Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries: Instituto de Matemática Pura e Aplicada, Rio de Janeiro
Adviser: C. Camacho

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Partial Differential Equations

Proceedings, Rio de Janeiro 1986

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Partial Differential Equations

Proceedings of ELAM VIII,
held in Rio de Janeiro, July 14–25, 1986

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VIII LATIN AMERICAN SCHOOL OF MATHEMATICS

The Latin American School of Mathematics (ELAM) is one of the most important mathematical events in Latin America. It is held every other year since 1968 in a different country of the region, and its theme varies according to the areas of interest of local research groups.

The VIII ELAM took place at the Instituto de Matemática Pura e Aplicada (IMPA/CNPq), Rio de Janeiro, Brazil, during the period of July 14-25, 1986.

The subject of the school was Partial Differential Equations with emphasis on Microlocal Analysis, Scattering Theory and the applications of Nonlinear Analysis to Elliptic Equations and Hamiltonian Systems.

The School was attended by mathematicians from many countries, ARGENTINA (12), BELGIUM (2), BRAZIL (125), CHILE (8), COLOMBIA(5), FRANCE (9), GERMANY (6), ISRAEL (1), ITALY (8), JAPAN (3), MEXICO (7), PERU (2), SOVIET UNION (1), SWEDEN (2), SWITZERLAND (2), UNITED STATES (19) and VENEZUELA (5).

This volume contains most of the conference delivered at the VIII ELAM.

We would like to express our gratitude to the ELAM International Committee, whose members are Professors Jacob Palis Junior (Brasil), José Adem (México) and Carlos Segóvia (Argentina), for having given us the opportunity to organize the VIII ELAM.

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Finally we must thank the members of IMPA's staff, in particular Suely Maiato and José Manoel do Outeiro, for their priceless organizational work and Wilson Góes for typing most of the papers contained in the present volume.

The Organizing Committee,

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3. M.BEN-ARTZI (Technion III, Haifa, Israel) - "The Limiting Absorption Principle for Differential Operators with Short-Range Perturbations".

4. P.BÉRARD (Univ.Savoi, France) - "On the Number of Bound States and Estimates on Some Geometric Invariants".

5. O.BOGOYAVLENSKY (Steklov Mathematical Institute, USSR) - "New Integrable Problems of Classical Mechanics".

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6. H.BRÉZIS (Univ.Paris VI, France) - "Liquid Crystal Degree Theory and Energy Estimates for S^k Valued Maps".

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39. D.ROBERT (Univ.Nantes, France) - "Semiclassical Analysis in Scattering Theory" and "Semiclassical Limit and Ergodicity".

40. L.RODINO (Univ.di Torino, Italy) - "Microlocal Analysis in Gevrey Classes".

41. J.SJÖSTRAND (Univ. of Lund, Sweeden) - "Estimates on the Number of Resonances for Semiclassical Schrödinger Operators".

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CONTENTS

CRITICAL POINTS AT INFINITY IN THE VARIATIONAL CALCULUS

A. BAHRI

in the honour of H. Lewy

1. ORBITS OF THE FLOW.

Let E be a space of variations (either a Hilbert space or a manifold modelled on a Hilbert space for sake of simplicity) and let :

(1) $$f : E \longrightarrow \mathbb{R}$$

be a functional which we will assume to be C^∞, for sake of simplicity also. Let :

(2) $(,)$ be the scalar product on E

and

(3) $\partial f(x)$ be the gradient of f at x in E.

We are concerned with finding a solution to the equation :

(4) $$\partial f(x) = 0 \quad x \in E \ .$$

For $a \in \mathbb{R}$, we introduce the level sets of the functional f at a :

(5) $$\begin{cases} f^a = \{x \in E \mid f(x) \leq a\} & \text{(sub-level set)} \\ f_a = \{x \in E \mid f(x) \geq a\} & \text{(upper-level set)} \\ {}^a f = \{x \in E \mid f(x) = a\} & \text{(level surface)} \ . \end{cases}$$

We also consider the differential equation :

(6) $$\frac{\partial x}{\partial s} = - \partial f(x) \ ; \quad x(0) = x_o \ .$$

Let :

(7) $x(s,x_o)$ be the solution of (6).

We then have the following very simple principle to solve (4) :

Proposition 1 : Let $b < a$. Assume f^b is not retract by deformation of f^a (e.g. $H_*(f^a, f^b; G) \neq 0$ for a certain coefficient group G ; and H_* is a homology theory ; or $\pi_*(f^a, f^b; G) \neq 0$; π_* is homotopy ...) Then :

either (4) has a solution $\bar{x}$ with $b \leqslant f(\bar{x}) \leqslant a$

or there exists an x_o such that $b < f(x_o) \leqslant a$ such that

(a) $\lim_{s \to +\infty} f(x(s, x_o)) \geqslant b$

(b) the closure of the set $\{x(s, x_o) ; s \geqslant 0\}$ is non compact .

