Lecture Notes in Mathematics 1630

Daniel Neuenschwander

Probabilities on the Heisenberg Group

Limit Theorems and Brownian Motion

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Cataloging-Data applied for
Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Neuenschwander, Daniel:
Probabilities on the Heisenberg group : limit theorems and Brownian motion. - Berlin ; Heidelberg ; New York ; Barcelona ; Budapest ; Hong Kong ; London ; Milan ; Paris ; Tokyo : Springer, 1996
(Lecture notes in mathematics ; 1630)
ISBN 3-540-61453-2
NE: GT

Mathematics Subject Classification (1991): 60-02, 60B15, 22E25, 47D06, 60F05, 60F15, 60F17, 60F25, 60G10, 60G17, 60G18, 60G42, 60H05, 60J25, 60J35, 60J45, 60J55, 60J60, 60J65, 62F35

ISSN 0075-8434
ISBN 3-540-61453-2 Springer-Verlag Berlin Heidelberg New York

Typesetting: Camera-ready $\TeX$ output by the author
SPIN: 10479780 46/3142-543210 - Printed on acid-free paper

Preface

Probability theory on algebraic and geometric structures such as e.g. topological groups has attracted much interest in the literature during the past decades and is a subject of growing importance. Stimuli which can not be overestimated for the research work which has and is currently been done in the field of probability theory on groups and related structures are the regular Oberwolfach conferences organized by L. Schmetterer, H. Heyer, and A. Mukherjea as well as the recent foundation of the "Journal of Theoretical Probability" also by A. Mukherjea.
In this work we will have, from the probabilistic point of view, a closer look at the so-called Heisenberg group. Its structure reflects the Heisenberg uncertainty principle as non-commutativity of the location and the momentum operator. In a certain sense, it is the simplest non-commutative Lie group, so it is clear that in generalizing classical results of probability theory to the non-commutative situation, one naturally passes by this group. Our aim will be to survey, under the limit theoretic aspect and its relation to Brownian motion, certain results which turned out to be valid on the Heisenberg group but which can not (or not yet) be generalized to the whole class of simply connected nilpotent Lie groups. For this wider framework, we refer (among others) to the forthcoming book of Hazod and Siebert (1995). So our work will to a certain degree be a complement to that book in the sense of some sort of a case study.
The second author of the above-mentioned book in preparation, Eberhard Siebert, untimely passed away in 1993. Without his fundamental contributions, the theory would at any rate not be at that level as it is now. It is one of the modest objectives of our book to underline the importance of Siebert's work in the development of probability theory on (in particular non-commutative) groups.
A word about applications: The Heisenberg group turned out to have many applications not only in mathematics itself (and there even in such remote fields such as combinatorics!), but also in physics (where it in fact comes from) and engineering science (signal theory). Due to the physical ignorance of the author, we have not tried to look for applications of the results presented in this work. The author would be delighted to hear one day about applications outside of "pure" mathematics!
It is my great pleasure to express my most sincere and heartfelt gratitude to my teacher and mentor Professor Henri Carnal for his constant benevolent support; to Professor Wilfried Hazod for his kind hospitality at the University of Dortmund; to Professor

René Schott for his kind hospitality at the University Henri Poincaré Nancy I; to Professor Yuri Stepanovič Hohlov and Professor Gyula Pap for many stimulating discussions; and last but not least to the Ingenieurschule Biel and its director Dr. Fredy Sidler for giving me the opportunity of taking a leave in order to continue my research activities and to begin with this work.

