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Preface

In recent years there have been considerable developments in symmetry methods (group methods) for differential equations as evidenced by the number of research papers devoted to the subject. This is no doubt due to the inherent applicability of the methods to nonlinear differential equations. Symmetry methods for differential equations, originally developed by Sophus Lie, are highly algorithmic. They systematically unify and extend existing ad hoc techniques to construct explicit solutions for differential equations, most importantly for nonlinear differential equations. Often ingenious techniques for solving particular differential equations arise transparently from the group point of view, and thus it is somewhat surprising that symmetry methods are not more widely used.

A major portion of this book discusses work that has appeared since the publication of the book *Similarity Methods for Differential Equations*, Springer-Verlag, 1974, by G.W. Bluman and J.D. Cole. The present book includes a comprehensive treatment of Lie groups of transformations and thorough discussions of basic symmetry methods for solving ordinary and partial differential equations. No knowledge of group theory is assumed. Emphasis is placed on explicit computational algorithms to discover symmetries admitted by differential equations and to construct solutions resulting from symmetries.

This book should be particularly suitable for physicists, applied mathematicians, and engineers. Almost all of the examples are taken from physical and engineering problems including those concerned with heat conduction, wave propagation, and fluid flows. A preliminary version was used as lecture notes for a two-semester course taught by the first author at the University of British Columbia in 1987–88 to graduate and senior undergraduate students in applied mathematics and physics.

Chapters 1 through 4 encompass basic material. More specialized topics are covered in Chapters 5 through 7.

Chapter 1 introduces the basic ideas of group transformations and their connections with differential equations through a thorough treatment of dimensional analysis and generalizations of the well-known Buckingham Pi-theorem. This chapter should give the reader an intuitive grasp of the subject matter of the book in an elementary setting.

Chapter 2 develops the basic concepts of Lie groups of transformations and Lie algebras necessary in subsequent chapters. A Lie group of transformations is characterized in terms of its infinitesimal generators which form a Lie algebra.

Chapter 3 is concerned with ordinary differential equations. It is shown how group transformations are used to construct solutions and how to find group transformations leaving ordinary differential equations invariant. We present a reduction algorithm that reduces an nth order differential equation, admitting a solvable r-parameter Lie group of transformations, to an $(n-r)$th order differential equation plus r quadratures. We derive an algorithm to construct special solutions (invariant solutions) that are invariant under admitted Lie groups of transformations. For a first order differential equation such invariant solutions include separatrices and singular envelope solutions.

Chapter 4 is concerned with partial differential equations. It is shown how one finds group transformations leaving them invariant, how corresponding invariant solutions are constructed, and how group methods are applied to boundary value problems.

Chapter 5 discusses the connection between conservation laws and the invariance of Euler–Lagrange equations, arising from variational problems, under Lie groups of transformations. Various formulations of Noether's theorem are presented to construct such conservation laws. This leads to generalizing the concept of Lie groups of point transformations of earlier chapters to Lie–Bäcklund transformations that account for higher order conservation laws associated with partial differential equations that have solutions exhibiting soliton behavior. We present algorithms to construct recursion operators generating infinite sequences of Lie–Bäcklund symmetries.

In Chapter 6 it is shown how group transformations can be used to determine whether or not a given differential equation can be mapped invertibly to a target differential equation. Algorithms are given to construct such mappings when they exist. In particular, we give necessary and sufficient conditions for mapping a given nonlinear system of partial differential equations to a linear system of partial differential equations and for mapping a given linear partial differential equation with variable coefficients to a linear partial differential equation with constant coefficients.

In Chapter 7 the concept of Lie groups of transformations is generalized further to include nonlocal symmetries of differential equations. We present a systematic method for finding a special class of nonlocal symmetries that are realized as local symmetries of related auxiliary systems (potential symmetries). The introduction of potential symmetries significantly extends the applicability of group methods to both ordinary and partial differential equations. Together with the mapping algorithms developed in Chapter 6, the use of potential symmetries allows one to find systematically non-invertible mappings that transform nonlinear partial differential equations to linear partial differential equations.

Chapters 2 through 7 can be read independently of Chapter 1. The material in Chapter 2 is essential for all subsequent chapters but a reader only interested in scalar ordinary differential equations may omit Sections 2.3.3 to 2.3.5. Chapter 4 can be read independently of Chapter 3. A reader interested in conservation laws (Chapter 5) needs to know how to find Lie groups of transformations admitted by differential equations (Sections 3.2.3, 3.3.4, 4.2.3, 4.3.3). Chapter 6 can be read independently of Chapters 3 and 5.

