Lecture Notes in Mathematics

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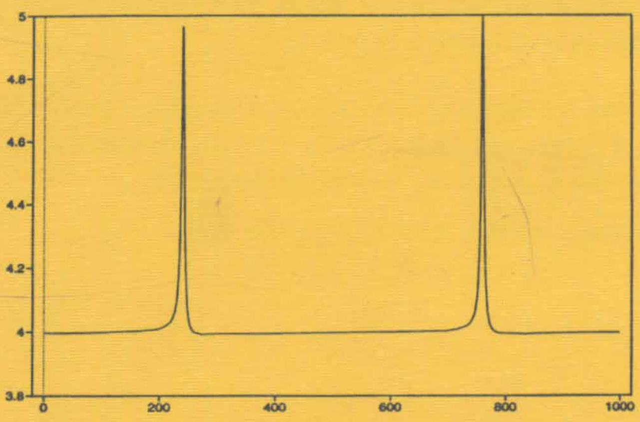
Lars B. Wahlbin

Superconvergence in Galerkin Finite Element Methods



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Preface.

These notes are from a graduate seminar at Cornell in Spring 1994. They are devoted mainly to basic concepts of superconvergence in second–order time–independent elliptic problems.

A brief chapter–by–chapter description is as follows: Chapter 1 considers one–dimensional problems and is intended to get us moving quickly into the subject matter of superconvergence. (The results in Sections 1.8 and 1.10 are new.) Some standard results and techniques used there are then expounded on in Chapters 2 and 3. Chapter 4 gives a few selected results about superconvergence in L_2–projections in any number of space dimensions. In Chapter 5 we elucidate local maximum–norm error estimates in second order elliptic partial differential equations and the techniques used in proving them, without aiming for complete detail. Theorems 5.5.1 and 5.5.2 are basic technical results; they will be used over and over again in the rest of the notes.

In Chapters 6 through 12 we treat a variety of topics in superconvergence for second order elliptic problems. Some are old and established, some are very recent and not yet published. Some of the earlier contributions have benefitted from later sharpening of tools, in particular with respect to local maximum–norm estimates.

In Chapter 6 we consider tensor–product elements. Using ideas of [Douglas, Dupont and Wheeler 1974b] we show that in some situations one–dimensional superconvergence results automatically translate to several dimensions. Chapter 7, "Superconvergence by local symmetry", presents recent fundamental results from [Schatz, Sloan and Wahlbin 1994]. Chapter 8 treats difference quotients for approximating derivatives of any order on translation invariant meshes. Here we follow the basic ideas of [Nitsche and Schatz 1974, Section 6]. In Chapter 9 we briefly comment on how, in many cases, results about superconvergence in linear problems automatically carry over to nonlinear problems. The essential idea is from [Douglas and Dupont 1975], which in turn is essentially the quadratic convergence of Newton's method. Chapter 10 is concerned with superconvergence on curved meshes which come about via isoparametric mappings of straight–lined meshes; it is based on [Cayco, Schatz and Wahlbin 1994]. Chapter 11 reverts back to the seventies. It is mainly concerned with the K–operator of [Bramble and Schatz 1974]; the presentation follows [Thomée 1977]. We give an application to boundary integral equations, [Tran 1993]. Also, we briefly mention a method for obtaining higher order accuracy in outflow derivatives, [Douglas, Dupont and Wheeler 1974a], and an averaging method of [Louis 1979]. Finally, in Chapter 12, we review the computational investigation of [Babuška, Strouboulis, Upadhyay and Gangaraj 1993] and comment on it in light of the theories of Chapters 6 and 7.

Previous treatises of superconvergence are [Chen 1982a] and [Zhu and Lin 1989]. Both are in Chinese and, although there is a description of Zhu and Lin's book in Mathematical Reviews, it is hard for me to judge how they compare with the present account. The use of local maximum–norm estimates seems common to all three. *For what appears to be a major difference of approach, see Remark 7.4.i.*

Surveys of superconvergence with a more limited scope have appeared in [Křížek and Neittaanmäki 1987a] and [Wahlbin 1991, Chapter VII].

