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Roberto Solis-Oba (Eds.)

Approximation and Online Algorithms

First International Workshop, WAOA 2003
Budapest, Hungary, September 2003
Revised Papers

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Preface

The Workshop on Approximation and Online Algorithms (WAOA 2003) focused on the design and analysis of algorithms for online and computationally hard problems. Both kinds of problems have a large number of applications arising from a variety of fields. The workshop also covered experimental research on approximation and online algorithms. WAOA 2003 took place in Budapest, Hungary, from September 16 to September 18. The workshop was part of the ALGO 2003 event, which also hosted ESA 2003, WABI 2003, and ATMOS 2003.

Topics of interest for WAOA 2003 were: competitive analysis, inapproximability results, randomization techniques, approximation classes, scheduling, coloring and partitioning, cuts and connectivity, packing and covering, geometric problems, network design, and applications to game theory and financial problems. In response to our call for papers we received 41 submissions. Each submission was reviewed by at least 3 referees, who judged the papers on originality, quality, and consistency with the topics of the conference. Based on these reviews the program committee selected 19 papers for presentation at the workshop and for publication in this proceedings. This volume contains the 19 selected papers and 5 invited abstracts from an ARACNE minisymposium which took place as part of WAOA.

We would like to thank all the authors who responded to the call for papers and the invited speakers who gave talks at the ARACNE minisymposium. We specially thank the local organizer Janos Csirik, the members of the program committee

- Susanne Albers (University of Freiburg)
- Evripidis Bampis (University of Evry)
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July 2003

Klaus Jansen and Roberto Solis-Oba
Program Chairs

Table of Contents

Contributed Talks

ARACNE Talks

Online Coloring of Intervals with Bandwidth

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Abstract. Motivated by resource allocation problems in communication networks, we consider the problem of online interval coloring in the case where the intervals have weights in $(0, 1]$ and the total weight of intersecting intervals with the same color must not exceed 1. We present an online algorithm for this problem that achieves a constant competitive ratio. Our algorithm is a combination of an optimal online algorithm for coloring interval graphs and First-Fit coloring, for which we generalize the analysis of Kierstead to the case of non-unit bandwidth.

1 Introduction

Online coloring of intervals is a classical problem whose investigation has led to a number of interesting insights into the power and limitations of online algorithms. Intervals are presented to the online algorithm in some externally specified order, and the algorithm must assign each interval a color that is different from the colors of all previously presented intervals intersecting the current interval. The goal is to use as few colors as possible. If the maximum clique size of the interval graph is ω, it is clear that ω colors are necessary (and also sufficient in the offline case [6]). Kierstead and Trotter presented an online algorithm using at most $3\,\omega - 2$ colors and showed that this is best possible [11]. Another line of research aimed at analyzing the First-Fit algorithm, i.e., the algorithm assigning each interval the smallest available color. It is known that First-Fit may need at least $4.4\,\omega$ colors [3] on some instances and is therefore not optimal. A linear upper bound of $40\,\omega$ was first presented by Kierstead [8] and later improved to $25.8\,\omega$ by Kierstead and Qin [10].

In this paper, we study a generalization of the online interval coloring problem where each interval has a weight in $(0, 1]$. Motivated by applications, we refer to the weights as bandwidth requirements or simply bandwidths. A set of intervals can be assigned the same color if for any point r on the real line, the sum of the bandwidths of its intervals containing r is at most 1. The special case where every interval has bandwidth 1 corresponds to the original online interval coloring problem. Our problem is thus a simultaneous generalization of online interval coloring and online bin-packing.

Our main result is an online algorithm that achieves a constant competitive ratio for online coloring of intervals with bandwidth requirements. The algorithm partitions

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the intervals into two classes and applies First-Fit to one class and the algorithm by Kierstead and Trotter [11] to the other class. In order to analyze First-Fit in our context, we extend the analysis by Kierstead [8] to the setting where the intervals have arbitrary bandwidth requirements in $(0, 1/2]$.

1.1 Applications

Besides its theoretical interest, investigating the online coloring problem for intervals with bandwidth requirements is motivated by several applications.

First, imagine a communication network with line topology. The bandwidth of each link is partitioned into channels, where each channel has capacity 1. The channels could be different wavelengths in an all-optical WDM (wavelength-division multiplexing) network or different fibers in an optical network supporting SDM (space-division multiplexing), for example. Connection requests with bandwidth requirements arrive online, and each request must be assigned to a channel without exceeding the capacity of the channel on any of the links of the connection. We assume that the network nodes do not support switching of traffic from one channel to another channel (which is the case if only add-drop multiplexers are used). Then a connection request from a to b corresponds to an interval $[a, b)$ with the respective bandwidth requirement, and the problem of minimizing the number of required channels to serve all requests is just our online coloring problem for intervals with bandwidth requirements.

