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Maxima and Minima Without Calculus

IVAN NIVEN

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The DOLCIANI MATHEMATICAL EXPOSITIONS series of the Mathematical Association of America was established through a generous gift to the Association from Mary P. Dolciani, Professor of Mathematics at Hunter College of the City University of New York. In making the gift, Professor Dolciani, herself an exceptionally talented and successful expositor of mathematics, had the purpose of furthering the ideal of excellence in mathematical exposition.

The Association, for its part, was delighted to accept the gracious gesture initiating the revolving fund for this series from one who has served the Association with distinction, both as a member of the Committee on Publications and as a member of the Board of Governors. It was with genuine pleasure that the Board chose to name the series in her honor.

The books in the series are selected for their lucid expository style and stimulating mathematical content. Typically, they contain an ample supply of exercises, many with accompanying solutions. They are intended to be sufficiently elementary for the undergraduate and even the mathematically inclined high-school student to understand and enjoy, but also to be interesting and sometimes challenging to the more advanced mathematician.

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TO BETTY

PREFACE

Our purpose is to bring together the principal elementary methods for solving problems in maxima and minima, except for two techniques that are treated adequately in standard textbooks. Calculus is deliberately omitted from our discussions, as are optimization processes through linear programming and game theory. In view of the many books and courses available on these subjects, our purpose is to complement these sources, not to compete with them. Thus there is a deliberate imbalance in this book, leaning toward methods in algebra and geometry that are not so widely known. Also, as the reader will readily note, our preference is often to solve geometric problems by reformulating them in an algebraic setting rather than by using purely geometric methods. Another author might do it another way, but, as the old saying goes, one man's fish is another man's poisson.

Calculus is such a systematically organized subject, providing as it does a step-by-step procedure for solving extremal problems, that its champions often regard alternative methods as trick procedures of limited usefulness. We attempt to counter this view by unifying these alternative procedures as much as possible. In this way the techniques can be perceived not simply as special devices of limited usefulness but as more general methods offering wider application. Thus we emphasize the line of argument that will accommodate many questions, rather than the brilliant shot that polishes off one and only one problem in splendid isolation.

Although calculus does provide a powerful and systematic technique for solving *some* problems in maxima and minima, the method is not universal. There are many questions that are awkward, if not impossible, by elementary calculus. Consider for example the question asked in calculus books of finding, among rectangles of a given perimeter, the one with largest area. The broader question of finding the *quadrilateral* of largest area among those of a given perimeter is not well suited to elementary calculus. Such questions are grist for our mill. Thus we follow a simple maxim: If a problem can be solved more simply by calculus, leave it to calculus.

Extremal questions are very close to problems in inequalities, so it is not surprising that this topic pops up quite regularly. However, our interest is not in inequalities per se, but only to the extent that they contribute to the solution of the extremal problems.

What background is needed to read this book? It is written for an audience at or near the maturity level of second- and third-year students in North American universities and colleges, assuming a good working knowledge of precalculus mathematics. Although calculus is not a prerequisite, a prior knowledge of that subject would enhance the reader's comprehension.

Although various techniques from geometry are introduced, there are three methods that are not used: orthogonal and other projections, vector analysis, and the geometry of complex numbers. These methods could have been used to simplify some of the solutions, but their introduction would have led us too far afield.

Chapter 1 contains some highlights of the background material needed, with the principal subject matter of the book starting in the second chapter. Although some readers will be able to proceed to Chapter 2 almost directly, Section 1.1 should be given some attention since it includes some basic agreements about language and notation.

The plan of the book is to proceed from easy problems to harder ones. For example, consider the isoperimetric problem in the plane: among all simple closed curves of a given length, which encloses the maximum area? This problem is solved in Chapter 4, in Section 4.3 to be specific, under the assumption that a solution exists. We return to the topic in Chapter 12, where the problem is solved without assuming the existence of a solution. From a logical standpoint these two chapters should be combined—in fact, with parts of Chapter 4 discarded because Chapter 12 is more general in its scope. However, the later chapter is not as easy to follow as the earlier one, which is much more elementary.

Chapters 2 to 6 are intended to be read in succession, each dependent on the earlier ones. These chapters are prerequisites for Chapters 7, 8 and 12, which can be read independently. Chapters 9, 10, and 11 also can be read independently, with Chapters 2 and 3 as needed background.

There are many problems for the reader scattered through the book. They are identified by a letter and a number; for example, E11 is the eleventh problem in Chapter 5. At the back of the book answers are given for all problems as needed, as well as solutions for most. The reader is urged, of course, to try the problems for herself or himself, turning to the solutions as a last resort. There are no exercises or drill problems, because the work is intended primarily as a resource book, not a textbook. The author has used parts of the material in the book, however, in an experimental course several times.

The notes at the ends of the chapters give not only sources of the material, but also suggestions for further reading. Although some references are listed in the body of the book, most are collected in one master list at the end, with the authors in alphabetical order. No attempt has been made to give a complete bibliography of the subject.

