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Nonequilibrium Problems in Many-Particle Systems

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Nonequilibrium Problems in Many-Particle Systems

Lectures given at the 3rd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini, Italy, June 15-27, 1992

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Preface

This volume contains the text of four sets of lectures delivered by L. Arkeryd, P.-L. Lions, P. A. Markowich and S. R. S. Varadhan at the third session of the Summer School organized by C.I.M.E. (Centro Internazionale Matematico Estivo). These texts are preceded by an introduction written by us which summarizes the present status in the area of Nonequilibrium Problems in Many-Particle Systems (this was the title given to the session) and tries to put the contents of the different sets of lectures in the right perspective, in order to orient the reader.

The lectures presented in this volume deal with the global existence of weak solutions for kinetic models and related topics, the basic concepts of non-standard analysis and their application to gas kinetics, the kinetic equations for semiconductors and the entropy methods in the study of hydrodynamic limits. The lectures were of high level and the school was by all standards a success.

We feel that this volume gives a coherent picture of an important field of applied mathematics which has undergone many important developments in the last few years.

Carlo Cercignani
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1. A sketch of the history of kinetic theory.

According to the atomic theory of matter, all bodies are made up of tiny constituents (particles, molecules, atoms) that, as long as we can ignore quantum effects, move according to the laws of classical mechanics. Thus, *e. g.*, if no external forces, such as gravity, are assumed to act on the particles, each of them will move in a straight line unless it happens to interact with another particle or a solid wall.

Although the rules generating the dynamics of these systems are easy to prescribe, the phenomena associated with this dynamics are not so simple. They are actually rather difficult to understand, especially if one is interested in the asymptotic behavior of the system for long times (ergodic properties) or in the case when the number of particles is very large (kinetic and hydrodynamical limits). Both aspects of the dynamics of molecules are relevant when dealing with a gas, but kinetic theory of gases[1] concentrates upon the problem of outlining the behavior of this system when the number of the particles is very large. This is due to the fact that there are about $2.7 \cdot 10^{19}$ molecules in a cubic centimeter of a gas at atmospheric pressure and a temperature of 0°C.

Given the enormous number of particles to be considered, it would of course be a perfectly hopeless task to attempt to describe the state of the gas by specifying the so-called microscopic state, *i. e.* the position and velocity of every individual particle, and we must have recourse to statistics. This is possible because in practice all that our observations can detect are changes in the macroscopic state of the gas, described by quantities such as density, velocity, temperature, stresses, heat flow, which are related to suitable averages of quantities depending on the microscopic state. At this point, however, a question of principle must be considered. If we knew the exact position and velocity of every molecule of the gas at a certain time instant, the further evolution of the system would be completely determined, according to the laws of mechanics; even if we assume that at a certain moment the positions and velocities of the molecules satisfy certain statistical laws, we are not entitled to expect that at any later time the state of the gas will conform to the same statistical assumptions, unless we prove that this is what mechanics predicts. In certain cases, it turns out that mechanics easily provides the required justification, but things are not so easy, and questions become much more complicated, if the gas is not in equilibrium, as is, *e. g.*, the case for air around a flying vehicle.

Questions of this kind have been asked since the appearance of the kinetic theory of gases; today the matter is relatively well understood and a rigorous kinetic theory

is emerging, as the contributions to this volume will illustrate. The importance of this matter stems from the need of a rigorous foundation of such a basic physical theory not only for its own sake, but also as a prototype of a mathematical construct central to the theory of non-equilibrium phenomena in large systems.

As is well known, James Clerk Maxwell (1831-1879) in 1866 developed an accurate method to deal with the nonequilibrium behavior of a gas[2], based on the transfer equations, and discovered the particularly simple properties of a model, according to which the molecules interact with a force inversely proportional to the fifth power of the distance (nowadays commonly called Maxwellian molecules). In the same paper he gave a better justification of a formula that he had previously discovered for the velocity distribution function for a gas in equilibrium.

With his transfer equations, Maxwell had come very close to an evolution equation for the distribution function, but this step must be credited to Ludwig Boltzmann[3] (1844-1906). The equation under consideration is usually called the Boltzmann equation and sometimes the Maxwell-Boltzmann equation (to recognize the important role played by Maxwell in its discovery).