A related scenario in a line network is that the connection requests have unit durations and the goal is to serve all connections in a schedule of minimum duration. In this case, the colors correspond to time slots, and the total number of colors corresponds to the schedule length. This is a special case of the call scheduling problem considered, for example, in [5, 4].

Finally, an interval could represent a time period during which a job must be processed, and the bandwidth of the interval could represent the fraction of a resource (machine) that the job needs during its execution. At any point in time, a machine can execute jobs whose bandwidths (here, resource requirements) sum up to at most 1. If jobs arrive online (before the actual schedule starts) and have to be assigned to a machine immediately, with the goal of using as few machines as possible, we again obtain our online interval coloring problem with bandwidth requirements.

1.2 Related Work

We have already discussed previous work on online coloring of intervals (without bandwidth requirements) in the beginning of the introduction. A survey of results for online graph coloring can be found in [9]. The problem of assigning colors to paths that represent connection requests in communication networks has been studied intensively due to its motivation by all-optical WDM networks. A survey of offline results can be found in [2]. Online path coloring results for trees and meshes are given in [1]. In these path coloring problems, the bandwidth requirement of each path is 1, implying that no two intersecting paths can get the same color.

2 Preliminaries

We are given a collection $\mathcal{I}$ of closed intervals over the real numbers $\mathbf{R}$, where each interval i is associated with a bandwidth requirement $b(i)$, where $0 < b(i) \leq 1$. In an online scenario these intervals are processed one by one in some externally determined sequence. Whenever an interval i arrives, the algorithm knows the previously processed intervals and the colors assigned to them, but it knows nothing about the unprocessed intervals arriving in the future. Based on this knowledge the algorithm irrevocably assigns a color $f(i)$ to the interval i, in such a way that for every color x and every point $r \in \mathbf{R}$ the sum of the bandwidths of the intervals containing r that are assigned color x is at most 1.

We compare the performance of an online algorithm with the number of colors used by an optimal offline algorithm, denoted by OPT. More precisely, we call an online algorithm A c-competitive if $A(\mathcal{I}) \leq c \cdot \text{OPT}(\mathcal{I})$ for all input sequences $\mathcal{I}$.

Let $\mathcal{I} = \{i_1, \ldots, i_\ell\}$ be the collection of intervals and let $b(i_t)$ denote the bandwidth requirement of interval i_t, i.e. $0 < b(i_t) \leq 1$ for $t = 1, \ldots, \ell$. See Fig. 1 for an example with $\ell = 8$ intervals depicted as rectangles. Their projections onto the real line are exactly the intervals over the real numbers, and the height of a rectangle shows the bandwidth requirement of the corresponding interval. Let $(\mathcal{I}, <)$ be the partial order of the intervals in $\mathcal{I}$, where the relation $i < j$ holds if and only if the right endpoint of interval i is less than the left endpoint of interval j, e.g. we have $i_4 < i_5$ in Fig. 1. Let $L \subseteq \mathcal{I}$ be a subset of intervals. For an interval $i \in \mathcal{I}$, we write $i < L$ ($L < i$) if for all intervals $j \in L$, $i < j$ ($j < i$) holds. The *neighborhood* of an interval $i \in \mathcal{I}$ is the set of intervals in $\mathcal{I}$ that are different from i and intersect i. It is denoted by $N(i)$. In the example, the neighborhood of interval i_4 is $N(i_4) = \{i_2, i_3\}$.

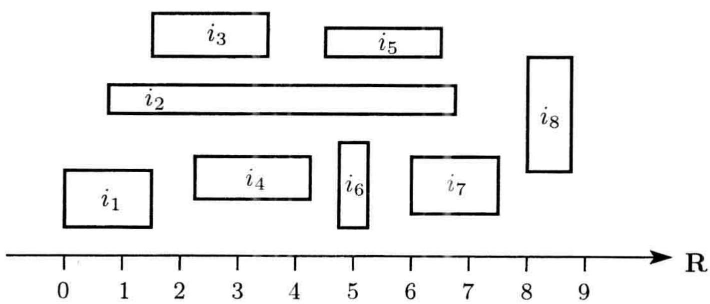


Fig. 1. The set $\mathcal{I} = \{i_1, \ldots, i_8\}$ of intervals.