A first version of the manuscript was read by members of the committee on the Dolciani series, and by G. D. Chakerian, Basil Gordon, and Roy Ryden. I was very fortunate to get their constructive suggestions, which have resulted in extensive improvements. I am also grateful to many people for suggesting topics, problems, and references that might have been overlooked; especially I mention M. S. Klamkin, L. H. Lange, and the late C. D. Olds in this connection.

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CONTENTS

CHAPTER 1

BACKGROUND MATERIAL

This chapter contains the definitions, notations, conventions and background results needed for an understanding of the book. Although for many readers it will suffice to skim this chapter, the first section is somewhat crucial since it contains agreements about the use of language and notation. But the really substantive discussions of maxima and minima begin with Chapter 2, so the reader is urged to move on to that as quickly as possible.

1.1 Language and Notation. If a and b are any real numbers, the assertion that a is greater than b means that $a - b$ is positive, and this can be written in several equivalent forms:

$$a > b, \quad a - b > 0, \quad b < a, \quad b - a < 0.$$

Similarly, the statement that a is greater than or equal to b means that $a - b$ is positive or zero, and we can write

$$a \geq b, \quad a - b \geq 0, \quad b \leq a, \quad b - a \leq 0.$$

The notation $\max(a, b, c)$ denotes the largest, or the *maximum*, among the real numbers a, b, c. For example

$$\max(2, 3, 5) = 5, \quad \max(2, 3, -5) = 3, \quad \max(3, 3, -5) = 3.$$

In general, let $a_1, a_2, \ldots, a_n$ be any finite collection of real numbers, not necessarily all distinct. The equation

$$\max(a_1, a_2, \ldots, a_n) = a_j,$$

where j is an integer among $1, 2, 3, \ldots, n$, means that all the inequalities

$$a_j \geq a_1, \quad a_j \geq a_2, \quad a_j \geq a_3, \ldots, a_j \geq a_n$$

hold. Similarly the *minimum* of a finite collection of real numbers is denoted by

$$\min(a_1, a_2, \ldots, a_n) = a_k,$$

and this means that all the inequalities

$$a_k \leq a_1, \quad a_k \leq a_2, \quad a_k \leq a_3, \ldots, a_k \leq a_n$$

hold.

For infinite sets of real numbers, there may or may not be a maximum or a minimum. As a simple example, there is no smallest positive real number because if r is any positive number, $r/2$ is smaller. If a and b are any given real numbers with $a < b$, the set of numbers x satisfying

$$a < x < b \tag{1}$$

has neither a maximum nor a minimum. However, this set of numbers does have a *least upper bound*, b, and a *greatest lower bound*, a. An *upper bound* of a set of real numbers is a number which is greater than or equal to any number in the set. Similarly a *lower bound* is a number which is less than or equal to any number in the set.

A set of real numbers is said to be *bounded* if there are constants c and k such that the inequalities $c \leq x \leq k$ hold for every number x in the set. The set is said to be *bounded above* if $x \leq k$ holds, and *bounded below* if $c \leq x$ holds, for every x in the set. Any bounded set of real numbers has a unique *least* upper bound and a unique *greatest* lower bound. This statement is not proved here, because for our purposes we need only the very special case where the sets are restricted to be intervals on the real line. (The x axis in analytic geometry is a well-known illustration of the "real line.") For example, the set of numbers x satisfying $a < x < b$ forms an *open interval*, denoted by (a, b).

The set of numbers x satisfying $a \leq x \leq b$ constitutes a *closed interval* denoted by $[a, b]$. This set of numbers has a maximum b, which is also the least upper bound, and a minimum a, which is also the greatest lower bound.

The notation $[a, b)$ denotes the interval consisting of all numbers x satisfying $a \leq x < b$. This set has a minimum a, but no maximum. It has least upper bound b and greatest lower bound a, as also does the set of x satisfying $a < x \leq b$. This latter interval is denoted by $(a, b]$. In this case the set has a maximum, b, but no minimum.

The words "supremum" and "infimum" are often used in the mathematical literature in place of "least upper bound" and "greatest lower bound," but we shall not use these terms.

It is sometimes convenient, in seeking the maximum or the minimum of a function $f(x)$, to look instead for the minimum or maximum of $-f(x)$. For example, if we know that the minimum value of $9 + x^2 - 2x$ over all real numbers x is 8, it follows that the maximum of $2x - 9 - x^2$ is -8. (These results follow readily from the identity $9 + x^2 - 2x = 8 + (x - 1)^2$.) It follows also that the minimum of $90 + x^2 - 2x$ is 89, and the minimum of $900 + 10x^2 - 20x$ is 890.

Reciprocals can be used in the same way. Continuing the example in the preceding paragraph, it follows also that the maximum value of $1/(9 + x^2 - 2x)$ over all real numbers x is $1/8$.