Biel-Bienne, May 1996

Daniel Neuenschwander

Contents

Introduction

From a historical point of view (cf. Heyer (1977), Introduction), the development of probability theory on other structures than Euclidean spaces may be traced back to Daniel Bernoulli, who in his astronomical investigations in 1734 assumed planets to be (uniformly distributed) random points on a sphere. Later on, during the first decades of the 20th century, several mathematicians continued to consider (also non-uniform) probability distributions on the circle and the sphere. We mention the names of Rayleigh, Pearson, Perrin, von Mises, Fisher, and Mardia. Nowadays this direction of research is called statistics of directional data. Von Mises and Lévy (1939) considered probability measures on the torus. The pioneering breakthrough to more general compact groups is due to Kawada and Ito (1940). Bochner, during the late fifties, began to study probabilities on locally compact abelian groups. An early overview of the theory is the book of Grenander (1963). Since then, the field has developed into several directions. One milestone was the paper of Hunt (1956), who considered continuous convolution semigroups on Lie groups. He was able to characterize their infinitesimal generators by an analogue of the classical Lévy-Hinčin formula. It turned out that convolution semigroups were a natural framework for studying limit theorems. The state of the art up to 1977 is most exhaustively described in Heyer's (1977) monograph. We also mention the important work of Stroock, Varadhan (1973) and Feinsilver (1978) concerning stochastic processes on Lie groups.
Since the late seventies it became clear that the simply connected nilpotent Lie groups play a special role in probability theory, in particular where limit theorems are concerned. Among the first papers along these lines are those of Crépel, Raugi (1978) and Raugi (1978) concerning the central limit theorem on simply connected nilpotent Lie groups, and that of Crépel, Roynette (1977), which gives a law of the iterated logarithm for the (three-dimensional) so-called Heisenberg group. The latter is the simplest example of a non-commutative simply connected nilpotent Lie group, in a certain sense even generally the simplest non-commutative non-discrete Lie group from the structural point of view. We will give a detailed description of it later. So we think that in aiming at the non-commutative situation, one must come by this group. In 1982, Hazod (1982) introduced a concept of stability on locally compact groups based on convolution semigroups and one-parameter automorphism groups. Later it was proved by Hazod and Siebert (1986) that strictly stable semigroups are concentrated on the contractible part of the corresponding one-parameter automorphism group, which is isomorphic to a simply connected nilpotent Lie group. On the other hand, Burrell and McCrudden (1974) have shown that any infinitely divisible probability measure on a simply connected nilpotent Lie group is embeddable into a continuous convolu-

tion semigroup. The theory of (semi-)stable semigroups on simply connected nilpotent Lie groups is in full growth at present; among the most important ones we mention the papers of Hazod (1982, 1984a, 1984b, 1986), Hazod, Siebert (1986, 1988), Drisch, Gallardo (1984), Nobel (1991), Hazod, Nobel (1989), Hazod, Scheffler (1993), Scheffler (1993, 1994, 1995a, 1995b), Carnal (1986), Kunita (1994a, 1994b, 1995), Neuenschwander, Scheffler (1996), Neuenschwander (1995a, 1995c, 1995d), and the theses of Nobel (1988), Scheffler (1992) and Neuenschwander (1991). Parallel to this stream of research several other aspects of probability theory on simply connected nilpotent Lie groups have been investigated. As some examples we mention further (weak and strong) limit theorems (Pap (1991a, 1991b, 1992, 1993, 1995), Berthuet (1979, 1986), Helmes (1986), Baldi (1986, 1990), Chaleyat-Maurel, Le Gall (1989), Ohring (1993), Neuenschwander (1992, 1995b, 1995e, 1995f), Neuenschwander, Scheffler (1995), Neuenschwander, Schott (1995)), the question of uniqueness of convolution semigroups (Pap (1994)), the explicit construction of Brownian motion with stochastic integrals (Roynette (1975)), as well as the development of a potential theory by Gallardo (1982).

The aim of this work is to give an account of certain limit theorems and of some aspects of Brownian motion under the limit-theoretic point of view on the (three-dimensional) Heisenberg group, with the major part grouped a little bit around the author's own results. The literature being fairly numerous, no claim to completeness is made. Let us also mention the important papers Gaveau (1977), Métivier (1980), and Helffer (1980). In general, only aspects which are (or up to now have been) special for the Heisenberg groups (or for simply connected step 2-nilpotent Lie groups) are taken into consideration in detail. Though the results are often available also for higher-dimensional Heisenberg groups or for all simply connected step 2-nilpotent Lie groups, we do not aim at maximal generality but will restrict ourselves to the simplest case of the three-dimensional Heisenberg group $\mathbb{H} = \mathbb{H}^1$ as a prototype in order to show as simply as possible the ideas and to unify the presentation, for otherwise the text would become too heterogeneous. For the more general formulations we can in most cases refer to the corresponding original works cited. The central component of Brownian motion on $\mathbb{H}$ is the so-called *Lévy stochastic area* process, arising as the area enclosed by the curve of a two-dimensional standard Brownian motion and the chord joining the endpoint to the origin. This process has of course many other interesting properties quite beyond the scope of this work and therefore excluded from it. At this place let us mention the relation of the stochastic area process to the Atiyah-Singer theorems (cf. Bismut (1984, 1988), Léandre (1988), Yor (1991)). There is another (equivalent) definition of $\mathbb{H}$ which is a little different from the one we use. With this other definition, "standard" Brownian motion has an interesting physical interpretation as the joint distribution of a Brownian motion on $\mathbb{R}$, another Brownian motion on $\mathbb{R}$ acting as a random constant field of forces, and the energy produced by the motion (cf. Hulanicki (1976)).