In the same paper, where he gave a heuristic derivation of his equation, Boltzmann deduced an important consequence from it, which later came to be known as the H-theorem. This theorem attempts to explain the irreversibility of natural processes in a gas, by showing how molecular collisions tend to increase entropy. The theory was attacked by several physicists and mathematicians in the 1890's, because it appeared to produce paradoxical results. However, within a few years of Boltzmann's suicide in 1906, the existence of atoms had been definitely established by experiments such as those on Brownian motion.

The paradoxes indicate, however, that some reinterpretation is necessary. Boltzmann himself had proposed that the H-theorem be interpreted statistically; later, Paulus Ehrenfest (1880-1933), together with his wife Tatiana, gave a brilliant analysis of the matter, which elucidated Boltzmann's ideas and made them highly plausible, at least from a heuristic standpoint. A rigorous analysis, however, had still to come.

In the meantime, the Boltzmann equation had become a practical tool for investigating the properties of dilute gases. In 1912 the great mathematician David Hilbert (1862-1943) indicated[4] how to obtain approximate solutions of the Boltzmann equation by a series expansion in a parameter, inversely proportional to the gas density. The paper is also reproduced as Chapter XXII of his treatise entitled *Gründzüge einer allgemeinen Theorie der linearen Integralgleichungen.* The reasons for this are clearly stated in the preface of the book ("Neu hinzugefügt habe ich zum Schluss ein Kapitel über kinetische Gastheorie. [...] erblicke ich in der Gastheorie die glänzendste Anwendung der die Auflösung der Integralgleichungen betreffenden Theoreme").

In about the same year (1916-1917) Sidney Chapman[5] (1888-1970) and David Enskog[6] (1884-1947) independently obtained approximate solutions of the Boltzmann equation, valid for a sufficiently dense gas. The results were identical as far as practical applications were concerned, but the methods differed widely in spirit and detail. Enskog presented a systematic technique generalizing Hilbert's idea, while Chapman simply extended a method previously indicated by Maxwell to obtain transport coefficients. Enskog's method was adopted by S. Chapman and T. G. Cowling in their book

The Mathematical Theory of Non-uniform Gases and thus became to be known as the Chapman-Enskog method.

Then for many years no essential progress in solving the equation came. Rather the ideas of kinetic theory found their way in other fields, such as radiative transfer, the theory of ionized gases and, subsequently, in the the theory of neutron transport. Almost unnoticed, however, the rigorous theory of the Boltzmann equation had started in 1933 with a paper[7] by Tage Gillis Torsten Carleman (1892-1949), who proved a theorem of global existence and uniqueness for a gas of hard spheres in the so-called space homogeneous case. The theorem was proved under the restrictive assumption that the initial data depend upon the molecular velocity only through its magnitude. This restriction is removed in a posthumous book by the same author[8].

In 1949 Harold Grad (1923-1986) wrote a paper[9], which became widely known because it contained a systematic method of solving the Boltzmann equation by expanding the solution into a series of orthogonal polynomials. In the same paper, however, Grad made a more basic contribution to the theory of the Boltzmann equation. In fact, he formulated a conjecture on the validity of the Boltzmann equation. In his words: "From the preceding discussion it is possible to see along what lines a rigorous derivation of the Boltzmann equation should proceed. First, from equilibrium considerations we must let the number density of molecules, N, increase without bound. At the same time we would like the macroscopic properties of the gas to be unchanged. To do this we allow m to approach zero in such a way that $mN = \rho$ is fixed. The Boltzmann equation for elastic spheres, [...] has a factor σ^2/m in the collision term. If σ is made to approach to zero at such a rate that σ^2/m is fixed, then the Boltzmann equation remains unaltered. [...] In the limiting process described here, it seems likely that solutions of Liouville's equation attain many of the significant properties of the Boltzmann equation."

In the 1950's there were some significant results concerning the Boltzmann equation. A few exact solutions were obtained by C. Truesdell[10] in U.S.A. and by V. S. Galkin[11,12] in Soviet Union, while the existence theory was extended by D. Morgenstern[13], who proved a global existence theorem for a gas of Maxwellian molecules in the space homogeneous case. His work was extended by L. Arkeryd[14,15] in 1972.