Let me next list some topics that are *not* included in these notes. The first is superconvergence in collocation finite element methods for differential equations. For this topic I have not even included references; the reader is referred to [Křížek and Neittaanmäki 1987a]. The second omitted topic is superconvergence in boundary integral methods or in integral equations (except for Section 11.5). Third, extrapolation methods, involving computation on two or more meshes, and fourth, the use of superconvergence in construction of smooth stress fields and the related use of superconvergence for a posteriori error estimation and adaptive refinement. For the second through fourth topics I have included a fair number of references so that the interested reader may, by glancing through the list of references, easily gain an inroad to the literature on the subjects. Finally, I have not considered superconvergence in Galerkin finite element methods for time-dependent problems. Here, though, I have included all references that I know of.

The references are likewise "complete" with respect to the mathematical literature for the main topics treated. Of course, many of these references touch only briefly on superconvergence. In preparing the references I have used Mathsci ($\simeq$ Mathematical Reviews since 1972 on line) which does not systematically cover the vast engineering literature. The number of references given is more than it makes sense to actually refer to in the text, unless one resorts to plain listing, which I have not done. I hope that nevertheless some readers will find the list of references useful. (As an example, if a reader interested in early history wants to find references to papers on superconvergence in finite element methods before 1970, our list gives: [Stricklin 1966], [Filho 1968], [Stricklin 1968], [Oganesyan and Rukhovetz 1969] and [Tong 1969].)

Basic discoveries continue to be made at present. Furthermore, the theory of superconvergence is very immature in carrying results up close to boundaries (or, internal lines of discontinuity). Today, such investigations are carried out on a case-by-case basis, and, of course, not all results hold all the way up to boundaries (see Section 1.7). Most results that are proven up to boundaries in several dimension pertain to axes-parallel parallelepipedes, or, locally, to straight boundaries. For these and other reasons I have decided to offer these notes essentially as they were written week-by-week during the seminar, rather than rework them into "text-book" form; such a textbook would most likely be out-of-date when it appears. The reader may be warned that, reflecting my lecturing style, proofs often appear before theorems; indeed, the "theorem" may be only an informal statement. Also, true to the principle that repetition is the mother of studies, there is a fair amount of such. E.g., symmetry considerations are first met with in two-point boundary value problems, then in L_2-projections and finally in multidimensional elliptic problems; difference quotients first ocur in two-point boundary value problems and later in many dimensions; and, tensor product elements are considered as well for L_2-projections as for elliptic problems.

Let me penultimately remark that there are three concepts which may be confused with superconvergence. "Supraconvergence" is a concept in finite-difference theory for irregular meshes, cf. [Kreiss, Manteuffel, Swartz, Wendroff and White 1986], [Manteuffel and White 1986], [Heinrich 1987, p. 107], and cf. also [Bramble 1970]. "Superapproximation" will be explained in these notes and "superlinear convergence" occurs in the theory of iterative methods, cf. e.g. [Ortega and Rheinboldt 1979, pp. 285 and 291].

Finally, I thank Arletta Havlik for her superior typing, Al Schatz for many stimulating discussions, Stig Larsson for timely help and the National Science Foundation for financial support.

Ithaca, January 1995
Lars B. Wahlbin

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simple in this chapter we make the following explicit assumption of quasi-uniformity of meshes, except in the case of continuous elements ($\mu = 0$) when no such restriction is made.

(1.1.7) If $\mu \geq 1$, then there exists a positive constant C_{QU} independent of h such that in any subdivision τ_h, we have $h \leq C_{QU} \min_i h_i$.

