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Edited by A. Dold and B. Eckmann

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Annie Cuyt

Padé Approximants for Operators: Theory and Applications

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To my darling husband.

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LIST OF NOTATIONS

	Significance
X	11
Y	11
O	2
I	2
Λ	1
$\otimes$	4
$F, G, \ldots$	nonlinear operators
$L(X^k, Y)$	3
$P, Q, R, S, T, U, V, W, \ldots$	abstract polynomials
∂P	6
$\partial_o P$	6
$F^{(k)}(x_o), P^{(k)}(x_o), \ldots$	6
$D(F)$	13
$B(x_o, r)$	5
$O(x^j)$	13
$C_k x^k$	15
$\frac{1}{Q_\star} . P_\star$	13
$\sim$	14
$P_{[n,m]}(x)$	17
$Q_{[n,m]}(x)$	17
$F_i(x)$	17
$\overline{F}_i(x)$	17

SUMMARY

In chapter I the concept of Padé approximants is generalized for nonlinear operators $F: X \to Y$ where X is a Banach space and Y is a commutative Banach algebra, starting from analyticity as is done in the classical theory. The generalization is such that the classical univariate Padé approximant ($X = \mathbb{R} = Y$) is a special case of the theory. We discuss the existence and unicity of a solution of the Padé-approximation problem of order (n,m) for F and prove that a lot of the properties for univariate Padé approximants remain valid: several covariance properties, recurrence relations, the epsilon algorithm , the qd-algorithm, the structure of the Padé table, criteria for regularity and normality of an entry of the Padé table. We are also able to prove a projection property and a product property.
In chapter II the multivariate Padé approximants ($X = \mathbb{R}^p$, $Y = \mathbb{R}$) are studied more extensively. We prove for instance the nontriviality of a solution of the Padé-approximation problem and the near-Toeplitz structure of the homogeneous system of equations. Also an extra covariance property and more recurrence relations are formulated. The multivariate Padé approximants introduced here are compared with other definitions of Padé approximants for multivariate functions given by different authors in the last few years. Our definition turns out to be an interesting generalization too.
Most of the applications are discussed in chapter III, except the acceleration of convergence of a table with multiple entry which is done by means of multivariate Padé approximants and therefore added to chapter II.
As far as the nonlinear operator equations are concerned, we treat the solution of nonlinear systems of equations, initial value problems, boundary value problems, partial differential equations and integral equations. An interesting procedure, especially in the neighbourhood of singularities, is the Halley-iteration which is newly introduced here. Its numerical stability for the solution of a system of nonlinear equations is formulated at the end of chapter III.

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CHAPTER I: ABSTRACT PADE APPROXIMANTS IN OPERATOR THEORY

§ 1. MOTIVATION

Padé approximants are a frequently used tool for the solution of mathematical problems: the solution of a nonlinear equation, the acceleration of convergence, numerical integration by means of nonlinear techniques, the solution of ordinary and partial differential equations. In the neighbourhood of singularities the use of Padé approximants can be very interesting.
Many attempts have been made to generalize the concept of Padé approximants in some sense; we refer to definitions of multivariate Padé approximants by Bose [7], Chisholm [11, 12, 13, 14], Karlsson and Wallin [32], Levin [34] and Lutterodt [37], to quadratic approximants and their generalizations [45, 21], to operator Padé approximants for formal power series in a parameter with non-commuting elements of a certain algebra as coefficients [4], to matrix-valued Padé approximants [3, 46], to Padé approximants for the operator exponential [17] and so on.

It would be important to generalize the concept of Padé approximants for nonlinear operators, following the ideas of the classical theory, for this would enable us to prove a lot of the classical properties for the generalizations as well and it would also enable us to use those generalizations for the solution of nonlinear operator equations. these are more general problems than the ones we solved with the aid of univariate Padé approximants; we mention nonlinear systems of equations, nonlinear initial value and boundary value problems, nonlinear partial differential equations and nonlinear integral equations.
Such a generalization is treated here.