The *density* $D(r \mid L)$ of L at a point $r \in \mathbf{R}$ is the sum of the bandwidths of the intervals in L that contain the point r, i.e. $D(r \mid L) = \sum_{r \in j \in L} b(j)$. The density $D(i \mid L)$ of an interval $i \in \mathcal{I}$ with respect to L is the minimum density $D(r \mid L)$ over all points r in the interval i, i.e. $D(i \mid L) = \min\{D(r \mid L) : r \in i\}$. With $D(L)$ we denote the maximum density of L, that is $D(L) = \max\{D(r \mid L) : r \in \mathbf{R}\}$. In our example shown in Fig. 1, the density $D(3 \mid \mathcal{I}) = b(i_2) + b(i_3) + b(i_4)$. The density of the interval i_3 with respect to $\mathcal{I}$ is $D(i_3 \mid \mathcal{I}) = b(i_2) + b(i_3)$ and it is achieved at the point 2 for example. Finally, the maximum density $D(\mathcal{I}) = b(i_2) + b(i_5) + b(i_6)$ is achieved at point 5.

Further, we need some notation for classifying intervals in terms of their left to right order $(\mathcal{I}, <)$. A chain C is a sequence of intervals, where $i < j$ holds for any two consecutive intervals i and j in this sequence C. Therewith, we define the *height* $h(i \mid L)$ of an interval i with respect to L to be the length of the longest chain C in L such that $C < i$. Similarly, the *depth* $d(i \mid L)$ is the length of the longest chain C in L such that $i < C$. The *centrality* of an interval i with respect to L is then given by $c(i \mid L) = \min\{h(i \mid L), d(i \mid L)\}$. Looking again at the example presented in Fig. 1, the height of interval i_4 with respect to $\mathcal{I}$ is $h(i_4 \mid \mathcal{I}) = 1$ since the longest chain consists only of the interval i_1. The corresponding depth $d(i_4 \mid \mathcal{I})$ equals 3 because of the chain $i_6 < i_7 < i_8$ of length 3. Hence, the centrality $c(i_4 \mid \mathcal{I})$ is given by $\min\{1, 3\} = 1$.

Notice that for a subset $L \subseteq N(j)$, if an interval i contains an endpoint of the interval j then $c(i \mid L) = 0$, and if $c(i \mid L) \geq 1$ then the interval i is contained in the interval j, i.e. $i \subset j$.

For our analysis we need two lemmas regarding the existence of an interval with high density in a clique of intervals. The first lemma is a generalization of a lemma that appeared in [12, 7] for the case of interval graphs, i.e. for the setting where all intervals have unit bandwidth.

Lemma 1. *Let $\mathcal{I}$ be a collection of intervals. If L is a clique in the intersection graph of $\mathcal{I}$ (meaning that any two intervals in L intersect), then there exists an interval $i \in L$ such that*

$$D(i \mid L) \geq \frac{D(L)}{2}\,.$$

Proof. Let $L = \{i_1, \ldots, i_\ell\}$. Since the intervals in L form a clique, there is a point $r \in \mathbf{R}$ that is contained in every interval in L and we have

$$D(L) = \max\{D(r \mid L) : r \in \mathbf{R}\} = \sum_{m=1}^{\ell} b(i_m).$$

We build the sequence S of bandwidths of the intervals in L from left to right. Whenever an interval i_m starts or ends, we append its bandwidth $b(i_m)$ to the sequence S. Thus, the length of the sequence S is 2ℓ since every bandwidth of intervals in L appears twice, once for the left endpoint and once for the right endpoint. Starting from the beginning of S, let S_1 be the largest initial subsequence of elements in S obeying the condition that their sum is less than $D(L)/2$. Likewise, let S_2 be the largest subsequence starting from the end of S (going backwards) under the condition that the sum of elements in S_2 is less than $D(L)/2$.

Now, we claim that there exists an interval i whose bandwidth appears neither in the subsequence S_1 nor in the subsequence S_2. This is the case, because otherwise all bandwidths of intervals would appear either in S_1 or in S_2, which, in turn, means that their total sum would be at least $D(L)$ contradicting the fact that each of the two subsequences sums up to less than $D(L)/2$.

