The second sort of disaster will be avoided by the distinction between *sets* and *classes*. Any collection of sets will be a *class*. Some collections of sets (such as the collections $\varnothing$ and $\{\varnothing\}$) will be sets. But some collections of sets (such as the collection of all sets not members of themselves) will be too large to allow as sets. These oversize collections will be called *proper classes*. The distinction will be discussed further presently.

In practice, avoidance of disaster will not really be an oppressive or onerous task. We will merely avoid ambiguity and avoid sweepingly vast sets. A prudent person would not want to do otherwise.

Exercises

1. Which of the following become true when "$\in$" is inserted in place of the blank? Which become true when "$\subseteq$" is inserted?

(a) $\{\varnothing\}$ — $\{\varnothing, \{\varnothing\}\}$.
(b) $\{\varnothing\}$ — $\{\varnothing, \{\{\varnothing\}\}\}$.
(c) $\{\{\varnothing\}\}$ — $\{\varnothing, \{\varnothing\}\}$.
(d) $\{\{\varnothing\}\}$ — $\{\varnothing, \{\{\varnothing\}\}\}$.
(e) $\{\{\varnothing\}\}$ — $\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$.

2. Show that no two of the three sets $\varnothing$, $\{\varnothing\}$, and $\{\{\varnothing\}\}$ are equal to each other.

3. Show that if $B \subseteq C$, then $\mathscr{P}B \subseteq \mathscr{P}C$.

4. Assume that x and y are members of a set B. Show that $\{\{x\}, \{x, y\}\} \in \mathscr{P}\mathscr{P}B$.

SETS—AN INFORMAL VIEW

We are about to present a somewhat vague description of how sets are obtained. (The description will be repeated much later in precise form.) None of our later work will actually depend on this informal description, but we hope it will illuminate the motivation behind some of the things we will do.

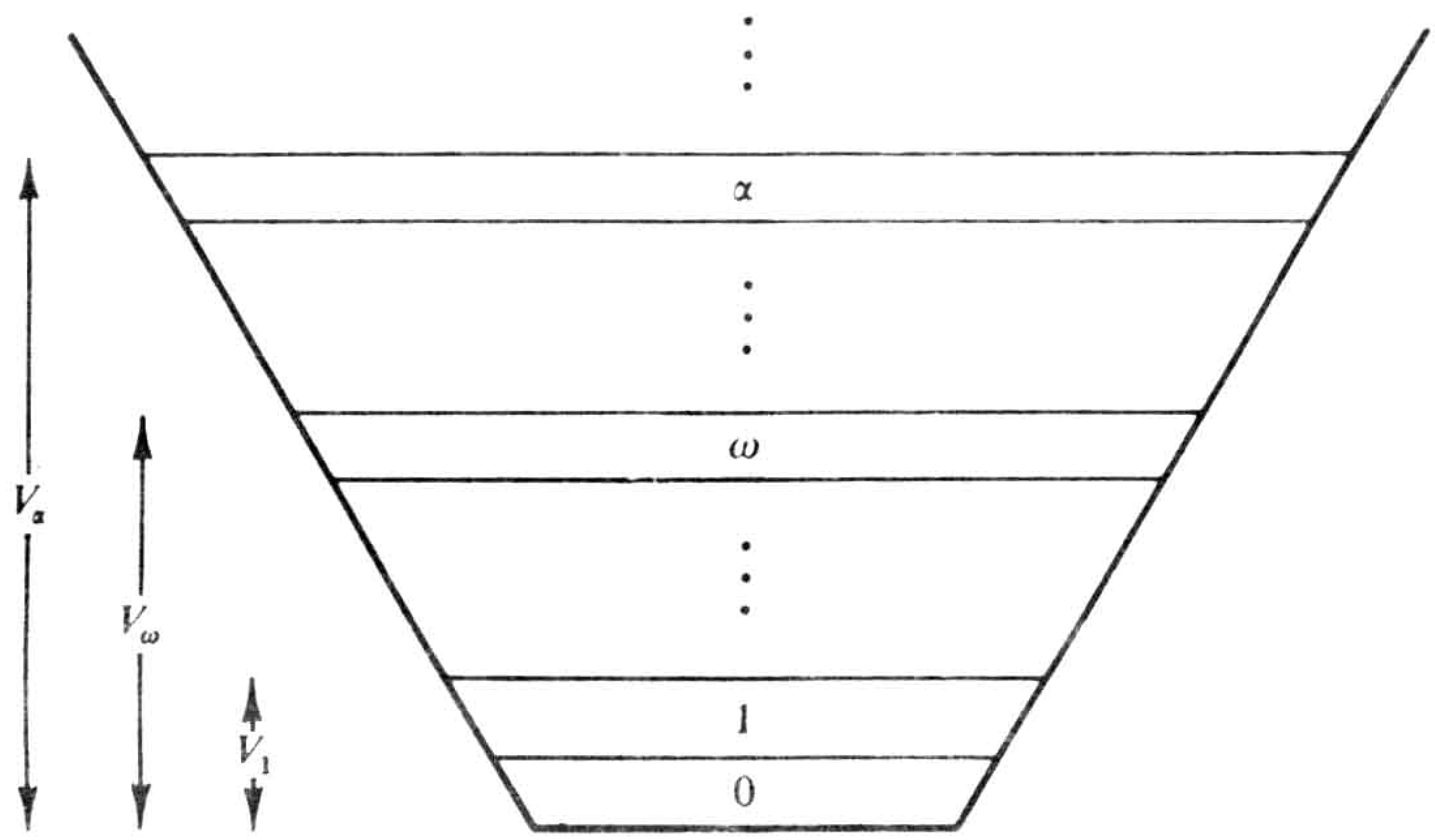

Fig. **2.** V_0 is the set A of atoms.

First we gather together all those things that are not themselves sets but that we want to have as members of sets. Call such things *atoms.* For example, if we want to be able to speak of the set of all two-headed coins, then we must include all such coins in our collection of atoms. Let A be the set of all atoms; it is the first set in our description.

We now proceed to build up a hierarchy

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$$

of sets. At the bottom level (in a vertical arrangement as in Fig. 2) we take $V_0 = A$, the set of atoms. The next level will also contain all sets of atoms:

$$V_1 = V_0 \cup \mathscr{P}V_0 = A \cup \mathscr{P}A.$$