In the 1960's, under the impact of the problems related to space research, the main interest was in the direction of finding approximate solutions of the Boltzmann equation and developing mathematical results for the perturbation of equilibrium[16,1]. Important methods developed by H. Grad[17] were brought to completion much later by S. Ukai, Y. Shizuta and K. Asano[18–20].

The problem of proving the validity of the Boltzmann equation was still completely open. In 1972, C. Cercignani[21] proved that taking the limit indicated by Grad in the passage quoted above (now currently called the Boltzmann-Grad limit) produced, from a formal point of view, a perfectly consistent theory, *i. e.* the so-called Boltzmann hierarchy. This result clearly indicated that the difficulties of the rigorous derivation of the Boltzmann equation were not of formal nature but were at least of the same order of difficulty as those of proving theorems of existence and uniqueness in the space inhomogeneous case. Subsequently, O. Lanford proved[22] that the formal derivation becomes rigorous if one limits himself to a sufficiently short time interval. The problem of a rigorous, globally valid justification of the Boltzmann equation is still open, except for the

case of an expanding rare cloud of gas in a vacuum, for which the difficulties were overcome by R. Illner and M. Pulvirenti[23-24], after that Illner and Shinbrot had provided the corresponding existence and uniqueness theorem for the Boltzmann equation[25].

Recently, R. Di Perna and P. L. Lions[26] have proved a global existence theorem for quite general data, but several important problems, such as proving that energy is conserved or controlling the local growth of density are still open. This result and the ideas related to it will be described in the contribution of P. L. Lions to this volume.

Before this basic result was obtained, the best results in the space inhomogeneous case were those of Arkeryd by means of the techniques of nonstandard analysis. These methods still play an important role in the exploration of open problems of kinetic theory and will be described in the contribution of L. Arkeryd.

The techniques of kinetic theory have become useful in many other fields, such as neutron transport in nuclear reactors, plasma physics, radiative transfer. One of the important recent applications is in the field of semiconductors. As a matter of fact, when the transport of charges in a semiconductor is considered on a sufficiently large time scale, then the motion of the carriers is decidedly influenced by the short range interactions with the crystal lattice, which can be described, in a classical picture of the electron gas, by particle collisions. This situation, which occurs in high-density integrated circuits, explains why there has been an increasing interest in understanding the mathematics of electron transport in submicron structures. The basic tool, in this situation, is given by the Boltzmann equation for carriers, which may exclude the short range interactions between these, which only play a role when the particle density is very large, but can incorporate the Pauli esclusion principle, if necessary. The recent mathematical developments in this field will be reviewed by P. Markowich.

The Boltzmann equation is one of the kinetic equations that can be considered. In dealing with semiconductors, *e. g.*, one frequently considers the field produced by the electrons themselves, which is related to the distribution function through the Maxwell equations because the charge and current density are proportional to moments of the distribution function. This, in the case of no collisions with the lattice, produces the Vlasov-Maxwell system of equations that in the simplest case (quasi steady electric field) reduces to the simpler Vlasov-Poisson system. If collisions are taken into account then one has to face the so called Boltzmann-Vlasov-Maxwell (or Boltzmann-Vlasov-Poisson) system. If the important effect of grazing collisions in a Coulomb field is taken into account, one obtains the so-called Landau equation. Finally, if the quantum aspects of electron transport are also considered one may obtain transport equations from the Schrödinger equation via the Wigner transform. All these equations will appear in the lectures by P. L. Lions and P. Markowich.

So far, we have discussed the Boltzmann (or kinetic) regime and the Vlasov (or mean field) regime for a many particle system in a non-equilibrium situation. It is also of primary importance to illustrate the hydrodynamical behavior of such a system. Real fluids are usually described by the Euler (or Navier-Stokes) equations, which are believed to be a reduced description of the particle system. Actually, if the right space-time scales are adopted and an appropriate limit is taken, Newton equations formally lead to the Euler equations. Unfortunately, very little is known on this problem from a rigorous point of view. Some progress, however, has been recently achieved by the so

called entropy method as will be illustrated in the lectures of R. S. Varadhan. Most of his analysis will be devoted to model systems which are ruled by a stochastic dynamics.