Another interesting subject concerning probabilities on the Heisenberg group (e.g.) is the definition and geometric characterization of an analogue of the Cauchy distribution (cf. Dunau, Sénateur (1986), (1988), Dani (1991)). The work of Neuenschwander (1993) has now been generalized to all positively graduated simply connected nilpotent Lie groups (cf. Neuenschwander, Schott (1996)).

The origin of the Heisenberg group lies in quantum mechanics. It became clear that the

Heisenberg commutation relation reflecting the Heisenberg uncertainty principle can be interpreted within the framework of Lie algebras. Consider the location operator $A : f \mapsto xf$ and the momentum operator $B : f \mapsto \frac{1}{2\pi i} f'$ on the space of infinitely differentiable functions with compact support on $\mathbb{R}$. Then $[A, B] := AB - BA = -\frac{1}{2\pi i} I$ (where I is the identity operator). Now the Heisenberg Lie algebra may be interpreted as the algebra generated by A, B, and I. So $\mathbb{H}$ can be described as $\mathbb{R}^3$, equipped with the group multiplication

$$\begin{aligned} x \cdot y &= x + y + \frac{1}{2}[x, y] \\ [x, y] &= (0, 0, x'y'' - x''y') \\ & \quad (x = (x', x'', x'''), y = (y', y'', y''')). \end{aligned}$$

The center of $\mathbb{H}$ is the line $\{0\} \times \{0\} \times \mathbb{R}$. Compared to the whole class of simply connected nilpotent Lie groups, $\mathbb{H}$ has several special features which are significant for probability theory; we mention:

- $\mathbb{H}$ is stratified,
- if k is the homogeneous dimension and ℓ the class of nilpotency, then $k - 2 - \ell = 0$,
- the center is 1-dimensional (so results known for $\mathbb{R}$ may be applied),
- $Aut(\mathbb{H})$ and the (contracting) one-parameter automorphism groups are explicitly known (cf. Folland (1989), pp.19ff., Drisch, Gallardo (1984)) (this can be used in the context of stable and semi-stable semigroups),
- $E([X, c]) = [E(X), c]$, hence $E(X \cdot (-E(X))) = 0$, i.e. $\mathbb{H}$-valued random variables may be centered with their expectation and $\{\prod_{n=1}^{N} X_n\}_{N \geq 1}$, where $\{X_n\}_{n \geq 1}$ are independent and $E(X_n) = 0$, is a martingale,
- $\prod_{j=1}^{n} x_j + \prod_{j=1}^{n} x_{n+1-j} = 2 \sum_{j=1}^{n} x_j$ (this has to be used in several places in order to apply results for the vector space-case),
- the density function of the central component of Brownian motion is explicitly known,
- the central component of standard Brownian motion on $\mathbb{H}$ - Lévy's stochastic area process - is a quadratic Brownian functional which has certain relations to winding numbers of two-dimensional Brownian motion (this was used by Shi (1995) to prove his results, which will be presented in 2.3.1).

So the Heisenberg group is much more than just a simple example of a simply connected nilpotent Lie group. $\mathbb{H}$ plays a role in several branches of mathematics and physics, see Folland (1989), Taylor (1986), and the very exhaustive survey article of Howe (1980). One can generalize several notions and facts from harmonic analysis and geometry to $\mathbb{H}$ (see e.g. Korányi (1983, 1985), Taylor (1986)). Relations to combinatorics of paths in the square lattice $\mathbb{Z}^2$ are discussed in Béguin, Valette, and Zuk (1995). There are