Every such interval i has a density $D(i \mid L) \geq D(L)/2$. This follows from the construction of the sequences S_1 and S_2 and from the fact that in a left-to-right scan through a clique of intervals all left interval endpoints precede all right interval endpoints. □

Lemma 2. *Let L be a clique of intervals each of which is assigned to a different color. If every interval in L has density at least $\rho \leq 1$ within its color class, then there exists an interval $i \in L$ such that the density of i with respect to the union of the present color classes is at least $\rho \cdot |L|/2$.*

Proof. This lemma can be proven similar to Lemma 1 by building a sequence S from a left-to-right scan of the intervals in L, except that this time we append the value ρ whenever an interval in L starts or ends. But we will prove it by induction on the size of L. In the base cases $|L| = 1$ and $|L| = 2$ any interval in L has the desired property, since $|L|/2 \leq 1$. For the induction step, let j be the interval of L with the smallest left endpoint and let k be the interval of L with the largest right endpoint. Using the induction hypothesis, we choose an interval $i \in L \setminus \{j, k\}$ such that the density of the interval i with respect to the union of color classes of intervals in $L \setminus \{j, k\}$ is at least $\rho \cdot (|L|/2 - 1)$. Since the intervals j and k intersect, the interval i is contained in $j \cup k$, and because the intervals j and k have density ρ within their color classes, the density of the interval i with respect to the union of all present color classes is at least $\rho \cdot |L|/2$. □

Finally, if $N = (n_1, \ldots, n_t)$ is a sequence of length t we denote the initial subsequence of N of length k by $N_k = (n_1, \ldots, n_k)$. The result $(n_1, \ldots, n_t, n)$ of appending n to the end of N is denoted by $N \circ n$.

3 The Algorithm

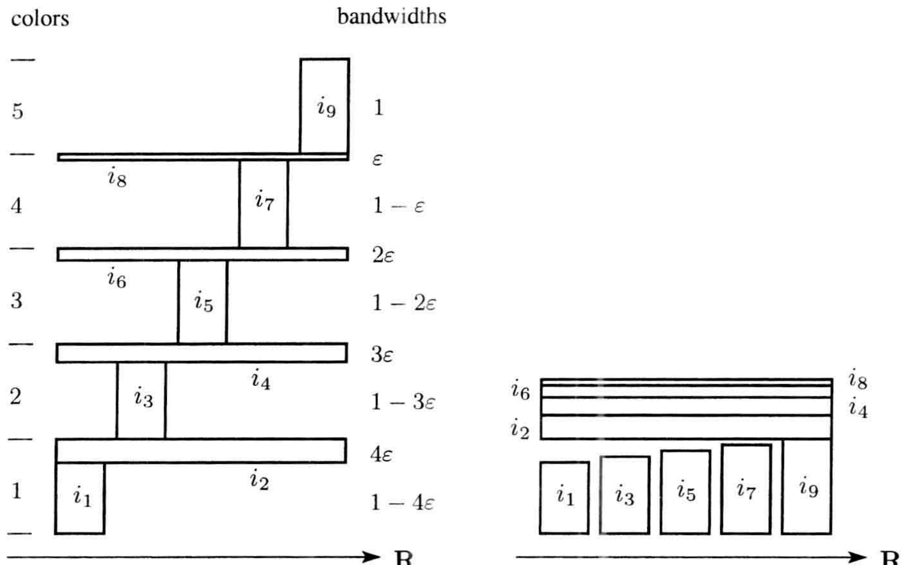
Our algorithm works with two sets C_1 and C_2 of disjoint colors. Let $\sigma = \sigma_1 \sigma_2 \cdots$ be any sequence of intervals. Each σ_i is associated with a bandwidth $b(\sigma_i)$, $0 < b(\sigma_i) \leq 1$. Whenever an interval σ_i arrives, we process it according to its bandwidth $b(\sigma_i)$.

If $b(\sigma_i) \leq 1/2$, the algorithm assigns the interval σ_i a color from the set C_1 of colors using the First-Fit principle. The First-Fit assignment is made by coloring the interval σ_i with color α, where α is the smallest color in C_1 such that for every $r \in \sigma_i$, the sum of the bandwidths of intervals previously colored α and containing r is at most $1 - b(\sigma_i)$.

If $b(\sigma_i) > 1/2$, the algorithm assigns the interval σ_i a color from the set C_2 using the optimal online interval coloring algorithm of Kierstead and Trotter [11]. Note that the intervals having bandwidth exceeding $1/2$ have the same property as intervals without bandwidth requirement, namely, if two intervals intersect, they have to receive different colors in a proper coloring.

We remark that the partition into two classes is indeed necessary in our approach, because First-Fit can be arbitrarily bad for intervals with bandwidths in $(0, 1]$ as shown in Fig. 2.

Theorem 1. *The algorithm solves the online problem of coloring intervals with bandwidths using at most 195 times as many colors as the optimal offline algorithm.*



(a) The First-Fit algorithm requires many colors for the sequence $i_1, i_2, \ldots, i_9$.

(b) The optimal offline algorithm needs only two colors.

Fig. 2. For bandwidth requirements in $(0, 1]$ the First-Fit algorithm can perform arbitrarily badly.