2. Basic equations and properties.

The Boltzmann equation is an evolution equation for the distribution function $f(x,\xi,t)$, which gives the probability density of finding a molecule at position x at time t with velocity ξ. If we assume that there is no body force (such as gravity) acting on the particles, we may write the Boltzmann equation in the following form

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} = Q(f,f) \tag{2.1}$$

where

$$Q(f,f) = \int\int (f'f'_* - ff_*)B(\xi - \xi_*, n)d\xi_* dn \tag{2.2}$$

Here $B(\xi - \xi_*, n)$ is a kernel associated with the details of the molecular interaction, f' f'_*, f_* are the same thing as f, except for the fact that the argument ξ is replaced by ξ', ξ'_*, ξ_*. The latter is an integration variable having the meaning of the velocity of a molecule colliding with the molecule of velocity ξ, whose evolution we are following, while ξ' and ξ'_* are the velocities of two molecules entering a collision which will bring them into a pair of molecules with velocities ξ and ξ_*. n is a unit vector defining the direction of approach of two colliding molecules. For details, we refer to the bibliography[1]. The collision term, although complicated, has many interesting properties, such as

$$\int Q(f,f)\phi(\xi)d\xi = \frac{1}{4}\int ff_*(\phi' + \phi'_* - \phi - \phi_*)B(|\xi - \xi_*|, n)d\xi_* d\xi dn \tag{2.3}$$

We now observe that the integral in Eq. (2.3) is zero independent of the particular functions f and g, if

$$\phi + \phi_* = \phi' + \phi'_* \tag{2.4}$$

is valid almost everywhere in velocity space. Since the integral appearing in the left hand side of Eq. (2.2) is the rate of change of the average value of the function ϕ due to collisions, the functions satisfying Eq. (2.4) are called "collision invariants". They play an important role in the discussion of the Boltzmann equation. It can be shown[1,27,28] that the most general solution of Eq. (2.4) is given by

$$\phi(\xi) = A + B \cdot \xi + C|\xi|^2 \tag{2.5}$$

Another important result is obtained by letting $\phi = \log f$ in Eq. (2.3). In fact the properties of the logaritmic function lead to the *Boltzmann inequality*:

$$\int_{R^3} \log f Q(f,f)d\xi \leq 0 \tag{2.6}$$

Further, the equality sign applies if, and only if, $\log f$ is a collision invariant, or, equivalently:

$$f = A\exp(-\beta|\xi - v|^2) \tag{2.7}$$

where A is a positive constant related to a, c, $|b|^2$ (β , v, A constitute a new set of constants). The function appearing in Eq. (2.7) is the so called ***Maxwell distribution*** or ***Maxwellian.*** It is a simple corollary, then, that the Maxwellians are the only functions for which $Q(f, f)$ vanishes.

3. The Vlasov equation and the mean-field limit.

In this section we exploit a few elementary facts concerning the Vlasov dynamics and its relation to the Newton equations.

Our starting point is a conservation law of the type:

$$\partial_t f(x,t) + \ \mathrm{div}(uf(x,t)) = 0 \tag{3.1}$$

where $f = f(x,t)$ is a probability density, $x \in \mathbf{R}^N$ and t is time; u is a vector field:

$$u = u(x,t) \in \mathbf{R}^N \tag{3.2}$$

which is a linear functional of f of the form

$$u(x,t) = \int dy K(x-y) f(y,t) \tag{3.3}$$

where K is a given, smooth, vector-valued kernel.

Consider now the N-particle system obeying the following ordinary differential system:

$$\frac{dx_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} K(x_i - x_j) \tag{3.4}$$

and the measure-valued function

$$\mu^N(t,dx) = \frac{1}{N}\sum_{j=1}^{N} \delta(x - x_j(t)) \tag{3.5}$$

where $x_j(t)$, $j = 1, 2, \ldots, N$ is a solution of Eq. (3.4) and δ denotes, as usual, the Dirac measure. If we let

$$\mu^N(t,\phi) = \frac{1}{N}\sum_{j=1}^{N} \phi(x_j(t)) = \int \mu^N(t,dx)\phi(x) \tag{3.6}$$

where ϕ is a bounded smooth function, an easy calculation shows that

$$\frac{d}{dt}\mu^N(t,\phi) = \mu^N(t, u\cdot\nabla\phi) \tag{3.7}$$

In other words $\mu^N(t,\cdot)$ is a weak solution to Eq. (3.1).