also certain applications in signal theory (cf. Schempp (1988)).
Now let us give a brief outline of the contents of this work:
Chapter 1 collects some facts from probability theory on simply connected nilpotent Lie groups G. The notion of a continuous convolution semigroup of probability measures is important. Since G is strongly root-compact (a notion which is due to Böge (1964)) and has no non-trivial compact subgroups, every infinitely divisible probability measure on G can be embedded into a continuous convolution semigroup. This is important for limit theorems and for defining the several types of domains of attraction. These are the contents of sections 1.1 and 1.2. In section 1.3 we present the necessary tools from potential theory as a preparation for the treatment of the Wiener sausage in 2.2.1 and the Lebesgue needle and related questions in 2.2.2.
In chapter 2 we study Brownian motion on $\mathbb{H}$ and its surroundings under the limit theoretic point of view in some detail. Pap (1994) proved that Gauss measures μ on simply connected nilpotent Lie groups determine uniquely the Gauss semigroup in which they may be embedded, but he left as an open problem if μ is also embeddable into a non-Gaussian continuous convolution semigroup. In section 2.1.1 we prove that for $\mathbb{H}$ this is indeed not the case. This may be viewed as a weak form of the Cramér-Lévy theorem telling that (on $\mathbb{R}$) Gaussian distributions have only Gaussian convolution factors. This result will be applied in 3.1.5 to formulate a "transfer principle" between limit theorems on $\mathbb{H}$ and on $(\mathbb{R}^3, +)$. Section 2.1.2 is devoted to a generalization of the Lindeberg theorem due to Pap and the Ljapunov theorem due to Ohring, while in section 2.1.3 we study the domain of normal attraction of Brownian motion on $\mathbb{H}$, work which has been done by Scheffler. An aspect which has to do with robust statistics (in the sense of outlier resistance) is studied in section 2.1.4: We show that after a certain so-called "intermediate" trimming procedure, domains of attraction of other stable semigroups merge into (loosely speaking) domains of attraction of Brownian motion. This explains in certain situations the influence of extremal terms in random products. We will come back to this topic in 3.1.3. Section 2.2.1 is devoted to a limit theorem for the "Wiener sausage" on $\mathbb{H}$ by Chaleyat-Maurel and Le Gall (1989) and its application to the absorbtion of Brownian motion on $\mathbb{H}$ by randomly thrown small sets on $\mathbb{H}$. In section 2.2.2 we present Gallardo's results concerning recurrence and the generalization of the so-called "Lebesgue needle". In section 2.3.1 we study some local and asymptotic results of iterated logarithm type, some of which are due to the author and to Schott. More precisely, we mention the asymptotic (Lévy-Berthuet-Baldi and Chung-Shi) laws of the iterated logarithm, give a new proof of the local (Lévy-Helmes) one, investigate the modulus of continuity, and carry over a qualitative form of the Erdös-Rényi law of large numbers for Brownian motion. Furthermore we present the results of Chaleyat-Maurel and Le Gall (1989) concerning the Hausdorff measure of the range of Brownian motion on $\mathbb{H}$ and the non-existence of multiple points. Section 2.3.2 treats the Crépel-Roynette law of the iterated logarithm for distributions having (roughly speaking) a $(2+\delta)$th absolute moment $(\delta > 0)$, an analogue of the classical Hartman-Wintner law of the iterated logarithm. In the course of the proof Crépel and Roynette gave an estimation of the speed of convergence in the central limit theorem on $\mathbb{H}$, which is of independent interest. In section 2.3.3 we apply the "subsequence-principle", which, in a general form, has been established by Chatterji