Next we turn to some geometric conventions. For any distinct points P and Q the notation PQ will be used in three senses, easily distinguishable by context: the *straight line* PQ, meaning the infinite line extending in both directions; the *line segment* PQ, namely, the portion of the line terminating at P at one end and Q at the other; and the *distance* PQ, which is a positive number for distinct points P and Q, so that $PQ = QP$. Thus a distance PQ is never negative, and $PQ = 0$ iff the points coincide. (The word "iff" is the shortened form of "if and only if.") As an illustration of the sense being readily determined from the context, such an equation as $PQ = RS$ clearly refers to the equality of two distances.

The *half-line*, or ray, PQ is the line beginning at P as an endpoint and extending from P to Q and indefinitely beyond Q.

A *triangle* consists of three noncollinear points, say A, B, C, together with the line segments AB, BC, AC. Thus the area of a triangle is positive, never zero. The *triangle inequality* states that the sum of the lengths of any two sides exceeds the third, for example, $AB + BC > AC$. More generally, given three distinct points P, Q, R

we have $PQ + QR \geq PR$, with equality iff the point Q lies on the line segment PR. We say that a point Q is an *interior point* of a line segment PR if it lies strictly between P and R on the segment.

For any integer $n \geq 3$, an *n-gon*, or *polygon of n sides*, consists of a set of n distinct points $P_1, P_2, \ldots, P_n$ lying in a plane, called the *vertices*, and the n line segments P_1P_2, P_2P_3, P_3P_4, $\ldots$, $P_{n-1}P_n$, P_nP_1, called the *sides*, satisfying the condition that the sides have no points in common except that each pair of adjacent sides has exactly one vertex in common. The sides collectively form the *perimeter* or *boundary* of the polygon, which effectively separates the exterior points from the interior points.

A polygon is *convex* if the line segment joining any two points on the polygon contains no exterior point, that is, no point lying outside the polygon. Thus a polygon is convex iff each of its interior angles is less than or equal to 180°. More generally, a set S of points is said to be *convex* iff for every pair of points A, B in S the entire line segment AB is contained in S.

1.2. Geometry and Trigonometry. The angle PCQ subtended at the center C of a circle by any arc PQ is twice the angle PKQ subtended by PQ at any point K on the complementary arc, as shown in Figure 1.2a. It follows that $\angle PKQ = \angle PHQ$ for any points H and K lying on the same arc from P to Q. Also, a quadrilateral $PQRS$ is inscribable in a circle if and only if the sum of a pair of opposite angles is 180°. The sum of all four interior angles of a quadrilateral is 360°. The sum of all n interior angles of an n-gon is $180(n - 2)$ degrees.

If P is any point on a semicircle with diameter AB, as shown in Figure 1.2b, then $\angle APB = 90°$. More briefly, the angle in a semicircle is a right angle. Conversely, given any curve from A to B such that $\angle APB = 90°$ for *every* point P on the curve, then *the curve is a semicircle.* This converse is not so widely known, so we give a proof. Impose a coordinate system with AB as the x axis and the origin at the midpoint C of the segment AB. Let c denote the length BC, so that the coordinates of B and A are $(c, 0)$ and $(-c, 0)$. If the coordinates of any point P on the curve are (x, y), then the slopes of the lines PB and PA are

$$\frac{y-0}{x-c} \quad \text{and} \quad \frac{y-0}{x+c}.$$

These are perpendicular lines, so the product of the slopes is -1, giving

$$\frac{y}{x-c} \cdot \frac{y}{x+c} = -1, \qquad \frac{y^2}{x^2-c^2} = -1.$$

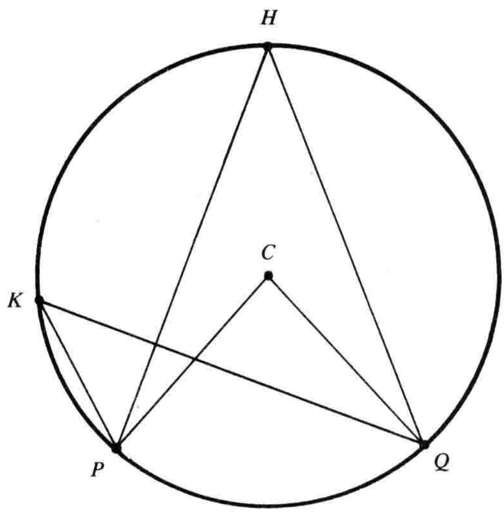

FIG. 1.2a

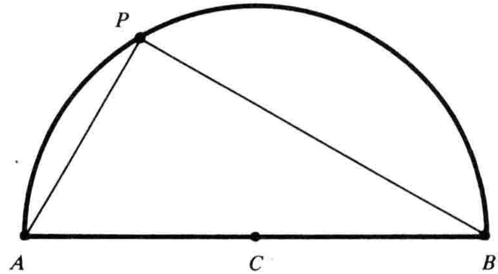

FIG. 1.2b