§ 2. INTRODUCTION

2.1. *Banach spaces and Banach algebras*

In ordinary analysis we work with the real or complex number system. Here we shall work in complete normed spaces which are generalizations of these number systems. Since linear spaces may consist of such interesting mathematical objects as vectors with a finite or infinite number of components or functions that satisfy given conditions, we shall be able to deal with a wide variety of problems.
In abstract terms, a linear vector space X over the scalar field Λ (where Λ is $\mathbb{R}$ or $\mathbb{C}$) is a set of elements with two operations, called addition and scalar multiplication, which satisfy certain conditions:

a) the set X is a commutative group with respect to the operation of addition (we shall denote the unit for the addition by 0)

b) for any scalars λ, μ in Λ and any elements x, y in X, the following rules hold:

$$\lambda x \in X$$
$$1.x = x$$
$$o.x = 0$$
$$(\lambda+\mu)x = \lambda x+\mu x$$
$$\lambda(x+y) = \lambda x+\lambda y$$

The algebraic structure of a linear space is similar to that of the real or complex number system. However, to deal with other concepts of theoretical and computational importance, such as accuracy of approximation, convergence of sequences, and so on, it is necessary to introduce additional structure into such spaces.

X is called a normed linear space if for each element x in X, a finite non-negative real number $\|x\|$, called the norm of x, is defined and the following conditions are satisfied:

a) $\|x\| = o$ if and only if $x = 0$

b) $\|\lambda x\| = |\lambda| \; \|x\|$

c) $\|x+y\| \leq \|x\| + \|y\|$

In the solution of many problems the basic issue is the existence of a limit $x^\star$ of an infinite sequence $\{x_i\}$ of elements of X. A normed linear space X is said to be complete if every Cauchy sequence of elements of X converges to a limit which is an element of X. Such a complete normed linear space is called a Banach space.

Some Banach spaces have the property that the product xy of two elements of the space is defined and is also an element of the space. Such a Banach space is called a Banach algebra if

$$\|xy\| \leq \|x\| . \|y\|$$

A Banach algebra is said to be commutative if

$$xy = yx$$

and we say that it has a unit for the multiplication, which we shall denote by I, if

$$x.I = x = I.x$$

The spaces $\mathbb{R}^p$ and $\mathbb{C}^p$ for example are Banach algebras with unit if the multiplication is defined component-wise.

2.2. *Linear and multilinear operators*

Many mathematical operations which transform one vector or function into another have certain simple algebraic properties. We shall now discuss such operators.

An operator L which maps a linear space X into a linear space Y over the same scalar field Λ so that for each x in X there is a uniquely defined element Lx in Y, is called linear if it is

a) additive: $L(x_1+x_2) = Lx_1 + Lx_2$

b) homogeneous: $L(\lambda x) = \lambda Lx$

If $X=\mathbb{R}^p$ and $Y=\mathbb{R}^q$ then a linear operator L has a unique representation as a qxp matrix. Another example of a linear operator is furnished by differentiation; the operator $D=\frac{d}{dt}$ maps $X=C'([0,1])$ into $Y=C([0,1])$ with

$$Dx(t) = \frac{dx}{dt} = y(t)$$

If X and Y are linear spaces over a common scalar field Λ, then the set of all linear operators from X into Y becomes a linear space over Λ if addition is defined by

$$(L_1+L_2)x = L_1x+L_2x$$

and scalar multiplication by

$$(\lambda L)x = \lambda(Lx)$$

The norm of a linear operator L is defined by

$$\|L\| = \sup_{\|x\|=1} \|Lx\|$$

and the operator L is called bounded if $\|L\| < \infty$.

We know that a continuous linear operator L from a Banach space X into a Banach space Y is bounded [41 pp. 38] and also that

$$\|Lx\| \le \|L\|.\|x\|$$

Clearly the set L(X,Y) of all bounded linear operators from a Banach space X into a Banach space Y is a Banach space itself. So we may consider linear operators which map X into L(X,Y). For such an operator B and for x_1 and x_2 in X, we would have

$$Bx_1 = L$$

a linear operator from X into Y, and

$$Bx_1x_2 = (Bx_1)x_2$$

an element of Y.

The operator B is called a bilinear operator from X into Y. Since the bounded linear operators from X into L(X,Y) form themselves a linear space L(X,L(X,Y)) which we shall denote by $L(X^2,Y)$, the foregoing process could be repeated, leading to a whole hierarchy of linear operators and spaces. These classes of operators play a fundamental role in the differential calculus in Banach spaces.