and Aldous, to the law of the iterated logarithm of Crépel-Roynette. This principle is a means of transferring limit theorems of i.i.d. random variables to limit theorems for subsequences of dependent random variables. The corresponding theorem for the Marcinkiewicz-Zygmund strong laws of large numbers will be mentioned in 3.2.1. Further (weak and strong) limit theorems are presented in chapter 3. Hazod (1993) has shown that on arbitrary contractible locally compact groups there exist so-called strictly *universal distributions* (in the sense of Doeblin), i.e. distributions which are partially attracted by every continuous convolution semigroup. Section 3.1.1 is in some sense a complement to that paper for the case where in the normalizing sequence also shifts are allowed: It is shown that a probability measure on $\mathbb{H}$ is universal (in this wider sense with shifts) iff it is universal on the underlying vector space $(\mathbb{R}^3, +)$. In section 3.1.2 we present the characterization of domains of attraction of (non-Gaussian) stable semigroups due to Scheffler. In section 3.1.3 we come back to the topic of 2.1.4: We characterize limits of "lightly" trimmed products in the domain of attraction of certain stable semigroups by means of some "Lévy construction" similar to that for stable laws themselves. In section 3.1.4 we present the results of Tutubalin (1964). Here, the normalization is performed such that the limit measure is a Gaussian distribution on $(\mathbb{R}^3, +)$ instead of $(\mathbb{H}, \cdot)$. Also, the norming maps are not endomorphisms of $(\mathbb{H}, \cdot)$ and the centering is taken with respect to " $+$ " rather than " $\cdot$ ". The topic of section 3.1.5 are triangular systems of probability measures on $\mathbb{H}$ which are not necessarily rowwise identically distributed. We show that limits of commutative infinitesimal triangular systems of probability measures on $\mathbb{H}$ satisfying some "local centering" condition are always infinitely divisible. As in the i.i.d. case for general simply connected nilpotent Lie groups, one can in this situation formulate a "transfer principle", saying that limit theorems for triangular systems on $(\mathbb{R}^3, +)$ have a canonical counterpart on $(\mathbb{H}, \cdot)$ if the measures within each row commute on $\mathbb{H}$. This transfer principle also holds the other way round if the limit measure is known to be Gaussian by the uniqueness property mentioned in 2.1.1. If it would be known that (as in the euclidean case) also on $\mathbb{H}$ any embeddable probability measure μ_1 determines uniquely the continuous convolution semigroup $\{\mu_t\}_{t\geq 0}$ in which it may be embedded, then this transfer principle would also hold the other way round in general. Furthermore, we show that also limits of non-commutative infinitesimal triangular systems of symmetric probability measures on $\mathbb{H}$ are infinitely divisible. While the classical (Kolmogorov) form of the strong law of large numbers is already known in fairly general situations (see e.g. Furstenberg (1963), Tutubalin (1969), Guivarc'h (1976)), we carry over in section 3.2.1 the strong law of large numbers in the form of Marcinkiewicz and Zygmund. We also apply the subsequence principle of 2.3.3 to this situation. In section 3.2.2 the convergence rates are precised; more accurately, the estimations of Baum-Katz and Hsu-Robbins-Erdös are carried over to $\mathbb{H}$. The Baum-Katz theorem is a precision of the Marcinkiewicz-Zygmund law of large numbers, while the Hsu-Robbins-Erdös theorem characterizes complete convergence in the law of large numbers by the finiteness of the second moment. Section 3.2.3 is devoted to the ergodic theorem and related results. Section 3.2.4 presents other (non-classical) versions of laws of the iterated logarithm for stable and semi-stable semigroups which are not yet covered by 2.3.2; these are due to the author and mainly to Scheffler. In section 3.2.5 we carry over the classical "three-series

theorem" due to Kolmogorov to symmetric random variables on $\mathbb{H}$.
Let us close the introduction with some suggestions for further research. In general, the results collected here are in a form which, up to now, has only been proved for $\mathbb{H}$ (or, somewhat more generally, the step 2-case). So it remains open to generalize them to nilpotent Lie groups of higher step.
Another challenging problem whose treatment is just at the beginning now is to try to get rid of the independence or even i.i.d. assumption, which, up to now, has mostly been imposed, and to consider more general processes on groups, also processes with several parameters. Papers which go into this direction are e.g. Watkins (1989), who generalized Donsker's invariance principle for mixing sequences to Lie groups, and the "approximate martingale" approach on abelian groups by Bingham (1993).
The convergence theory for continuous convolution semigroups (which model processes with independent stationary increments) on groups and their generating distributions is now developped quite well. But a vast field still to be examined would be to find corresponding theorems for convolution hemigroups (which model processes with independent, but not necessarily stationary increments) and their generating families. Important first steps in this context were undertaken by Feinsilver (1978), Siebert (1982), Heyer, Pap (1996), and Pap (1996a, 1996b).
Also the generalization to (infinite-dimensional) Hilbert-Lie groups and to p-adic groups has just been begun (cf. Coşkun (1991), Riddhi Shah (1991, 1995), Telöken (1996)).
In theoretical physics, the so-called quantum and braided Heisenberg groups are of growing interest (see e.g. Feinsilver, Schott (1996), section 5.3). There are several approaches to define stochastic processes on these structures (see e.g. Feinsilver, Franz, Schott (1995a), (1995b), Franz, Schott (1996), and the literature cited there). So the question arises if results valid for the ordinary Heisenberg group carry over to this context in some form.
Finally, a subject which, to our knowledge, has not at all been touched so far are random subsets of groups (with the "Minkowski multiplication" $A \cdot B = \{a \cdot b : a \in A, b \in B\}$ as operation). The reason is two-fold: First, limit theorems for random subsets of $\mathbb{R}^d$ with Minkowski addition are often only valid for convex sets, or at least their proof passes by this special case. Now what is a reasonable analogue of convexity on a group? The second reason is that for random sets on $\mathbb{R}^d$, the Minkowski addition of convex subsets corresponds to the addition of their support functions, so one can use the corresponding Banach space results. See e.g. Molchanov (1993), chapter 2 and the literature cited there. This method does not seem to be applicable at all in the non-commutative case.