For the proof of this theorem we need two lemmas which we will prove below. Fix a sequence σ and let $\mathcal{I}$ be the set of all intervals appearing in σ. Let $\mathcal{I}_1 \subset \mathcal{I}$ be the intervals with bandwidth at most $1/2$, and let $\mathcal{I}_2 \subset \mathcal{I}$ be the intervals with bandwidth exceeding $1/2$. We have $\mathcal{I} = \mathcal{I}_1 \dot{\cup} \mathcal{I}_2$.

Lemma 3. *If the number of colors that the algorithm uses from the set C_1 is at least 2, then this number is less than 192 times the maximum density $D(\mathcal{I}_1)$ of intervals in $\mathcal{I}_1$.*

Lemma 4. *The number of colors from the set C_2 that are used by the algorithm is less than 3 times the number of colors used by an optimal offline algorithm for coloring the intervals in $\mathcal{I}_2$.*

Proof. (of Theorem 1) Let OPT denote the number of colors used by an optimal offline algorithm. Clearly, this number is at least as large as the maximum density of the intervals, i.e. $D(\mathcal{I}) \leq \text{OPT}$. With $\text{OPT}(\mathcal{I}_1)$ and $\text{OPT}(\mathcal{I}_2)$ we denote the number of colors assigned by an the optimal offline algorithm to the intervals in $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Since $\text{OPT}(\mathcal{I}_1)$ and $\text{OPT}(\mathcal{I}_2)$ count the number of colors for restricted sets of intervals, $\text{OPT}(\mathcal{I}_1) \leq \text{OPT}$ and $\text{OPT}(\mathcal{I}_2) \leq \text{OPT}$ hold true.

By Lemma 3, the algorithm colors the intervals in $\mathcal{I}_1$ using the First-Fit algorithm with less than $192 \cdot \text{OPT}(\mathcal{I}_1)$ many colors. This is true, because $\text{OPT}(\mathcal{I}_1) \geq D(\mathcal{I}_1)$ and if the First-Fit algorithm needs just one color for coloring the intervals in $\mathcal{I}_1$, then also the optimal offline algorithm needs one color.

By Lemma 4, the algorithm colors the intervals in $\mathcal{I}_2$ using an optimal online algorithm for coloring interval graphs with less than 3 times as many colors as an optimal offline algorithm.

Thus, the number C of colors used by the algorithm is

$$\begin{aligned} C &< 192 \cdot \mathrm{OPT}(\mathcal{I}_1) + 3 \cdot \mathrm{OPT}(\mathcal{I}_2) \\ &\leq 192 \cdot \mathrm{OPT} + 3 \cdot \mathrm{OPT} \\ &= 195 \cdot \mathrm{OPT}. \end{aligned}$$

Hence, the algorithm is 195-competitive. □

Let us start with the proof of Lemma 4.

Proof. (of Lemma 4) The optimal online interval coloring algorithm uses at most $3 \cdot \omega(\mathcal{I}_2) - 2$ colors for coloring the intervals in $\mathcal{I}_2$ [11]. Since the bandwidth of the intervals in $\mathcal{I}_2$ is greater than $1/2$, any two intersecting intervals must receive different colors. Therefore any offline algorithm needs at least $\omega(\mathcal{I}_2)$ many colors for coloring the intervals in $\mathcal{I}_2$. □

Now we will prove Lemma 3, using a rather technical induction.

Proof. (of Lemma 3)

In this proof we generalize a proof of Kierstead, who proved in [8] that First-Fit uses less than $40\,\omega$ colors for interval coloring, i.e. the case where all bandwidths are 1. We extend the proof to the case where all intervals have bandwidth at most $1/2$.

The main idea of the proof is the following: If the First-Fit coloring uses $x \geq 192 \cdot y$ colors, there is a point $r \in \mathbf{R}$ where the density $D(r \mid \mathcal{I}_1)$ exceeds y. For $y \geq D(\mathcal{I}_1)$ we would arrive at a contradiction, since—by definition of $D(\mathcal{I}_1)$—there will not be any point whose density with respect to $\mathcal{I}_1$ exceeds $D(\mathcal{I}_1)$. Therefore the number of colors used by the First-Fit algorithm is $x < 192 \cdot D(\mathcal{I}_1)$.

Let $f(i)$ denote the color that First-Fit assigns to interval i, and let $C(q) = \{i \in \mathcal{I} : f(i) = q\}$ be the set of intervals with color q. For a range of colors we denote by $C[p,q] = \cup_{p \leq r \leq q} C(r)$ all intervals whose colors are within that range.