We shall assume that (1.1.3) has a unique solution u for any f in $L_2(I)$, say. Following [Schatz 1974], cf. also [Hildebrandt and Weinholz 1964] and [Schatz and Wang 1994], we then know that there exists $h_0 > 0$ such that for $h \leq h_0$, given any $f \in L_2(I)$ there is a unique solution u_h to (1.1.6). Furthermore, under our general conditions on smooth enough data, we have the following estimates for the error $e = u - u_h$:

$$\|e\|_{L_2(I)} + h\|e\|_{H^1(I)} \leq Ch^r\|u\|_{H^r(I)} \tag{1.1.8}$$

and

$$\|e\|_{H^{-s}(I)} \leq Ch^{r+s}\|u\|_{H^r(I)}, \text{ for } s \leq r-2. \tag{1.1.9}$$

Here

$$\|v\|_{H^{-s}(I)} = \sup_{\substack{w \in H^s(I) \\ \|w\|_{H^s(I)}=1}} (v, w). \tag{1.1.10}$$

Correspondingly we have in maximum–norm,

$$\|e\|_{L_\infty(I)} + h\|e\|_{W^1_\infty(I)} \leq Ch^r\|u\|_{W^r_\infty(I)}, \tag{1.1.11}$$

and

$$\|e\|_{W^{-s}_\infty(I)} \leq Ch^{r+s}\|u\|_{W^r_\infty(I)}, \text{ for } s \leq r-2, \tag{1.1.12}$$

where now

$$\|v\|_{W^{-s}_\infty(I)} = \sup_{\substack{w \in W^s_1(I) \\ \|w\|_{W^s_1(I)}=1}} (v, w). \tag{1.1.13}$$

The constants C occurring are independent of h and u. They do depend on the coefficients a_i and on μ and r; in the case of $\mu \geq 1$ they may also depend on C_{QU} in (1.1.7).

The $\overset{\circ}{H}{}^1$–estimate in (1.1.8) can be found in this generality in [Schatz 1974], after use of standard approximation theory. The L_2–estimate and the negative norm estimates (1.1.9) follow by standard duality arguments, cf. (2.2.4) below. The L_∞–estimate (1.1.11) for $\mu = 0$ is in [Wheeler, M. F. 1973]. The case of general μ, with the quasi–uniformity condition (1.1.7), is in [Douglas, Dupont and Wahlbin 1975]. From this the W^1_∞–estimate follows in the quasi–uniform case. The W^1_∞–estimate in the case $\mu = 0$, although "well–known folklore", we have not been able to locate without restrictions on the meshes. We will therefore give a proof, in Remark 1.3.2 below. The negative norm estimates in (1.1.12) follow in a standard duality fashion from (1.1.11).

It is not hard to convince one–self that the powers of h occurring in (1.1.11) are the best possible in general. E.g.,

(1.1.14)
$$\begin{aligned}\min_{\chi\in\Pi_{r-1}(0,h)} &\|x^r-\chi\|_{L_\infty(0,h)}\\ &= h^r \min_{\chi\in\Pi_{r-1}(0,h)} \left\|\left(\frac{x}{h}\right)^r - \frac{1}{h^r}\chi(x)\right\|_{L_\infty(0,h)}\\ &= h^r \min_{\chi\in\Pi_{r-1}(0,h)} \left\|\xi^r - \frac{1}{h^r}\chi(h\xi)\right\|_{L_\infty(0,1)}\\ &= h^r \min_{\tilde\chi\in\Pi_{r-1}(0,1)} \|\xi^r - \tilde\chi(\xi)\|_{L_\infty(0,1)} = Ch^r.\end{aligned}$$

A <u>superconvergent</u> "point" for function values of order σ is now a family of points $\xi = \xi(h)$ such that

(1.1.15)
$$|e(\xi)| \le Ch^{r+\sigma},$$

where $\sigma > 0$ and $C = C(u, a_2, a_1, a_0)$ (and possibly also depending on G_{QU} in (1.1.7)). Similarly, $\eta = \eta(h)$ is superconvergent of order σ for first derivatives if

(1.1.16)
$$|e'(\eta)| \le Ch^{r-1+\sigma},$$

with $\sigma > 0$.

In principle, we could talk about superconvergence points for a particular problem or for a particular solution. Generally what we have in mind, though, is some class of problems with (locally) smooth coefficients and solutions, and we then wish to determine points which are superconvergent for the whole class.

Furthermore, we point out that what is described in (1.1.15) and (1.1.16) is only so–called "natural" superconvergence. That is, u_h (or u'_h) is simply evaluated at the point ξ (or η) and then compared with $u(\xi)$ (or $u'(\eta)$). In these notes we shall also see many examples of superconvergence involving postprocessing of u_h (trivial, or not so trivial). In fact, in terms of implementation on a computer, many postprocessing methods are simpler to implement than is evaluating u_h at a (non–node) point.