Chapter 1

Probability theory on simply connected nilpotent Lie groups

1.1 Continuous convolution semigroups of probability measures

Let G be a locally compact group, e the neutral element, $G^* := G\backslash\{e\}$. $(M^1(G), *, \xrightarrow{w})$ is the topological semigroup of (regular) probability measures on G, equipped with the operation of convolution and the weak topology (cf. Heyer (1977), Theorem 1.2.2). A continuous convolution semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures on G (c.c.s.[1] for short) is a continuous semigroup homomorphism

$$([0, \infty], +) \ni t \mapsto \mu_t \in (M^1(G), *, \xrightarrow{w}),$$

$$\mu_0 = \varepsilon_e$$

(ε_x denotes the Dirac probability measure at $x \in G$.) For simply connected nilpotent Lie groups the request $\mu_0 = \varepsilon_e$ is no restriction, since in any case μ_0 has to be an idempotent element of G and is thus the Haar measure ω_K on some compact subgroup $K \subset G$ (cf. Heyer (1977), 1.5.6); however, simply connected nilpotent Lie groups have no non-trivial compact subgroups (cf. Nobel (1991), 2.2). Let $M^b(G)$ be the Banach algebra of bounded Radon measures on G, equipped with the norm $||.||$ of total variation. For $\mu \in M^b(G)$ one defines

$$\exp \mu := \varepsilon_e + \sum_{k=1}^{\infty} \frac{\mu^{*k}}{k!}.$$

A *Poisson semigroup* is a c.c.s. of the form

$$\{\exp t(\eta - ||\eta||\varepsilon_e)\}_{t\geq 0}.$$

A probability measure on G is called Poisson, if it lies in a Poisson semigroup on G. Let G be a Lie group, $C_b(G)$ the space of all bounded complex-valued functions on

[1]Subsequently, the term "c.c.s." will always mean a continuous convolution semigroup of probability measures.

G, $C_0(G)$ the subspace of all complex-valued continuous functions on G vanishing at infinity, $C_b^\infty(G)$ the space of bounded complex-valued C^∞-functions on G, $\mathcal{D}(G)$ the subspace of complex-valued C^∞-functions with compact support. For a non-negative measure η on G, its adjoint measure $\tilde{\eta}$ is given by $\int_G f(x)\tilde{\eta}(dx) = \int_G f(x^{-1})\eta(dx)$ $(f \in C_b(G))$. The measure η is called symmetric if $\eta = \tilde{\eta}$. For a measurable map $\Phi : G \to G$, the measure $\Phi(\eta)$ is given by $\int_G f(x)\Phi(\eta)(dx) := \int_G F(\Phi(x))\eta(dx)$. A ($G$-valued) random variable X is called symmetric if its law $\mathcal{L}(X) \in M^1(G)$ is symmetric. For $\mu \in M^1(G)$ define the (right) convolution operator $T_\mu : C_b(G) \to C_b(G)$ by

$$T_\mu f(x) := \int_G f(xy)\mu(dy).$$

Now for a c.c.s. $\{\mu_t\}_{t\geq 0}$ and $f \in C_0(G)$ the *infinitesimal generator* $\mathcal{N}$ is defined as

$$\begin{aligned} \mathcal{N}f &:= \lim_{t\to 0+} \frac{1}{t}(T_{\mu_t}f - f) \\ &= \frac{d}{dt}\Big|_{t=0+} T_{\mu_t}f. \end{aligned}$$