The following natural question arises. Assume that, at time zero

$$\mu^N(0,\phi) \to \int f(0,x)\phi(x)dx, \text{ as } N \to \infty \quad ; \tag{3.8}$$

then is the same true at time t? *I. e.*, denoting by $f(x,t)$ the solution of Eq. (3.1) with initial datum $f(0,x)$, we ask whether

$$\mu^N(t,\phi) \to \int f(t,x)\phi(x)dx, \text{ as } N \to \infty \quad . \tag{3.9}$$

This convergence can indeed be proved. Actually, it is nothing else than a continuity property of the solutions of Eq. (3.1) with respect to initial conditions, in the topology of the weak convergence of measures.

Thanks to (3.9) we can say that Eq. (3.1) has been rigorously derived, in the so called mean-field approximation, starting from the particle dynamics (3.4).

The above analysis can be slightly modified to include the Vlasov equation, which, in conservation form, reads as follows:

$$\partial_t f(x,\xi,t) + \ \mathrm{div}_{x,\xi}(Uf(x,\xi,t)) = 0 \tag{3.10}$$

where:

$$U = (\xi, K * \rho) \in \mathbf{R}^{2d} \qquad (\rho = \int f(x,\xi)d\xi) \tag{3.11}$$

where d (=2,3) is the number of dimensions of the physical space and * denotes convolution.

So far, we have assumed that K is smooth. An important case for applications is, however, the kernel

$$K(x) = \alpha \frac{x}{|x|^{d-1}} \tag{3.12}$$

where $\alpha \in \mathbf{R}$ is a constant. In this case Eq. (3.10) is called Vlasov-Poisson for obvious reasons. It describes a gas of charged particles (or plasma) in the mean field approximation.

Magnetic effects can also be considered. In this case we are led to the Vlasov-Maxwell equations which can be studied in a relativistic framework as well.

Due to the singularity of the kernel (3.12), the validity of the Vlasov-Poisson equation has not been established as yet and the mere existence and uniqueness of smooth solutions in dimension 3 has only recently been achieved.

An analysis of the existence theory of the Vlasov-Poisson and Vlasov-Maxwell equations will be presented by P. L. Lions, while a practical application of these equations in the context of the semiconductor theory will be illustrated by P. Markowich.

4. Kinetic theory and fluid dynamics.

In this section we compare the microscopic description supplied by kinetic theory with the macroscopic description supplied by continuum gas dynamics. To this end we introduce the definitions of density ρ, of bulk velocity v (with components v_i), of random velocity c, of stress tensor with components p_{ij}, of heat flow vector q:

$$\rho = \int_{R^3} f d\xi \tag{4.1}$$

$$v = \frac{\int_{R^3} \xi f d\xi}{\int_{R^3} f d\xi} \tag{4.2}$$

$$c = \xi - v \tag{4.3}$$

$$p_{ij} = \int_{R^3} c_i c_j f d\xi; \qquad (i,j = 1,2,3) \tag{4.4}$$

$$\rho e = \frac{1}{2} \int_{R^3} |c|^2 f d\xi; \tag{4.5}$$

$$q_i = \frac{1}{2} \int_{R^3} c_i |c|^2 f d\xi \tag{4.6}$$

Then, using the fact that 1, ξ_i and ξ^2 are collision invariants, we multiply the Boltzmann equation by these functions and integrate with respect to ξ, to obtain:

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\rho v_i) = 0, \tag{4.7}$$

$$\frac{\partial}{\partial t}(\rho v_j) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\rho v_i v_j + p_{ij}) = 0, \qquad (j = 1,2,3) \tag{4.8}$$

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho|v|^2 + \rho e) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}[\rho v_i(\frac{1}{2}|v|^2 + e) + \sum_{j=1} v_j p_{ij} + q_i] = 0. \tag{4.9}$$

These equations are the balance equations of mass, momentum and energy well-known in continuum mechanics. It is not worthless to mention, at this point, that Eqs. (4.7-9) are *not* fluid-dynamical equations. Actually they cannot even be solved without first solving the Boltzmann equation to determine p_{ij} and q_i. There are situations. however, where the distribution function can be shown to be very close to a Maxwellian, so that q_i and the anisotropic part of p_{ij} are negligible, and, by taking

$$q_i = 0, \qquad p_{ij} = p\delta_{ij}, \tag{4.10}$$

we can describe the gas by means of the Euler equations. How to pass from the kinetic regime (described by the Boltzmann equation) to the hydrodynamical regime (described

by the Euler equations) is one of the interesting problems related with the Boltzmann equation, that we shall now sketch.