Every topic is illustrated by examples. Almost all sections have many exercises. It is essential to do these exercises in order to obtain a working knowledge of the material. Each chapter ends with a Discussion section that puts its contents in perspective by summarizing major results, by referring to related works, and by introducing related material in subsequent chapters.

Within each section and subsection of a given chapter, definitions, theorems, lemmas, and corollaries are numbered separately as well as consecutively. For example, Definition 2.2.3-1 refers to the first definition and Theorem 2.2.3-1 to the first theorem in Section 2.2.3; Definition 1.4-1 refers to the first definition in Section 1.4. Exercises appear at the conclusion of a section; Exercise 1.3-4 refers to the fourth problem of Exercises 1.3.

We thank Greg Reid for helpful suggestions that improved Chapter 7, Alex Ma for his assistance, and Doug Jamison, Mei-Ling Fong, Sheila Hancock, Joanne Congo, Marilyn Lacate, Joan de Niverville, and Rita Sieber for their patience and care in typing various drafts of the manuscripts.

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Contents

Introduction

In the latter part of the 19th century Sophus Lie introduced the notion of continuous groups, now known as Lie groups, in order to unify and extend various specialized solution methods for ordinary differential equations. Lie was inspired by lectures of Sylow given at Christiania (present-day Oslo) on Galois theory and Abel's related works. [In 1881 Sylow and Lie collaborated in a careful editing of Abel's complete works.] Lie showed that the order of an ordinary differential equation can be reduced by one, constructively, if it is invariant under a one-parameter Lie group of point transformations.

Lie's work systematically relates a miscellany of topics in ordinary differential equations including: integrating factor, separable equation, homogeneous equation, reduction of order and the methods of undetermined coefficients and variation of parameters for linear equations, solution of the Euler equation, and the use of the Laplace transform. Lie (1881) also indicated that for linear partial differential equations, invariance under a Lie group leads directly to superpositions of solutions in terms of transforms.

A *symmetry group of a system of differential equations* is a group of transformations which maps any solution to another solution of the system. In Lie's framework such a group depends on continuous parameters and consists of either point transformations (*point symmetries*) acting on the system's space of independent and dependent variables, or more generally, contact transformations (*contact symmetries*) acting on the space including all first derivatives of the dependent variables. Elementary examples of Lie groups include translations, rotations, and scalings. An autonomous system of first order ordinary differential equations, i.e. a stationary flow, essentially defines a one-parameter Lie group of point transformations. Unlike discrete groups, for example reflections, Lie showed that for a given differential equation the admitted continuous group of point transformations, acting on the space of its independent and dependent variables, can be determined by an explicit computational algorithm (*Lie's algorithm*).

In this book the applications of continuous groups to differential equations make no use of the global aspects of Lie groups. These applications use connected local Lie groups of transformations. Lie's fundamental theorems show that such groups are completely characterized by their *infinitesimal generators*. In turn these form a *Lie algebra* determined by structure constants.

Lie groups, and hence their infinitesimal generators, can be naturally *extended* or "*prolonged*" to act on the space of independent variables, dependent variables and derivatives of the dependent variables up to any finite order. As a consequence, the seemingly intractable nonlinear conditions of group invariance of a given system of differential equations reduce to linear homogeneous equations determining the infinitesimal generators of the group. Since these *determining equations* form an overdetermined system of linear homogeneous partial differential equations, one can usually determine the infinitesimal generators in closed form. For a given system of differential equations, the setting up of the determining equations is entirely routine. Symbolic manipulation programs exist to set up the determining equations and in some cases explicitly solve them [cf. Schwarz (1985, 1988), Kersten (1987)].

If a system of partial differential equations is invariant under a Lie group of point transformations, one can find, constructively, special solutions, called *similarity solutions* or *invariant solutions,* which are invariant under some subgroup of the full group admitted by the system. These solutions result from solving a reduced system of differential equations with fewer independent variables. This application of Lie groups was discovered by Lie but first came to prominence in the late 1950s through the work of the Soviet group at Novosibirsk, led by Ovsiannikov (1962, 1982). Invariant solutions can also be constructed for specific boundary value problems. Here one seeks a subgroup of the full group of a given partial differential equation which leaves boundary curves and conditions imposed on them invariant [cf. Bluman and Cole (1974)]. Such solutions include *self-similar* (*automodel*) *solutions* which can be obtained through *dimensional analysis* or more generally from invariance under groups of scalings. Connections between invariant solutions and separation of variables have been studied extensively by Miller (1977) and co-workers.