A k-linear operator L on X is an operator $L:X^k \to Y$ which is linear and homogeneous in each of its arguments separately. If $x_1=\ldots=x_k=x$, we shall use the notation

$$Lx^k = Lx_1\ldots x_k$$

We write $L(X^k,Y)$ for the set of all bounded k-linear operators from X into Y.

We define a o-linear operator on X to be a constant function, i.e. for y fixed in Y, we have

$$Lx = y \text{ for all } x \text{ in } X$$

The set $L(X^o,Y)$ is identified with Y.

If $L \in L(X^k,Y)$ and $x_1,\ldots,x_\ell \in X$ with $k \ge \ell \ge 1$ then

$$Lx_1\ldots x_\ell$$

is a bounded $(k-\ell)$-linear operator.

In general the elements $Lx_1 \dots x_k$ and $Lx_{i_1} \dots x_{i_k}$ with $(x_1,\dots,x_k)$ in X^k and $(i_1,\dots,i_k)$ a permutation of $(1,\dots,k)$ are different, so that actually k! k-linear operators are associated with a given k-linear operator L.
But if

$$L\, x_1 \dots x_k = L\, x_{i_1} \dots x_{i_k}$$

for all $(x_1,\dots,x_k)$ in X^k and for all permutations $(i_1,\dots,i_k)$ of $(1,\dots,k)$ [41 pp. 103-104] then the k-linear bounded operator L is called symmetric.
If Y is a Banach algebra, multilinear operators can also be obtained by forming tensor-products.

Definition I.2.1.:

Let $F : X \to Y$ and $G : X \to Y$ be operators.

The product F.G is defined by $(F.G)(x) = F(x).G(x)$ in Y.

Definition I.2.2.:

Let $X_1, \dots, X_p, Z_1, \dots, Z_q$ be vector spaces and let

$F : X_1 \times \dots \times X_p \to Y$ be bounded and p-linear and

$G : Z_1 \times \dots \times Z_q \to Y$ be bounded and q-linear.

The tensorproduct $F \otimes G : X_1 \times \dots \times X_p \times Z_1 \times \dots \times Z_q \to Y$

is bounded and (p+q)-linear when defined by

$(F \otimes G)\, x_1 \dots x_p\, z_1 \dots z_q = F\, x_1 \dots x_p \,.\, Gz_1 \dots z_q$

[23 pp. 318].

2.3. *Fréchet-derivatives*

An operator F from X into Y is called nonlinear if it is not a linear operator.
Now suppose that F is an operator that maps a Banach space X into a Banach space Y.
If L in L(X,Y) exists such that

$$\lim_{\|\Delta x\| \to o} \frac{\| F(x_o + \Delta x) - F(x_o) - L\Delta x \|}{\|\Delta x\|} = o$$

then F is said to be Fréchet-differentiable at x_o, and the bounded linear operator

$$L = F'(x_o)$$

is called the first Fréchet-derivative of F at x_o.
Note that the classical rules for differentiation, like the chain rule still hold for Fréchet differentiation. In practice, to differentiate a given nonlinear operator F, we attempt to write the difference $F(x_o+\Delta x) - F(x_o)$ in the form

$$F(x_o+\Delta x) - F(x_o) = L(x_o,\Delta x)\Delta x + \eta(x_o,\Delta x)$$

where $L(x_o,\Delta x)$ is a bounded linear operator for given x_o and Δx with

$$\lim_{\|\Delta x\|\to o} L(x_o,\Delta x) = L \in L(X,Y)$$

and

$$\lim_{\|\Delta x\|\to o} \frac{\|\eta(x_o,\Delta x)\|}{\|\Delta x\|} = o$$

To illustrate this process, consider the operator F in $C([0,1])$ defined by

$$F(x) = x(t) \int_0^1 \frac{t}{s+t} x(s)\, ds \qquad 0 \le t \le 1$$

The difference $F(x_o+\Delta x) - F(x_o)$ equals

$$x_o(t) \int_0^1 \frac{t}{s+t} \Delta x(s)\, ds + \Delta x(t) \int_0^1 \frac{t}{s+t} x_o(s)\, ds + \Delta x(t) \int_0^1 \frac{t}{s+t} \Delta x(s)\, ds$$

So the operator $L(x_o,\Delta x)$ equals

$$x_o(t) \int_0^1 \frac{t}{s+t}[\ ]\, ds + [\ ] \int_0^1 \frac{t}{s+t} x_o(s)\, ds + [\ ] \int_0^1 \frac{t}{s+t} \Delta x(s)\, ds$$

where [ ] is a place holder and is used to indicate the position of the argument of the operator $L(x_o,\Delta x)$.