5. Scaling properties.

A point of great relevance in the study of the Boltzmann equation is the analysis of the scaling properties: a large system, as we shall presently see, can be more conveniently described in terms of fluid-dynamical equations, when it is considered on a suitable space-time scale.

Let us consider a gas obeying the Boltzmann equation, confined in a large box Λ_ϵ of side ϵ^{-1}, ϵ being a parameter to be sent to zero. Let $f^\epsilon = f^\epsilon(x,\xi,t), x \in \Lambda_\epsilon$ be the number density of the particles. We assume that the total number of particles is proportional to the volume of the box, *i. e.* we normalize f^ϵ as follows:

$$\int_{\Lambda^\epsilon \times \mathbf{R}^3} f^\epsilon(x,\xi) dx d\xi = \epsilon^{-3} \tag{5.1}$$

We also assume that the time evolution is given by the Boltzmann equation

$$\frac{\partial f^\epsilon}{\partial t} + \xi \cdot \frac{\partial f^\epsilon}{\partial x} = \mathcal{A}Q(f^\epsilon, f^\epsilon) \tag{5.2}$$

and look at the behavior of the system on the scale of the box; in this case we have to use appropriate space and time variables, because in terms of the variable x, the box is of size ϵ^{-1}, while we would like to regard it as of order unity. Thus we introduce the new independent and dependent variables

$$r = \epsilon x, \qquad \tau = \epsilon t; \qquad (r \in \Lambda) \tag{5.3}$$

$$\hat{f}(r,\xi,t) = f^\epsilon(x,\xi,t) \tag{5.4}$$

Clearly, $\hat{f}$ describes the gas on the scale of the box and is normalized to unity:

$$\int_{\Lambda \times \mathbf{R}^3} \hat{f}(r,\xi) dr d\xi = 1 \tag{5.5}$$

The picture of the (same) system in terms of the variables r and τ is called *macroscopic*, while the picture in terms of x and t is called *microscopic*. Note that on the macroscopic scale the typical length for the kinetic phenomena described by the Boltzmann equation, *i. e.* the mean free path, turns out to be of order ϵ (since it is of order unity on the scale described by x). Thus sending the size of the box to infinity like ϵ^{-1} or the mean free path to zero like ϵ are equivalent limiting processes.

In terms of the macroscopic variables, Eq. (5.2) reads as follows:

$$\frac{\partial \hat{f}}{\partial \tau} + \xi \cdot \frac{\partial \hat{f}}{\partial r} = \epsilon^{-1} \mathcal{A}Q(\hat{f}, \hat{f}) \tag{5.6}$$

Thus, on the scale of the box, the mean free path (inversely proportional to the factor in front of Q) is reduced by a factor ϵ. This means that the average number of collisions diverges when $\epsilon \to 0$ and the collisions become dominant. For Eq. (5.6) to hold,

$Q(\hat{f}, \hat{f})$ must be small of order ϵ, so that $\hat{f}$ is expected to be close to a Maxwellian, whose parameters are, in general, space and time dependent. In this case the macroscopic balance equations (5.7- 9) can be closed through Eqs. (5.10) to obtain the Euler equations for a perfect compressible fluid. These considerations can be made rigorous and appropriate references will be given below.

For the present time, let us mention other physical considerations concerning our scaling. To this end, let us consider a small portion of fluid in a neighborhood of a point $r \in \Lambda$ (fig. 1). By the scaling transformation this portion is magnified into a large system of particles, which is seen to evolve on a long time scale. It will have a tendency to "thermalize" so that its distribution will quickly become a local Maxwellian with parameters $A(\epsilon^{-1}r)$, $\beta(\epsilon^{-1}r)$, $v(\epsilon^{-1}r)$ suitably related to the fluid-dynamical fields ρ , e, v. These will evolve according to the Euler equation on a much slower scale of times.