In a celebrated paper, Noether (1918) showed how the symmetries of an action integral (*variational symmetries*) lead constructively to conservation laws for the corresponding Euler–Lagrange equations. For example, conservation of energy follows from invariance under translation in time; conservation of linear and angular momenta, respectively, from invariances under translations and rotations in space. Such variational symmetries leave the Euler–Lagrange equations invariant. They can be determined through Lie's algorithm.

The applicability of symmetry methods to differential equations is further extended by considering invariance under *Lie–Bäcklund transformations* (*Lie–Bäcklund symmetries*). Here the infinitesimal generators depend on derivatives of the dependent variables up to any finite order. The possibility of the existence of such symmetries was recognized by Noether (1918). Lie–Bäcklund transformations are discussed in some detail in Olver (1986) and Ibragimov (1985). Lie–Bäcklund transformations cannot be represented in closed form by integrating a finite system of ordinary differential equations as is the case for Lie groups of point transformations. However their infinitesimal generators can be computed for a given differential equation by a simple extension of Lie's algorithm. The invariance of a partial differential equation under a Lie–Bäcklund symmetry usually leads to invariance under an infinite number of such symmetries connected by recursion operators [Olver (1977)]. The theory and computation of recursion operators are discussed comprehensively in Olver (1986). Lie–Bäcklund symmetries can be shown to account for the conserved Runge–Lenz vector for the Kepler problem and the infinity of conservation laws for the Korteweg–de Vries equation and other nonlinear partial differential equations exhibiting soliton behavior.

Another application of symmetry methods to differential equations is to discover related differential equations of simpler form. By comparing the Lie groups admitted by a given differential equation and another differential equation (target equation) one can find, constructively, necessary conditions for a mapping of the given equation to the target equation. If the target equation is characterized completely in terms of a Lie symmetry group then one can algorithmically determine if an invertible mapping exists between the equations. In particular, one can constructively answer such questions as: Can a given nonlinear system of partial differential equations be mapped invertibly to a linear system? Can a given linear partial differential equation with variable coefficients be mapped into one with constant coefficients?

One can extend the classes of symmetries admitted by differential equations beyond *local symmetries* (which include point, contact, and Lie–Bäcklund symmetries) to *nonlocal symmetries* by considering a system related to a given differential equation. Here one starts by finding a conservation law for the given differential equation. This leads to a related system through the introduction of auxiliary dependent variables (*potentials*). A Lie group of point transformations admitted by the system is a symmetry group of the given differential equation since it maps any solution of the given equation to another solution. For partial differential equations such symmetries are often nonlocal symmetries (*potential symmetries*). This extension of local symmetries to potential symmetries considerably widens the applicability of symmetry methods to the construction of solutions of both ordinary and partial differential equations.

1

Dimensional Analysis, Modelling, and Invariance

1.1 Introduction

In this chapter we introduce the ideas of invariance concretely through a thorough treatment of dimensional analysis. We show how dimensional analysis is connected to modelling and the construction of solutions obtained through invariance for boundary value problems for partial differential equations.

Often for a quantity of interest one knows at most the independent quantities it depends upon, say n in total, and the dimensions of all $n+1$ quantities. The application of dimensional analysis usually reduces the number of essential independent quantities. This is the starting point of modelling where the objective is to reduce significantly the number of experimental measurements. In the following sections we will show that dimensional analysis can lead to a reduction in the number of independent variables appearing in a boundary value problem for a partial differential equation. Most importantly we show that for partial differential equations the reduction of variables through dimensional analysis is a special case of reduction from invariance under groups of scaling (stretching) transformations.

1.2 Dimensional Analysis—Buckingham Pi-Theorem

The basic theorem of dimensional analysis is the so-called *Buckingham Pi-theorem,* attributed to the American engineering scientist Buckingham (1914, 1915a, b). General references on the subject include those of Birkhoff (1950), Bridgman (1931), Barenblatt (1979), Sedov (1959), and Bluman (1983a). A historical perspective is given by Görtler (1975). For a detailed mathematical perspective see Curtis, Logan, and Parker (1982).

The following assumptions and conclusions of dimensional analysis constitute the Buckingham Pi-theorem.