K D Joshi

Foundations of Discrete Mathematics

Foundations of
DISCRETE MATHEMATICS

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JOHN WILEY & SONS
New York Chichester Brisbane Toronto Singapore

First published in 1989 by
WILEY EASTERN LIMITED
4835/24, Ansari Road, Daryaganj
New Delhi 110 002, India

Distributors:

Australia and New Zealand:
JACARANDA WILEY LTD.
GPO Box 859, Brisbane, Queensland 4001, Australia

Canada:
JOHN WILEY & SONS CANADA LIMITED
22 Worcester Road, Rexdale, Ontario, Canada

Europe and Africa:
JOHN WILEY & SONS LIMITED
Baffins Lane, Chichester, West Sussex, England

South East Asia:
JOHN WILEY & SONS, INC.
05-04, Block B, Union Industrial Building
37 Jalan Pemimpin, Singapore 2057

Africa and South Asia:
WILEY EXPORTS LIMITED
4835/24, Ansari Road, Daryaganj
New Delhi 110 002, India

North and South America and rest of the world:
JOHN WILEY & SONS, INC.
605 Third Avenue, New York, NY 10158, USA

Library of Congress Cataloging in Publication Data

Joshi, K.D.
Foundations of discrete mathematics/K.D. Joshi.
p. cm.
Includes index
I. Mathematics – 1961– 2. Electronic data processing—Mathematics. 3. Combinatorial analysis. I. Title.
QA39.2.J67 1989 88-14811
510—dc19 CIP

ISBN 0-470-21152-0 John Wiley & Sons, Inc.
ISBN 81-224-0120-1 Wiley Eastern Limited

Printed in India at Prabhat Press, Meerut,

To my discreet wife
SWARADA
for her continuous support

List of Standard Symbols

Symbol	*Meaning*
ϕ	the empty set
$\mathbb{N}$	the set of natural numbers (= positive integers)
$\mathbb{Z}$	the set of all integers
$\mathbb{Z}_m$	the set of residue classes modulo a positive integer m
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^n$	the set of ordered n-tuples of real numbers
S^1	the set of complex numbers of length 1
$\lvert X \rvert$	cardinality of a set X
$P(X)$	the power set of a set X
$P_r(X)$	the set of all r-subsets of X, i.e. $\{A \subset X : \lvert A \rvert = r\}$
$n!$	'n factorial' $(= 1, 2, 3 \ldots (n-1)n)$.
$\binom{n}{m}$	'n choose m' $\left(= \dfrac{n(n-1)\ldots(n-m+1)}{m!}\right)$
D_n	dihedral group of order $2n$
S_n	symmetric group on n symbols
A_n	alternating group on n symbols
$p(n)$	number of partitions of an integer n
P_n	number of partitions of a set with n elements. (= nth Bell number)
F_n	nth Fibonacci number
C_n	nth Catalan number
H_n	nth harmonic number
$o(G)$	order of group G
$\subset$	contained in (and possibly equal to)
$\subsetneq$	contained in but not equal to
$\mid$	divides

Suggested Course Coverage

This book is intended to be covered in two one-semester courses with 14 weeks of instruction per semester. Because of the diversity of preferences on the part of the instructors and the diversity of the backgrounds, the calibres and the needs on the part of students, it is impossible to prescribe a uniform style or schedule of coverage. However, the following is a pattern which may be applied under 'average' conditions.

The core of the text lies in Chapters 2 to 7. The level of the material and the style of presentation is kept elementary. Also the answers to most exercises are given at the end of the book. Because of these features, coupled with some initiation from the instructor, it is expected that an average sincere student should be able to read and understand most of the material largely on his own. In that case each section can be covered generally in one week (assuming 3 to 4 hours of instruction per week). The instructor is strongly urged not to duplicate the proofs given here but instead to supplement them with numerical and diagrammatic illustrations (which are somewhat lacking in this book), with comments about the subtle points in the proofs and with alternate proofs wherever possible. Depending on the level of the class, the instructor may also wish to skip some of the exercises and/or to supplement them with simpler exercises designed to give computational drill.

The suggested schedule of coverage is as follows :

First Course : Spend about 2 weeks on Chapter 1. Then spend about one week per section in Chapters 2, 3 and 4.

Second Course : Spend about one week per section in Chapters 5, 6 and 7. Allow a little extra time for Sections 6.2, 6.3 and 6.4.