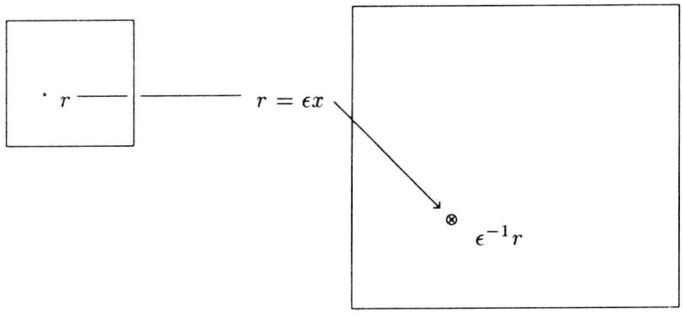


Fig. 1

Thus we have illustrated two different time scales. The fast one, which we call *kinetic*, is of the order of the time necessary to reach a local equilibrium, a process described by the Boltzmann equation. The slow scale, which we call *fluid-dynamical*, describes the time evolution of the parameters of the local Maxwellian.

It would be even more natural to apply the same considerations to the Newton equations. But, although one might expect that such dynamics should yield, under the above scaling, the Euler equations, our ignorance of the long time behavior of the Hamiltonian systems is such that, at the moment, we are quite far from a rigorous derivation of the equations of hydrodinamics from the basic laws of Classical Mechanics. As we shall see in the lectures by R. S. Varadhan, however, the hydrodynamics of a class of Hamiltonian systems can be derived if we assume that some ergodic properties are satisfied, at least as far as a smooth solution of the Euler equations exists.

It may be worth, at this point, to underline how different is the hydrodynamic behavior of a gas obeying the Boltzmann equation and thus the state law of perfect gases, from the behavior arising from a particle system describing a real gas and thus

a more complicated state law, including the effects of the interaction potential between molecules. In other words, as a consequence of the Boltzmann-Grad limit, the local equilibrium of a Boltzmann gas is that of a free gas, while, in general, the local equilibrium of a gas is a Gibbs state for an interacting particle system. Although the latter is the local equilibrium taking place in real fluids, the mathematical analysis of the hydrodynamics arising from the Boltzmann equation is technically easier and has produced more results.

Let us now analyse another scaling, which clarifies the nature of the Boltzmann-Grad limit. We now require the number of particles in Λ_ϵ to be of the order of ϵ^{-2}, *i. e.* we replace Eq. (5.1) by

$$\int_{\Lambda^\epsilon \times \mathbf{R}^3} f^\epsilon(x,\xi) dx d\xi = \epsilon^{-2} \tag{5.7}$$

In order to keep the normalization to unity of $\hat{f}(r,\xi,t)$, expressed by Eq. (5.5) we change the scaling from Eq. (5.4)

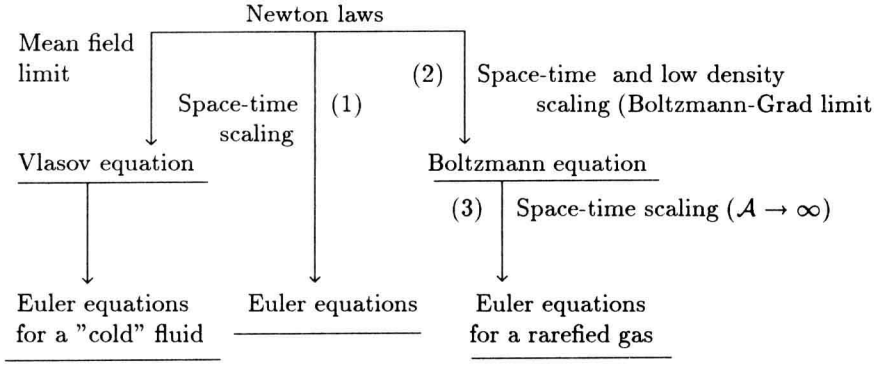
$$\hat{f}(r,\xi,t) = \epsilon^{-1} f^\epsilon(x,\xi,t) \tag{5.8}$$

Then we obtain, in place of Eq. (5.6)

$$\frac{\partial \hat{f}}{\partial \tau} + \xi \cdot \frac{\partial \hat{f}}{\partial r} = \mathcal{A} Q(\hat{f},\hat{f}) \tag{5.9}$$

Hence the Boltzmann equation is invariant for the space-time scaling (5.3), provided that the particle number goes as the power 2/3 of the volume. This invariance property suggests that the Boltzmann equation can be derived from the BBGKY hierarchy via a space time scaling with the total number of particles proportional to ϵ^{-2}; this is what can be checked at a formal level[1,21]. It is also clear why the Boltzmann-Grad limit is frequently called the *low density* limit; in fact, in this limit, the particle number in a large box divided by the volume of the box goes to zero. The number of collisions per unit (macroscopic) time stays finite, while it diverges in the hydrodynamical limit, as we saw before.