1.2. Nodal superconvergence for function values in continuous elements ($\mu = 0$).

We shall give the argument of [Douglas and Dupont 1974]. Let $G(\xi;\cdot)$ be the Green's function for (1.1.1), cf. e.g. [Birkhoff and Rota 1969, Theorem 10, p.52–53], so that

(1.2.1)
$$e(\xi) = A(e, G(\xi;\cdot)).$$

From (1.1.3) and (1.1.6), $A(e,\chi) = 0$, for $\chi \in \mathring{S}_h$ and thus

(1.2.2)
$$e(\xi) = A(e, G(\xi;\cdot) - \chi), \text{ for any } \chi \in \mathring{S}_h.$$

Let now $\xi = x_i$, a meshpoint. Since $G(\xi;\cdot)$ is continuous and (uniformly in ξ) smooth on both sides of $x = \xi$, and since $\mu = 0$, we have from standard approximation theory that, for a suitable χ,

(1.2.3)
$$\|G(\xi;\cdot) - \chi\|_{H^1(I)} \le Ch^{r-1}.$$

Since by (1.1.8) also $\|e\|_{H^1(I)} \le Ch^{r-1}\|u\|_{H^r(I)}$ we thus obtain from (1.2.2),

$$|e(x_i)| \le Ch^{2r-2}\|u\|_{H^r(I)}, \tag{1.2.4}$$

for x_i a meshpoint. Thus, provided $r \ge 3$ (i.e., continuous piecewise quadratics or higher elements are used), we have superconvergence at the knots. There is no restriction on the geometries of the meshes allowed.

We state the above as a theorem.

Theorem 1.2.1. *Under the assumptions of Section 1.1, for $\mu = 0$ (with no mesh restrictions),*

$$|(u - u_h)(x_i)| \le Ch^{2r-2}\|u\|_{H^r(I)}. \tag{1.2.5}$$

Let us mention that [Douglas and Dupont 1974] gives an explicit example (with $a_2 \in C^{r-1}(I)$ only, though) showing that the power h^{2r-2} is sharp.

1.3. Reduction to a model problem.

In Sections 1.5–1.11 we shall investigate superconvergence of, typically, order one (i.e., $\sigma = 1$ in (1.1.15) or (1.1.16)). It turns out to be convenient to reduce the investigations to the case $a_2 \equiv 1$, $a_1 = a_0 \equiv 0$. The argument follows [Wahlbin 1992] which in turn was based on [Douglas, Dupont and Wahlbin 1975]. Let thus

$$A(u - u_h, \chi) = 0, \quad \text{for } \chi \in \overset{\circ}{S}_h, \tag{1.3.1}$$

(cf., (1.1.3) and (1.1.6)) and let $\widetilde{u}_h \in \overset{\circ}{S}_h$ be another approximation to u given by

$$(u' - \widetilde{u}_h', \chi') = 0, \quad \text{for } \chi \in \overset{\circ}{S}_h. \tag{1.3.2}$$

Then, with $\theta = \widetilde{u}_h - u_h \in \overset{\circ}{S}_h$,

$$\begin{aligned}(\theta', \chi') &= ((\widetilde{u}_h - u)', \chi') - ((u_h - u)', \chi') \\ &= -\Big(a_2(u_h - u)', \frac{1}{a_2}\chi'\Big) \\ &= -\Big(a_2(u_h - u)', \Big(\frac{1}{a_2}\chi\Big)'\Big) + \Big(u_h - u, \Big(\frac{a_2'}{a_2}\chi\Big)'\Big) \\ &= -\Big(a_2(u_h - u)', \Big(\frac{1}{a_2}\chi\Big)' - \psi'\Big) + \Big(a_1(u_h - u), \psi'\Big) \\ &\quad + \Big(a_0(u_h - u), \psi\Big) + \Big(u_h - u, \Big(\frac{a_2'}{a_2}\chi\Big)'\Big), \quad \text{for } \chi, \psi \in \overset{\circ}{S}_h.\end{aligned} \tag{1.3.3}$$