In order to find a point with high density in $\mathcal{I}_1$, we construct a sequence I of intervals $i_1 \supset \cdots \supset i_t$ and disjoint blocks $B_1, \ldots, B_t$ of intervals such that $i_m \in B_m$ and the sum of densities $D(i_1 \mid B_1) + \cdots + D(i_m \mid B_m)$ is at least some number s_m, for $m = 1, \ldots, t$. In this construction, every block B_m consists of intervals from the neighborhood of the interval i_{m-1} whose colors are within a certain range. More formally, $B_m = C[x - 192 \cdot r_m + 1, x - 192 \cdot r_{m-1}] \cap N(i_{m-1})$ where $r_m < s_m$, for $m = 1, \ldots, t$. Since the intervals in I form a decreasing sequence, the density of the interval i_t is large, i.e. $D(i_t \mid C[x - 192 \cdot r_t + 1, x]) \geq s_t$. The numbers r_m and s_m will be chosen in such a way that $192 \cdot r_m$ and $192 \cdot s_m$ are integral.

In the induction step we would like to find an interval $i_{t+1} \subset i_t$ with high density in B_{t+1}. However, there may not be any interval with high density in B_{t+1}, or the only intervals with high density in B_{t+1} may overlap i_t. Before we explain below how to deal with these cases, we introduce some more notation.

Let $I = (i_1, \ldots, i_t)$ be a sequence of intervals and let $N = (n_1, \ldots, n_t)$ be a sequence of real numbers, where each n_m is of the form 2^{k-2} for some integer $k \geq 0$.

In particular, we have $n_m \geq 1/4$ for all $m = 1, \dots, t$. The following parameters are defined on the pair (N, I) for $1 \leq m \leq t$:

- Let s_m be the sum of the first m numbers of the sequence N, i.e. $s_m = \sum_{j=1}^{m} n_j$, and set $s_0 = 0$. Note that $4 \cdot s_m$ is integral, since all n_j's are of the form 2^{k-2}.
- Let $r_m = s_{m-1} + n_m/2 < s_m$, and set $r_0 = 0$. Observe that $8 \cdot r_m$ is integral for the same reason as above.
- Let B_m be the disjoint blocks of intervals defined by $B_m = C[x - 192 \cdot r_m + 1, x - 192 \cdot r_{m-1}] \cap N(i_{m-1})$ for $m \geq 1$, where $i_0 = \mathbf{R}$. B_m consists of all intervals in $\mathcal{I}_1$ that intersect the previous interval i_{m-1} of the sequence I and have a color in the given range.
- Let c_m be the centrality, and d_m be the density of the interval i_m within the intervals in B_m, i.e. $c_m = c(i_m \mid B_m)$ and $d_m = D(i_m \mid B_m)$.

We call a pair (N, I) *admissible* if the following conditions hold for $1 \leq m \leq t$:

(1) The interval i_m is contained in the set B_m, i.e. $i_m \in B_m$.
(2) The centrality of the interval i_m with respect to B_m is at least 2, i.e. $c_m \geq 2$, if $m > 1$.
(3) The number $n_{m+1} \geq n_m 4^{2-c_{m+1}}$, if $m < t$.
(4) The density of the interval i_m with respect to B_m is at least n_m, i.e. $d_m \geq n_m$.

Notice that if (N, I) is an admissible pair, then the density of the interval i_t is at least s_t, i.e. $D(i_t \mid \mathcal{I}_1) \geq s_t$. Since the blocks B_m are disjoint, and the intervals i_m are contained in each other by (2), the density of the interval i_t is at least the sum of the densities d_m, which in turn are at least n_m by (4). By definition s_t constitutes exactly this sum.

We shall prove the lemma by first showing that there exists an admissible pair (N, I), and if (N, I) is such an admissible pair and the number x of used colors fulfills $x \geq 192 \cdot s_t$ then there exists an admissible pair (N', I') such that $s'_{t'} > s_t$. Here and below primes indicate elements and parameters of the new admissible pair.

For the basis of the induction we need an admissible pair. Let i be any interval colored x.

If $b(i) = 1/2$, then $N = (1/2)$ and $I = (i)$ is an admissible pair consisting of $t = 1$ interval. Its parameters are as follows: n_1 and therefore s_1 equal $1/2$, and $r_1 = n_1/2 = 1/4$. Since the interval $i_1 = i$ is colored x, it is contained in the block $B_1 = C[x - 48 + 1, x]$. Further its density d_1 in B_1 is at least $1/2 = n_1$, since the interval i_1 itself contributes a bandwidth $b(i_1) = 1/2$. The conditions (2) and (3) do not play a role for $t = 1$.