$\mathcal{N}$ exists at least on $\mathcal{D}(G)$ (cf. Heyer (1977), Theorem 4.2.8). The whole domain of definition will be denoted by $D_\mathcal{N} \subset C_0(G)$. For $f \in C_b(G)$, the generating distribution $\mathcal{A}$ is defined as

$$\begin{aligned} \mathcal{A}f &:= \lim_{t\to 0+} \frac{1}{t} \int_G [f(x) - f(e)]\mu_t(dx) \\ &= \frac{d}{dt}\Big|_{t=0+} \int_G f(x)\mu_t(dx). \end{aligned}$$

It exists on the whole of $C_b^\infty(G)$ (cf. Siebert (1981), p.119). If $f \in D_\mathcal{N} \cap C_b^\infty(G)$, then $\mathcal{A}f = \mathcal{N}f(0)$.

Now let G be a simply connected nilpotent Lie group. This means that G is a Lie group with Lie algebra $\mathcal{G}$ such that $\exp : G \to \mathcal{G}$ is a diffeomorphism and that the *descending central series* is finite, i.e. there is some $r \in \mathbb{N}_0$ such that

$$\mathcal{G}_0 \underset{\neq}{\supset} \mathcal{G}_1 \underset{\neq}{\supset} \cdots \underset{\neq}{\supset} \mathcal{G}_r = \{0\},$$

where

$$\mathcal{G}_0 := \mathcal{G}, \quad \mathcal{G}_{k+1} := [\mathcal{G}, \mathcal{G}_k] \quad (0 \leq k \leq r-1).$$

G is then called step r-nilpotent. We may (and will from now on often) identify G with $\mathcal{G} = \mathbb{R}^d$ via exp. If it will be necessary to distinguish between objects (elements, functions, ...) on G resp. $\mathcal{G}$, then for an object Ξ on G the corresponding object on $\mathcal{G}$ will be denoted by $^\circ\Xi$. So G may be interpreted as $\mathbb{R}^d$ equipped with a Lie bracket $[\cdot,\cdot] : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ which is bilinear, skew-symmetric, and satisfies the *Jacobi identity*

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0.$$

Write $ad(x)(y) := [x, y]$. The group product is then given by the *Campbell-Hausdorff formula* (cf. Serre (1965)), where due to the nilpotency only the terms up to order r arise:

$$x \cdot y = \sum_{n=1}^{r} z_n,$$

$$z_n = \frac{1}{n} \sum_{p+q=n} (z'_{p,q} + z''_{p,q}),$$

$$z'_{p,q} = \sum_{\substack{p_1+p_2+\ldots+p_m=p \\ q_1+q_2+\ldots+q_{m-1}=q-1 \\ p_i+q_i \geq 1 \\ p_m \geq 1}} \frac{(-1)^{m+1}}{m} \frac{ad(x)^{p_1} ad(y)^{q_1} \ldots ad(x)^{p_m}(y)}{p_1! q_1! \ldots p_m!},$$

$$z''_{p,q} = \sum_{\substack{p_1+p_2+\ldots+p_{m-1}=p-1 \\ q_1+q_2+\ldots+q_{m-1}=q \\ p_i+q_i \geq 1}} \frac{(-1)^{m+1}}{m} \frac{ad(x)^{p_1} ad(y)^{q_1} \ldots ad(y)^{q_{m-1}}(x)}{p_1! q_1! \ldots q_{m-1}!}.$$

The first few terms are

$$x \cdot y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}\{[[x, y], y] + [[y, x], x]\} + \ldots.$$

It is clear that the neutral element e is 0 and that $x^{-1} = -x$. We will implicitly use the relation $[x + y, y] = [x, y]$. If $x_1, x_2, \ldots \in G$ we will use, for ordered products, the notation $\prod_{j=1}^{n} x_j := x_1 \cdot x_2 \cdot \ldots \cdot x_n$. The simply connected step 2-nilpotent Lie groups play a special role. Here $[.,.]$ is simply a bilinear skew-symmetric map $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying $[[x, y], z] = 0$. The most prominent examples are the so-called *Heisenberg groups* $\mathbb{H}^d$, given by $\mathbb{R}^{2d+1}$ and the Lie bracket

$$\begin{aligned} [x, y] &:= (0, 0, \langle x', y' \rangle - \langle x'', y' \rangle) \\ &\quad (x = (x', x'', x'''), y = (y', y'', y''') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \cong \mathbb{R}^{2d+1}). \end{aligned}$$