Let us now consider $(\frac{1}{a_2}\chi)' - \psi'$, and let us treat in detail the continuous case ($\mu = 0$). Let then ψ be the natural (Lagrange) interpolant (at x_i, x_{i+1}; and the appropriate number of equispaced points in the interior of I_i) to $\frac{1}{a_2}\chi$. By standard approximation theory,

$$\Big\|\Big(\frac{1}{a_2}\chi\Big)' - \psi'\Big\|_{L_1(I_i)} \le Ch_i^{r-1}\Big\|\Big(\frac{1}{a_2}\chi\Big)^{(r)}\Big\|_{L_1(I_i)}, \tag{1.3.4}$$

and since $\chi^{(r)} \equiv 0$ we have by Leibniz' formula and inverse estimates (cf. Section 2.1),

$$\left\| \left(\frac{1}{a_2} \chi \right)' - \psi' \right\|_{L_1(I_i)} \le C h_i \|\chi\|_{W_1^1(I_i)}, \tag{1.3.5}$$

where C depends on a_2. Thus, summing over all intervals,

$$\left\| \left(\frac{1}{a_2} \chi \right)' - \psi' \right\|_{L_1(I)} \le C h \|\chi\|_{W_1^1(I)}. \tag{1.3.6}$$

This result is a special case of "superapproximation", cf. [Nitsche and Schatz 1974]. Employing a quasi-interpolant à la [deBoor and Fix 1973] (see e.g. [Wahlbin 1991, Lemma 3.2, p.367]), (1.3.6) is true for any $\overset{\circ}{S}_h$ considered by us, cf. (1.1.7). We shall give some detail on this in Section 2.3 below. From (1.3.3) we now have using also (1.1.11),

$$|(\theta', \chi')| \le C h^r \|u\|_{W_\infty^r(I)} \|\chi\|_{W_1^1(I)}. \tag{1.3.7}$$

We proceed to estimate $\|\theta'\|_{L_\infty(I)}$. We have

$$\|\theta'\|_{L_\infty(I)} = \sup_{\|\psi\|_{L_1(I)}=1} (\theta', \psi). \tag{1.3.8}$$

Let

$$\$_h = \$_h^{\mu-1,r-1} = \left\{ \chi(x) : \chi \in C^{\mu-1}(I); \chi\big|_{I_i} \in \Pi_{r-2} \right\}, \tag{1.3.9}$$

and let P denote the L_2–projection into $\$_h$. For each ψ occurring in (1.3.8) we have

$$(\theta', \psi) = (\theta', P\psi) \tag{1.3.10}$$

since $\theta' \in \$_h$. Now set

$$\phi(x) = \int_0^x P\psi(y)dy - xMV(P\psi) \tag{1.3.11}$$

where $MV(P\psi) = \int_0^1 P\psi dy$. Then $\phi \in \overset{\circ}{S}_h$, and $\phi' = P\psi - MV(P\psi)$. Since $(\theta', 1) = 0$ we have from (1.3.10) and (1.3.7)

$$|(\theta', \psi)| = |(\theta', \phi')| \le C h^r \|u\|_{W_\infty^r(I)} \|\phi\|_{W_1^1(I)}. \tag{1.3.12}$$

Since $\|\phi\|_{W_1^1(I)} \le C\|P\psi\|_{L_1(I)}$ and the L_2–projection into $\$_h$ is bounded in L_1 (easy if $\mu = 0$ so that P is completely local; for the cases $\mu \ge 1$ see [Douglas, Dupont and Wahlbin 1975] or [Wahlbin 1991, Lemma 3.5] or, Theorem 3.2.3 below) we thus have $\|\phi\|_{W_1^1(I)} \le C\|\psi\|_{L_1(I)} = C$. Hence

$$\|\theta'\|_{L_\infty(I)} \le C h^r \|u\|_{W_\infty^r(I)}. \tag{1.3.13}$$

We state this result as a theorem.