There are several directions where one can make the content of Proposition 1 more precise and in the same time more general :

1st precision : In case $\bar{x}$ exists, the basic assumption which is used to study the situation near $\bar{x}$ is that ∂f is Fredholm at $\bar{x}$. Then, if $\bar{x}$ is a non degenerate critical point of f, one computes the Morse index of f at $\bar{x}$.

We then have :

Assume the injection $f^b \to f^a$ is not a homotopy equivalence, then $\bar{x}$ cannot contribute to the difference of topology between f^b and f^a if the Morse index of f at $\bar{x}$ is infinite.

Thus a critical point $\bar{x}$ is relevant in the calculus of variations in the large if it comes with some properness assumption (Fredholm structure) and if it has a finite Morse index.

In case the Fredholm structure is available, there is a way to drop the assumption of non degeneracy (see Marino and Prodi [1]). Otherwise, very few is known.

2nd precision : In the second case of the alternative provided by Proposition 1, we have :

(8) $$\int_0^{+\infty} |\partial f(x(s, x_o))|^2 \, ds < f(x_o) - b$$

(9) $\int_0^{\tau} \partial f(x,(s, x_o))ds$ cannot be included in a compact set for $\tau \in [0, +\infty[$.

We then have a sequence (s_n) or either $(x(s_n, x_o))$ such that :

(10) $$b \leqslant f(x(s_n, x_o)) \leqslant a \quad ; \quad \partial f(x(s_n, x_o)) \longrightarrow 0 \quad .$$

Classically, in the calculus of variations, there is an assumption, called the (C) condition, introduced by Palais and Smale forbidding (10). The content of this condition is the following :

(11) for any sequence (x_n) such that $b \leq f(x_n) \leq a$ and $\partial f(x_n) \to 0$, there is a convergent subsequence.

This condition forbids the second case in Proposition 1 and thus allows to find solutions to (4) via the study of the difference of topology in the level sets.
However the condition (C) forbids more tant (8)-(9). Indeed, in (8)-(9), we deal with sequences which lie on the same orbit of the flow ; while with the (C) condition, we deal with arbitrary sequences (both of them satisfying $b \leq f(x_n) \leq a$; $\partial f(x_n) \to 0$).

Thus, to do variational calculus, we need :

(1) either a weaker condition than (C) or (11) : namely, we can impose on the x_n's to lie on the same deformation line,

(2) or to study the difference of topology induced in the level sets by theses orbits of the flow which satisfy (8)-(9).

The first case (1) is just an improvement of condition (C).
The second case (2) is concerned with getting rid of any condition of this type, by considering the flow as a dynamical system, having possibly singularities "at infinity".
This goal of getting rid of the condition (C) has been set by S. Smale in his book : "Mathematics of the time" [2].

It is far from being concretly achieved in all variational problems of interest.

3rd precision : The invariants.

The orbits of the flow satisfying (8)-(9) are of course relevant to the flow itself , i.e. to the differential equation (6).

On the other hand, the variational calculus is concerned with the function f ; in fact, equation (4) means that it is impossible to decrease strictly with respect to f a whole neighbourhood in E of the critical point x.

From this correct interpretation of the calculus of variations, we may replace (6) by any other differential equation corresponding to a (pseudo)-gradient for f (in particular consider another scalar product on E) or even more consider any decreasing (with respect to f), globally defined deformation of the level sets.

Thus we see immediately that there is some ambiguity, if we do not make some further specification in case ② .

We discuss here this ambiguity :

1 - First of all, there is an intrinsic notion relevant to f and not ∂f (or either deformation). This notion is the difference of topology of the level sets when crossing the critical level, say c.
i.e. if this critical level c is isolated, which we will assume for sake of simplicity , then for $\varepsilon > 0$ small enough, the homotopy type of the space $f^{c+\varepsilon}/f^{c-\varepsilon}$ is an invariant.

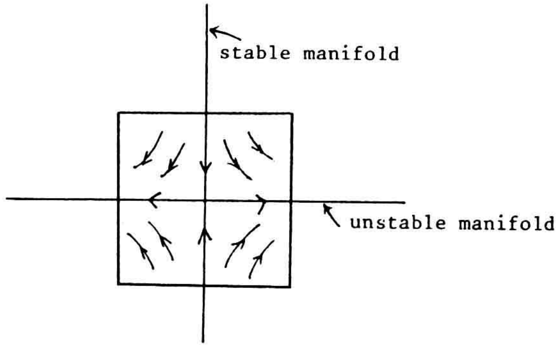
2 - There is a second invariant which is of dynamical type.