If $b(i) < 1/2$, then because of First-Fit there is a point $r \in i$, where intervals colored $x-1$ intersect, whose total bandwidth exceeds $1/2$. Otherwise the interval i would have been colored $x - 1$. These intervals form a clique L and according to Lemma 1, there exists an interval $j \in L$ with density $D(j \mid L) > 1/4$. Then $I = (j)$ and $N = (1/4)$ is an admissible pair. Again we have to check the conditions. The numbers n_1 and s_1 are $1/4$, whereas $r_1 = n_1/2 = 1/8$. Since the interval $i_1 = j$ is colored $x - 1$, it is contained in the block $B_1 = C[x - 24 + 1, x]$. The condition (4) is fulfilled, because the density of the interval is $d_1 > 1/4 = n_1$.

Now fix an admissible pair (N, I). We begin the process of constructing (N', I'). Let $B = C[x - 192 \cdot s_t + 1, x - 192 \cdot r_t] \cap N(i_t)$.

The bandwidth of the interval i_t is $b(i_t) \leq 1/2$. Since $i_t \in B_t$, the color $f(i_t)$ of the interval i_t is larger than $x - 192 \cdot r_t$. According to First-Fit, there exists for each color $\alpha \in [x - 192 \cdot s_t + 1, x - 192 \cdot r_t]$ a point $r_\alpha \in i_t$ such that there are intervals with color α intersecting at r_α, whose bandwidths sum up to more than $1/2$. Otherwise the interval i_t would have received a smaller color. Since $s_t - r_t = n_t/2$, there are at least $96 \cdot n_t$ places r_α where intervals with total bandwidth greater than $1/2$ intersect, and according to Lemma 1, we find at least $96 \cdot n_t$ intervals of different colors each with density exceeding $1/4$ within its color class. Let M denote this set of intervals.

We partition M into level sets according to the left-to-right ordering of the intervals. Let $L_k = \{i \in M : h(i \mid M) = k \leq d(i \mid M)\}$ and $R_k = \{i \in M : h(i \mid M) > k = d(i \mid M)\}$. For an interval $i \in M$, $i \in L_k \cup R_k$ or $i \in M \setminus \bigcup_{j<k}(L_j \cup R_j)$ implies that the centrality $c(i \mid M)$ of i with respect to M is at least k.

By the pigeon hole principle (at least) one of the following four cases holds true. Otherwise M would contain fewer than $96 \cdot n_t$ intervals.

(i) There is an interval $i \in M$ with high centrality $c(i \mid M) \geq 3 + \lceil \log_4 n_t \rceil$.
(ii) There is a large central level set, i.e. $|L_k| \geq 8n_t 4^{2-k}$ or $|R_k| \geq 8n_t 4^{2-k}$ for some k such that $2 \leq k \leq 2 + \lceil \log_4 n_t \rceil$.
(iii) There is a large first level set, i.e. $|L_1| \geq 8n_t$ or $|R_1| \geq 8n_t$.
(iv) There is an extra large outer level set, i.e. $|L_0| \geq 28n_t$ or $|R_0| \geq 28n_t$.

- Case (i):

  If there is an interval $i \in M$ with high centrality, we extend the sequence I by this interval i. The length of the sequence increases by 1. Hence, we set $t' = t + 1$, $i'_{t+1} = i$, and $n'_{t+1} = 1/4$, because the interval i'_{t+1} has density greater than $1/4$ within its color class.

  We have to show that $N' = N \circ n'_{t+1}$ and $I' = I \circ i'_{t+1}$ form an admissible pair (N', I'). For condition (1) we have to observe that the interval i'_{t+1} is chosen from B which is a subset of B'_{t+1}, because $r'_{t+1} = s_t + n'_{t+1}/2 > s_t$. Hence, $i'_{t+1} \in B'_{t+1}$. Condition (2) holds, because $c'_{t+1} \geq 3 + \lceil \log_4 n_t \rceil \geq 2$ since $n_t \geq 1/4$. The number $n'_{t+1} = 1/4 = 4^{-1} = n_t 4^{2-3-\log_4 n_t} \geq n_t 4^{2-c'_{t+1}}$ fulfills the condition (3), and the density $d'_{t+1} \geq 1/4 = n'_{t+1}$, which establishes condition (4). The density of the interval $i'_{t'}$ with respect to $\mathcal{I}_1$ is at least $s'_{t'} = s_t + 1/4$.