Preface

This book is intended to be more than just a textbook of discrete mathematics. Its ultimate goal is to make a strong case for the incorporation of discrete mathematics into the basic core curriculum of undergraduate mathematics. This is a rather ambitious task and deserves some elaboration.

Mathematics can be broadly divided into two parts; the continuous mathematics and the discrete mathematics depending upon the presence or absence of the limiting process. (The distinction is brought out more fully in Chapter 1.) Discrete mathematics is conceptually easier and more akin to human experience than continuous mathematics. Ironically, it is these very qualities which give the impression that discrete mathematics is elementary, indeed trivial, and does not deserve to be studied at the collegiate level. So it is the continuous mathematics which has long dominated the scene. This is also consistent with the history of applications of mathematics. Apart from the age-old applications of arithmetic, algebra, geometry and trigonometry, nearly all real-life applications of mathematics till the end of the nineteenth century were through physics. Even today, what is generally understood by 'applied mathematics' consists of topics such as mechanics of solids and fluids, heat transfer, electromagnetic theory etc. In the view of the nineteenth century physics, energy and the other physical variables were assumed to be continuous. This explains why the concept of a limit acquired such a paramount position, a natural corollary being the standardisation of the undergraduate mathematics curriculum so as to include a sequence of two or three calculus courses, followed by one or two courses in differential equations, numerical analysis, complex analysis and so on. There is hardly any room for discrete mathematics in this programme. Whatever little discrete mathematics is needed is considered mostly a matter of common sense or something which one just picks up along the way.

This picture began to change in the twentieth century. Applications of mathematics to physics continued to flourish, especially during the first half where they got a big boost because of the theory of relativity. But applications to other fields, such as statistics, operations research, economics, design of circuits, logic, computer science etc. also got in full swing. An even more important development was the change that took place in the conception of mathematics. Its focal point changed from the concept of a number to the concept of a set (more on this point in Chapter 3). By its very nature this new non-numerical mathematics was more amenable to the methods of discrete mathematics than to those of continuous mathematics. This resulted in a tremendous increase in the real-life applications of discrete mathematics. Things such as finite fields, which at one time were considered to be too abstract to have any practical relevance were shown to have down-to-earth applications (e.g. in designing economical codes).

As a result, the attitude towards discrete mathematics is changing slowly but surely. The skepticism that once prevailed about its relevance is waning. Rich research contributions are being made to it. At the pedagogical level, specialised elective courses catering to various branches of discrete mathematics such as applied algebra, combinatorics, graph theory, linear programming have been started. Many books are being written on these subjects.

Welcome as all these signs are, the undergraduate curricula are yet to accord discrete mathematics its due place in the mainstream of mathematics. The 'core' courses continue to be dominated, almost exclusively, by continuous mathematics. Where specialised courses in discrete mathematics such as those mentioned above do exist, they are generally treated as peripheral, of interest only to certain classes of students (mostly majoring outside mathematics). They are rarely considered to be an integral part of 'basic' mathematics. As a result, we have the paradoxical situation that although in a compulsory calculus course a student is thoroughly drilled into finding maxima and minima, these methods are rarely used in an actual problem, whereas the methods that *are* actually used in practice (e.g. the simplex method) have to be picked up from specialised, 'elective' courses!

Ironically, again, the very strengths of discrete mathematics have come in the way of its entry into the mathematics core. One of the strong points of discrete mathematics is its powerful applications to fields like computer science, engineering and operations research. Indeed, a good deal of discrete mathematics owes its development to problems in some of these areas. In this respect the discrete mathematics behaves no differently from continuous mathematics. Many concepts of the classical continuous mathematics also originated from applications to physical sciences. However, in the case of continuous mathematics, the process of isolating the underlying mathematical thought from a particular application began a long time ago. As a result, it is possible today to give coherent courses in continuous mathematics (already named above), in which, although there are sufficient hints about the applicability of the various concepts, the emphasis is on the mathematical aspects of those concepts and not on any particular applications.