In dynamical systems, an invariant set for a flow comes with a stable and unstable manifold, at least when it is non degenerate (or either hyperbolic), which holds generically (otherwise some perturbative argument is necessary using some kind of Fredholm structure ; these days, even degenerate cases are studied, see for instance the work of Yomdin in algebraic geometry and Cappell-Weinberger in algebraic topology).

What is invariant in a hyperbolic situation is not the stable and unstable manifolds but rather their dimension and the qualitative behaviour of the flow on the boundary of an isolating block in the sense of C.C. Conley [3].

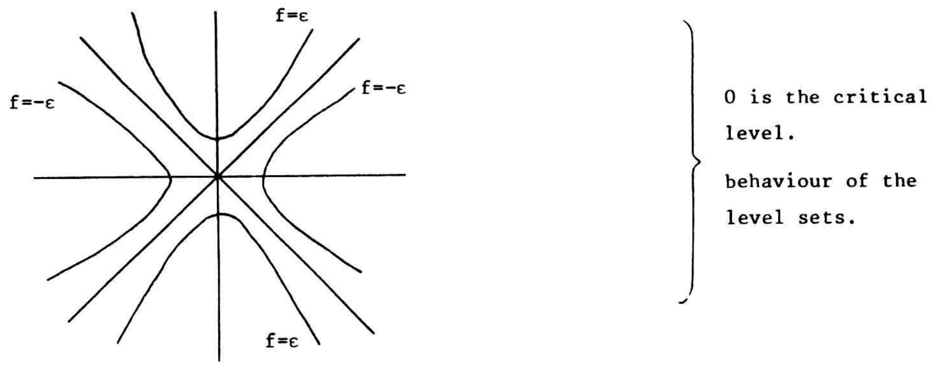
To give the simplest picture of this invariant, the best example is the situation nearby a non degenerate critical point of a function f on a finite dimensional manifold. We then have, by Morse lemma, the following local situation (see M. Hirsch [4], for instance).

(12)



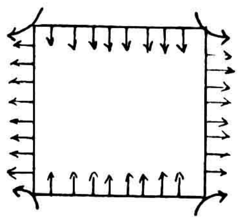
dynamics of the flow and of any pseudo-gradient hyperbolic flow

(13)



In (12), we can retain the behaviour of the flow on the boundary of the box :

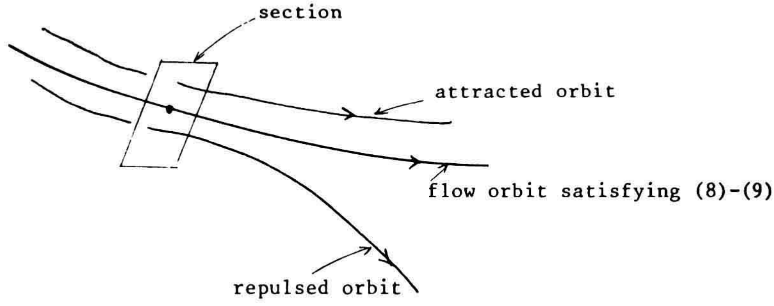
(14)



C.C Conley [3] has introduced invariants related to this Morse decomposition, namely the homology of what comes in and what comes out.
C.C. Conley introduced these invariants for a general hyperbolic flow. In such a genral situation, these invariants do not determine the invariant set and its hyperbolic structure (i.e. stable and unstable manifolds) inside the isolating blowk. However, when the flow is pseudo-gradient and the critical (or rest) point is isolated (without assumption of degeneracy), these invariants completely determine the nature of the critical point inside.

This notion of Morse decomposition and isolating block can be extended to the situation of (8)-(9), giving rise to Conley invariants related to this situation, which provide with a second set of invariants, more precise than the difference of topology in the level sets.
Qualitatively, we draw the following picture of the flow, under (8)-(9).

(15)



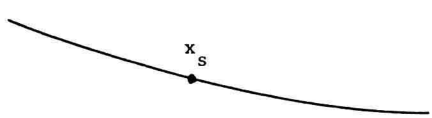
The analysis of the flow on sections provides with this second set of invariants.

4th precision : The global nature of the deformation.

With this phenomenon of the failure of the (C)-condition, there is something important which enters into account :

Consider the case of a usual non degenerate critical point for a functional f, then there is no way one can decrease with respect to the functional a whole neighbourhood of this critical point.