We summarize the content of this discussion in the graph below.



As we said before the limit corresponding to arrow (1) is not completely understood as yet and the best attempts in this direction will be discussed in the lectures by R. S. Varadhan. The limit corresponding to arrow (2) has been proved for short times and globally only for an expanding rare cloud of gas. The limit corresponding to arrow (3) is well understood for times up to the occurence of the first singularity in the fluid-dynamical equations[29,30,31,32]. The mean field limit (4) is well understood for smooth bounded interaction potentials[33,34,35].

Finally, it is also possible to derive (again before shocks develop) a hydrodynamical regime for the Vlasov dynamics (arrow (5))[36,37]. We also mention that the limit (3) is just one of a large class for which the incompressible Euler and Navier-Stokes equations can also be derived[38,39,40].

REFERENCES

1. C. Cercignani, *The Boltzmann equation and its applications*, Springer, New York (1988).
2. J. C. Maxwell, *Philosophical Transactions of the Royal Society of London* **157**, 49-88 (1867).
3. L. Boltzmann, *Sitzungsberichte Akad. Wiss.*, Vienna, part II, **66**, 275-370 (1872).
4. D. Hilbert, *Mathematische Annalen* **72**, 562-577 (1912).
5. S. Chapman, *Proceedings of the Royal Society* (London) **A93**, 1-20 (1916/17).
6. D. Enskog, *Kinetische Theorie der Vorgänge in mässig verdünnten Gasen, I. Allgemeiner Teil*, Almqvist & Wiksell, Uppsala (1917).
7. T. Carleman, *Acta Mathematica* **60**, 91-146 (1933).
8. T. Carleman, *Problèmes Mathématiques dans* la *Théorie Cinétique des Gaz*, Almqvist & Wiksell, Uppsala (1957).
9. H. Grad, *Comm. Pure Appl. Math.* **2**, 331-407 (1949).
10. C. Truesdell, *Jour. Rat. Mech. Anal.* **5**, 55-128 (1956).
11. V. S. Galkin *PMM (in Russian)* **20**, 445-446 (1955).
12. V. S. Galkin, *PMM* **22**, 532-536 (1958).
13. D. Morgenstern, *Proceedings of the National Academy of Sciences* (U.S.A.) **40**, 719-721 (1954).
14. L. Arkeryd, *Arch. Rat. Mech. Anal.* **45**, 1-16 (1972).
15. L. Arkeryd, *Arch. Rat. Mech. Anal.* **45**, 17-34 (1972).
16. C. Cercignani, *Mathematical Methods in Kinetic Theory*, Plenum Press, New York (1969).
17. H. Grad, *Proceedings of the American Mathematical Society Symposia on Applied Mathematics* **17**, 154-183 (1965).
18. S. Ukai, *Proceedings of the Japan Academy* **50**, 179-184 (1974).
19. T. Nishida and K. Imai, *Publications of the Research Institute for Mathematical Sciences, Kyoto University* **12**, 229-239 (1977).
20. Y. Shizuta and K. Asano, *Proceedings of the Japan Academy* **53**, 3-5 (1974).
21. C. Cercignani, *Transp. Theory Stat. Phys.*, 211-225 (1972).
22. O. Lanford, III, in *Dynamical Systems, Theory and Applications*, J. Moser, Ed., **LNP 35**, 1, Springer, Berlin (1975).
23. R. Illner and M. Pulvirenti, *Commun. Math. Phys.* **105**, 189-203 (1986) .
24. R. Illner and M. Pulvirenti, *Comm. Math. Phys.* **121**, 143-146 (1989).
25. R. Illner and M. Shinbrot, *Comm. Math. Phys.*, **95**, 217-226 (1984).