Now $L(x_o,\Delta x)$ is a continuous function of Δx; so we may set $\Delta x = 0$ to obtain $F'(x_o) = L$:

$$F'(x_o) = x_o(t) \int_0^1 \frac{t}{s+t}[\ ]\, ds + [\ ] \int_0^1 \frac{t}{s+t} x_o(s)\, ds$$

where now [ ] indicates the position of the argument of the linear operator $F'(x_o)$.

Suppose that an operator F from X into Y is differentiable at x_o and also at every point of the open ball $B(x_o,r)$ with centre x_o and radius $r > o$. For each x in $B(x_o,r)$ $F'(x_o)$ will be an element of the space $L(X,Y)$. Consequently F' may be considered to be an operator defined in a neighbourhood of x_o. We know that F' will be differentiable at x_o if a bounded linear operator B from X into $L(X,Y)$ exists such that

$$\lim_{\|\Delta x\|\to o} \frac{\|F'(x_o+\Delta x) - F'(x_o) - B\Delta x\|}{\|\Delta x\|} = o$$

Such a bounded linear operator B is known to be a bilinear operator and if it exists, it is called the second derivative of F at x_o and denoted by $F''(x_o) = B$. Thus the second derivative of an operator F is obtained by differentiating its first derivative F'. Now it is possible to give an inductive definition of higher derivatives of an operator F.

2.4. *Abstract polynomials*

If L is a k-linear operator from a Banach space X into a Banach algebra Y, then the operator P from X into Y defined by

$$P(x) = Lx^k \text{ for } x \text{ in } X$$

is a nonlinear operator. In this way we can define abstract polynomials.

Definition I.2.3.:

An <u>abstract polynomial</u> is a nonlinear operator $P : X \to Y$ such that

$P(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_o$ with

$A_i \in L(X^i, Y)$ and A_i symmetric [41 p. 107].

The degree of $P(x)$ is n. We also introduce the following notations.
If there exists a positive integer j_1 such that for all $o \le k < j_1$: $A_k x^k \equiv 0$ and $A_{j_1} x^{j_1} \not\equiv 0$ then $\partial_o P = j_1$ is called the order of the abstract polynomial P.
If there exists a positive integer j_2 such that for all $j_2 < k \le n$: $A_k x^k \equiv 0$ and $A_{j_2} x^{j_2} \not\equiv 0$ then $\partial P = j_2$ is called the exact degree of the abstract polynomial P.
Abstract polynomials are differentiated as in elementary calculus: if $P(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_o$ then the Fréchet-derivatives of P at x_o are

$$P'(x_o) = n\, A_n\, x_o^{n-1} + \ldots + 2\, A_2\, x_o + A_1 \in L(X, Y)$$

$$P^{(2)}(x_o) = n(n-1)\, A_n\, x_o^{n-2} + \ldots + 2\, A_2 \in L(X^2, Y)$$

$$\vdots$$

$$P^{(n)}(x_o) = n!\, A_n \in L(X^n, Y)$$

We emphasis the fact that for an operator $F: X \to Y$, the k^{th} Fréchet-derivative at x_o, $F^{(k)}(x_o)$, is a symmetric k-linear and bounded operator [41 pp. 110]. Examples of abstract polynomials and k^{th} Fréchet-derivatives of a nonlinear operator can be found in § 3. of this chapter.
We can easily prove the following important lemmas for abstract polynomials.

Lemma I.2.1.:

Let the abstract polynomial P be given by $P(x) = \sum_{i=o}^{n} A_i x^i$.

If $P(x) \equiv 0$ then $A_i = 0$ for $i = o,\ldots,n$.

Lemma I.2.2.:

Let V be an abstract polynomial and U a continuous operator with $D(U) \neq \emptyset$.

If $U(x).V(x) \equiv 0$ then $V(x) \equiv 0$.

Proof:

Since $D(U) \neq \emptyset$, we can find x_o in X such that $U(x_o)$ is regular.

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