With a flow line going to $+\infty$, the situation is different :



Indeed, consider a point x_s on this flow line : then there is no problem to decrease with respect to the functional a whole neighbourhood to x_s.
But, in fact, there is more : namely, one can decrease in most cases where we studied critical points at infinity a whole neighbourhood of the flow line ; in particular, the flow line itself.
What really matters is how "large" this neighbourhood can be taken, with respect to the structure of the flow in section and also (it is somewhat the same thing) how "large" it is with respect to the "end of the orbit". There is a real ambiguity here, of the same kind than one encounters with the ends of analytic functions. What is defined is a way to arrive to them (when they are "accessible") ; not really themsleves ; i.e. the way to arrive to them defines them. This is why the global nature of the

deformation, in particular the flow when necessary, is an important tool.

(For a discussion on the ends and the accessibility, see Moise, "Geometric topology in dimension 2 and 3".)

5th precision : The representation device.

Assume there are natural ends for the flow lying in some manifold V. This manifold might be given by the sequences (x_n) violating the (C)-condition. This manifold depends then only on the notion of pseudo-gradient, not on the precise pseudo-gradient chosen.
Consider then a bundle F over V with fiber a space of parameters $\Lambda = \Lambda_1 \times H$ where Λ_1 is, for sake of simplicity, finite dimensional and H is a neighbourhood of zero in a Hilbert space (or so).
Let $(\Lambda_1 \times H)_x$ be the fiber at x.
Assume now there is a way to represent the functions x of E such that $|\partial f(x)| < \varepsilon$; $|f(x) - c| < \varepsilon$ in F. Calling this representation R, we have a functional defined on F ; namely f(Rx).
Assume also that over any point u in V and for any $\lambda_1 \in \Lambda_1$, we can minimize this functional for the variations in H.
We then have a functional on Γ, a bundle over V, with fiber Λ_1 ; and we are reduced to a variational problem of finite dimensional type, which depends on the representation R.
In the simplest case, $\Lambda_1 = (\mathbb{R}^+)^p$, $p \in \mathbb{N}$ and when $\varepsilon \to 0$, $(\lambda_1, \dots, \lambda_p) \to (0, \dots, 0)$ in $(\mathbb{R}^+)^p$.
It is then natural to look at the functional f(Rx) over V in a neighbourhood of (0,...,0) in the fiber. Then V might be thought as the space of variations at infinity (not the critical set at infinity) ; and the behaviour of the functional in a neighbourhood of Γ will select, under good conditions, the part of V which is critical "at infinity".
This relates this kind of variational problems to the study of the singularities. This is the way C.H. Taubes proceeds [13] ; and he proves that the singularities do not interfere with certain homology or homotopy classes.

After these five precisions, we introduce the notion of critical point at infinity (see Abbas Bahri [5]).

Definition 1 : A critical point at infinity is an orbit of a (pseudo)-gradient flow for the functional f, starting at a point x_0 for $s \geqslant 0$, such that $f(x(s,x_0)) \to c \in \mathbb{R}$, whose closure in E is non compact.
Thus a critical point at infinity is related to a (pseudo)-gradient.
If it is of hyperbolic type, or if we are dealing with a hyperbolic set of

critical points at infinity, there are invariants (in particular Conley invariants) related to such a critical point at infinity.
Finally, in case there is an appropriate extension of the variational problem nearby infinity (i.e. the bundle F) with a normal form of f, then there might exist a space of representation for these critical points at infinity.
In any case, the notion of critical point at infinity is not intrinsic to the variational problem.

2. AN ABSTRACT DEFORMATION LEMMA.

Assume we know that, with $b < a$,

(16) f^b is not retract by deformation of f^a ,

but we do not know the condition (C) to hold on $[b,a]$.
There is then a way to analyze the possible defect of compactness, which amounts to a deformation lemma we present now.
Assume also we have a function $g : E \to \mathbb{R}$ such that if

(17) (x_n) is a sequence such that $b \leqslant f(x_n) \leqslant a$; $\partial f(x_n) \to 0$ and $(g(x_n))$ is bounded, there is a convergent subsequence.

Thus, if a sequence (x_n) violates the (C)-condition, $g(x_n)$ goes to $+\infty$.
In general, there are many possible choices of g and the best one is in some sense the function g (if it exists) which measures how a sequence violating the (C)-condition leaves the compact sets of E (examples will be provided later on).

Let :

(18) $$\varphi(x) = \frac{|\partial f|}{|\partial g|}(x)(f(x)+g(x)) \quad \text{if} \quad \partial g(x) \neq 0 \quad ; \quad \varphi(x) = +\infty \quad \text{if} \quad \partial g(x) = 0 \quad .$$

We assume

(19) $$\partial g(x) = 0 \Longrightarrow \partial f(x) \neq 0 \quad \text{if} \quad b \leqslant f(x) \leqslant a \quad .$$