- Case (ii):

  If there is a large central set, we can without loss of generality assume that $|L_k| \geq 8n_t 4^{2-k}$. The other case is exactly symmetric. Note that L_k is a clique in $\mathcal{I}_1$ since $i, j \in M$ and $i < j$ implies that $h(i \mid M) < h(j \mid M)$. Furthermore, each interval in L_k has a density greater $1/4$ within its color. Thus, by applying Lemma 2 with $\rho = 1/4$ we can find an interval $i \in L_k$ with density $D(i \mid B) > n_t 4^{2-k}$. We extend the sequence I by this interval i and set the values $t' = t + 1$, $i'_{t+1} = i$, and $n'_{t+1} = n_t 4^{2-k}$. Note that $n'_{t+1} \geq 1/4$ since $n'_{t+1} = n_t 4^{2-k} \geq n_t 4^{2-2-\lceil \log_4 n_t \rceil} \geq n_t 4^{-\log_4 n_t - 1} = 1/4$.

Again, we have to verify that $N' = N \circ n'_{t+1}$ and $I' = I \circ i'_{t+1}$ form an admissible pair (N', I'). The condition (1) is satisfied, because the interval i'_{t+1} is chosen from B, which is a subset of B'_{t+1}, since $r'_{t+1} = s_t + n'_{t+1}/2 > s_t$. The centrality $c'_{t+1} = k \geq 2$ and the number $n'_{t+1} = n_t 4^{2-k}$ fulfill the conditions (2) and (3). The condition (4) concerning the density of the interval i'_{t+1} holds, because $d'_{t+1} > n_t 4^{2-k} = n'_{t+1}$.

Hence, the density of the interval $i'_{t'}$ with respect to $\mathcal{I}_1$ is at least $s'_{t'} = s_t + n_t 4^{2-k}$.

- Case (iii)

  If there is a large first level set, we can without loss of generality assume that $|L_1| \geq 8n_t$. For the same reason as in Case (ii) the set L_1 is a clique in $\mathcal{I}_1$, and each interval in L_1 has a density greater $1/4$ within its color class. Hence, by Lemma 2 (again with $\rho = 1/4$) there exists an interval $i \in L_1$ with density $D(i \mid B) > n_t$.

  Since this interval i with high density in B'_{t+1} might have a centrality of one only, thus violating the condition (2), we replace the interval i_t with the interval i. We set $t' = t$, $i'_t = i$, and $n'_t = 2n_t$.

  Then the new block $B'_t \supseteq B_t \cup B$, since $r'_t = s_{t-1} + n'_t/2 = s_t$. Also the interval $i'_t \subset i_t$, because i'_t has centrality 1 in the subset M of the neighborhood $N(i_t)$. Hence, $i'_t \in B'_t$ establishing the condition (1). The condition (2) simply holds, because we have $c'_t \geq c_t \geq 2$, if $t > 1$. For condition (3) a rough estimate suffices, namely $n'_t > n_t \geq n_{t-1} 4^{2-c_t} \geq n'_{t-1} 4^{2-c'_t}$. The density condition (4) is satisfied, because $d'_t \geq D(i \mid B) + d_t > 2n_t = n'_t$.

  In total, the density of the interval $i'_{t'}$ with respect to $\mathcal{I}_1$ is at least $s'_{t'} = s_t + n_t$.

- Case (iv)

  If there is an extra large outer level set, we can without loss of generality assume that $|L_0| \geq 28n_t$. Each interval in L_0 has density greater $1/4$ within its color class. Let e be the leftmost point that is contained in the interval i_t as well as in all intervals in L_0. We define $T := B_t \cup B$. Then, the density $D(e \mid T)$ of the point e with respect to T is at least the density of the interval i_t with respect to B_t plus the density of e within the set L_0. Hence, $D(e \mid T) \geq D(i_t \mid B_t) + D(e \mid L_0) > n_t + 7n_t = 8n_t$, and according to Lemma 1 we find an interval j_1 in the clique of intervals in T containing e, such that $D(j_1 \mid T) \geq 4n_t$.

  Additionally, we choose an interval j_2 from B_t such that its centrality $c(j_2 \mid B_t) = c_t - 1$ and $j_2 < i_t$. Note that such an interval must exist, because of the centrality of the interval i_t in B_t.

  Now we distinguish two cases according to the left interval endpoints of j_1 and j_2. Either the left endpoint of j_2 lies further left, which implies that the centrality $c(j_1 \mid T) \geq c(j_2 \mid T) \geq c_t - 1$, or the left endpoint of the interval j_1 lies further left, in which case the interval j_2 is contained in j_1, and thus the density $D(j_2 \mid T) \geq D(j_1 \mid T) \geq 4n_t$.

  In either case there exists an interval $i \in \{j_1, j_2\}$ in T such that $D(i \mid T) \geq 4n_t$ and $c(i \mid T) \geq c_t - 1$. In order to fulfill the condition (2) we have to replace either the interval i_t or the interval i_{t-1} with the new interval i depending on the centrality $c(i \mid T)$.