Theorem 1.3.1. *Let u_h be the projection into $\overset{\circ}{S}_h$ of u based on the form A, which satisfies the assumptions of Section 1.1. Let $\widetilde{u}_h \in \overset{\circ}{S}_h$ be given by $(\widetilde{u}_h' - u', \chi') = 0$, for $\chi \in \overset{\circ}{S}_h$. For $\mu \ge 1$ assume quasiuniformity as in (1.1.7). Then*

$$\|(u_h - \widetilde{u}_h)'\|_{L_\infty(I)} \le C h^r \|u\|_{W_\infty^r(I)}. \tag{1.3.14}$$

We shall next prove an analogue of this for function values.

Theorem 1.3.2. *With assumptions as in Theorem 1.3.1 and, in addition,* $r \geq 3$, *we have*

$$\|u_h - \widetilde{u}_h\|_{L_\infty(I)} \leq Ch^{r+1}\|u\|_{W^r_\infty(I)}. \tag{1.3.15}$$

Proof: In the case $\mu = 0$ we have from Theorem 1.2.1 that $\theta(x_i) \equiv (u_h - \widetilde{u}_h)(x_i) = 0(h^{2r-2}) \leq Ch^{r+1}$. For $x \in I_i$, $\theta(x) = \theta(x_i) + \int_{x_i}^x \theta'(y)dy$ and (1.3.15) follows in this case from Theorem 1.3.1.

For the general case of $\mu \geq 1$ we write

$$\|\theta\|_{L_\infty(I)} = \sup_{\|v\|_{L_1(I)}=1} (\theta, v). \tag{1.3.16}$$

For each such v, let $-w'' = v$ in I, $w(0) = w(1) = 0$. Then

$$(\theta, v) = (\theta', w') = (\theta', P(w')) \tag{1.3.17}$$

where P is the L_2–projection into $\$_h = \$_h^{\mu-1,r-1}$. Since $MV(P(w')) = MV(w') = 0$, $Q = \int_0^x P(w') \in \overset{\circ}{S}_h$ with $Q' = P(w')$ so that from (1.3.3), with $e = u_h - u$,

$$\begin{aligned}(\theta, v) &= (\theta', Q') \\ &= -\left(a_2 e', \left(\frac{1}{a_2}Q\right)' - \psi'\right) + (a_1 e, \psi') + (a_0 e, \psi) \\ &\qquad + \left(e, \left(\frac{a_2'}{a_2}Q\right)'\right) \\ &\equiv I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{1.3.18}$$

Integrating by parts, and using (1.1.11) and superapproximation,

$$\begin{aligned}|I_1| &= \left|\left(e, \left(a_2\left(\frac{1}{a_2}Q\right)' - \psi'\right)'\right)\right| \\ &\leq C\|e\|_{L_\infty(I)} \left\|\left(\frac{1}{a_2}Q\right)' - \psi'\right\|_{W_1^1(I)} \\ &\leq Ch^{r+1}\|Q\|_{W_1^2(I)}.\end{aligned} \tag{1.3.19}$$

Similarly, using the W_∞^{-1} error estimate of (1.1.12),

$$|I_2 + I_3 + I_4| \leq Ch^{r+1}\|Q\|_{W_1^2(I)}. \tag{1.3.20}$$

Under our quasi–uniformly assumption (1.1.7) it is easy to see that, since P is stable in L_1, it is also stable in $W_1^1(I)$. Thus $\|Q\|_{W_1^2(I)} \leq C\|w\|_{W_1^2(I)} \leq C\|v\|_{L_1} = C$. It follows from this and (1.3.16)–(1.3.18) that

$$\|\theta\|_{L_\infty(I)} \leq Ch^{r+1}. \tag{1.3.21}$$

This completes the proof of the theorem. □

Remark 1.3.1. In various special cases, better results can be had. E.g., if $r \geq 3$, if $a_2 \equiv 1$ and $a_1 \equiv 0$, then (1.3.14) in Theorem 1.3.1 may be replaced by

$$\|(u_h - \widetilde{u}_h)'\|_{L_\infty(I)} \leq Ch^{r+1}\|u\|_{W^r_\infty(I)}.$$