This notation (for $d = 1$) will be kept throughout this work. The center of $\mathbb{H}^d$ is the line $\{0\} \times \{0\} \times \mathbb{R}$. The group that we will consider in this work is the three-dimensional Heisenberg group $\mathbb{H}^1$. From now on, we will denote it simply by $\mathbb{H}$. For the central component we will use the notation $q(x) := x'''$ $(x = (x', x'', x''') \in \mathbb{H} \cong \mathbb{R}^3)$. On the other hand we put $p(x) := (x', x'')$. Observe that by the Cauchy-Schwarz inequality we have $||[x, y]|| \leq ||x|| \cdot ||y||$ $(x, y \in \mathbb{H})$. This relation will also be used implicitly. $\mathbb{H}$ is the simplest one among the simply connected nilpotent Lie groups, even the simplest non-commutative non-discrete Lie group among all Lie groups. So we think that, generally, in going to the non-commutative situation in Lie group theory, one naturally passes by such groups. Every simply connected step 2-nilpotent Lie group is a quotient of a free simply connected step 2-nilpotent Lie group (cf. Taylor (1986), p.156). We mention that all so-called *groups of type H* (cf. Kaplan (1980)) arising in the context of composition of quadratic forms are simply connected step 2-nilpotent. It turns out that the nilpotent part of the Iwasawa decomposition of a semisimple Lie

group of real rank 1 is of type H (cf. Korányi (1985), Proposition 1.1).
Let G be a simply connected step r-nilpotent Lie group. A positive graduation of $G \cong \mathcal{G}$ is a vector space decomposition

$$G \cong \mathcal{G} \cong \bigoplus_{j=1}^{r} V_j$$

such that $[V_i, V_j] \subset V_{i+j}(i+j \leq r)$ and $[V_i, V_j] = \{0\}(i+j > r)$. G is called *stratified* if it admits a positive graduation such that V_1 generates $\mathcal{G}$ as a Lie algebra. The number

$$k := \sum_{i=1}^{r} i \dim(V_i)$$

is called the *homogeneous dimension* of G. Clearly, $\mathbb{H}$ is stratified with homogeneous dimension $k = 4$. For $t > 0$, let the dilatations $\delta_t : G \to G$ on a positively graduated simply connected nilpotent Lie group G be given by

$$G \cong \bigoplus_{j=1}^{r} V_j \ni (x_1, x_2, \ldots, x_r) \mapsto (tx_1, t^2x_2, \ldots, t^r x_r) \in \bigoplus_{j=1}^{r} V_j \cong G$$

(see also Folland, Stein (1982)). Then a *homogeneous norm* on G is a continuous function $|.| : G \to [0, \infty[$ satisfying the properties

(i) $|0| = 0, |x| > 0 (x \in G\backslash\{0\})$,
(ii) $|\delta_t(x)| = t|x| \quad (t > 0, x \in G)$.

It is well-known that all homogeneous norms are equivalent (in the sense that $|.|_1 \leq C|.|_2 \leq D|.|_1$ uniformly on G (cf. Goodman (1977), Lemma 1)). There always exists a homogeneous norm $|.|$ such that

(iii) $|x| = |-x|$ (symmetry),
(iv) $|x \cdot y| \leq |x| + |y|$ (subadditivity)

(cf. Hebisch, Sikora (1990)). For $G = \mathbb{H}$ the homogenous norm

$$|x| := (||(x', x'')||^4 + 16x'''^2)^{1/4} \tag{1.1}$$

is subadditive and symmetric (cf. Korányi (1985), (1.4)). For any homogeneous norm $|.|$ on G there exist constants $C_1, C_2, c > 0$ such that

$$\begin{aligned} |x \cdot y| &\leq C_1(|x| + |y|), \\ |-x| &\leq C_2|x|, \\ ||x_j|| &\leq c|x|^j \end{aligned} \tag{1.2}$$

(cf. Goodman (1977), Lemma 2, Pap (1992), (2)).
Let $p > 0, c \in G$, and X a G-valued random variable. Since

$$E|X \cdot c|^p \leq C_1^p E(|X| + |c|)^p,$$

it follows that

$$E|X|^p < \infty \Longrightarrow E|X \cdot c|^p < \infty.$$