In the case of discrete mathematics, there is an inherent difficulty in separating the underlying mathematics from its applications. It is a fact that many results of discrete mathematics, when stripped of their applications, appear either too trivial or too abstract. In either case, their introduction into the core courses in mathematics is not looked at favourably by the pedagogists of mathematics. A good case in point is the well-known pigeon-hole principle or the double counting argument or some results in graph theory. When ingeneously applied, they work wonders. But when seen all by themselves, they appear so trivial that one wonders if they deserve even to be stated explicitly. As examples of the other kind, we have the theories of groups and fields. Both are replete with profound results. But without some down-to-earth applications (such as Polya's theory of counting or coding theory), they are likely to be disposed of as sterile intellectual exercises. Many mathematicians are either unaware of these applications or tend to dispose them of as too esoteric. Consequently, although both the group theory and the field theory are highly respected branches of graduate mathematics, they are still not considered to be parts of 'everybody's mathematics' the way double integrals and differential equations are.

Because discrete mathematics is so wedded to its own applications, it is not surprising that most of the currently available books on discrete mathematics are applications-oriented. As a rule, they are written for non-mathematicians such as computer scientists or engineers. In such books mathematics is only a means and not the goal and often gets a treatment which is at best utilitarian. As a result, even though some of these books are widely acclaimed outside the mathematical circles, they often fail to appeal to a traditional mathematician. He is not readily convinced that their mathematical contents deserve to be studied mathematically regardless of their particular applications.

Another strong point of discrete mathematics is its infinite variety of interesting problems. Recently a number of books of the 'problems and solutions' type have appeared on discrete mathematics, especially on combinatorics and graph theory. Such books are an intellectual treat for those who love to solve problems. They are also excellent sources of references. In books of this kind, it is possible to pack within a few pages a huge amount of information which, in a conventionally written book, would probably occupy several times as much space. A mature reader, who is in a position to supply the missing details, can learn a lot from such books. Naturally, these books lack coherence and cannot be used as texts in regular courses. There is very little motivation or elaboration of the ideas involved. If not taken in the proper spirit, these books are liable to give the impression that discrete mathematics is a scattered bunch of intellectual puzzles rather than a coherent subject capable of a systematic study.

In short, the dilemma of discrete mathematics is that on one hand, in order to firmly establish it as a fundamental branch of mathematics, it must be delinked from problems and applications of a particular kind. But on the other hand, in doing so there is a danger that the very heart of the subject may be lost.

In the present book, I have made an attempt to find a way out of this dilemma. Keeping in mind the role of problems in discrete mathematics, in the first chapter I have given a fairly long list of some typical problems and a few comments about them. Their solutions are, however, intentionally postponed. Instead, these problems are used from time to time to provide motivation for the various stages of development of the theory. The theory itself is developed systematically, in the traditional definition-theorem-proof pattern so characteristic of mathematics. Following the practice of modern mathematics I have taken a set as a starting point. The general theme is to show how a suitable set empowered with a suitable additional structure provides a convenient mathematical model for a given problem. I am aware that the average reader of this book may not be mathematically mature. I have therefore made a conscious effort to develop his mathematical maturity. This is done through comments about the nature of discrete mathematics, its place in mathematics as a whole and its relationship with continuous mathematics, a review of mathematical logic, and through general comments about the process of abstraction. A special emphasis is laid on motivating the definitions and results.

The applications of discrete mathematics have not been entirely ignored. In fact, I have given quite a few of them. However, as indicated above, the emphasis is more on the *applicability* of discrete structures rather than on any particular applications. So I have generally avoided applications of a technical nature which would demand a knowledge of some other fields, such as computer science or economics. In the few places where technical applications do occur (e.g. the applications of Boolean algebra to switching circuits), the relevant background has been developed. For all other applications, I have chosen real-life problems which can be understood and appreciated even by a layman. (Admittedly, some of these problems appear rather contrived.) One can of course garb a problem, at least superficially, so that it looks like a problem in some other field. For example, in a problem of putting balls into boxes we can think of a ball as a piece of data and a box as a memory register and the problem now becomes a problem in computer science! Whatever be the selling value of such tricks, I have generally refrained from them.

Inasmuch as the introduction of discrete mathematics into the undergraduate curricula is still in its infancy, the contents and the degree of coverage of the topics in a book like this are open to debate. Although there are dozens of different texts on calculus. they more or less cover the same topics and to the same degree of depth. A similar standardisation is also slowly taking place in textbooks covering particular aspects of discrete mathematics such as combinatorics, graph theory or applied algebra. The present book, however, is meant to give a unified rather than a piece-meal treatment of discrete mathematics. Naturally, I could not go as deep as a book specialising in any one area. Still I have attempted to reach a reasonable degree of depth.

The original plan was to include all the meterial in a single book '*Introduction to Discrete Structures*'. But this proved impracticable in view of the size. So it was necessary to split it into two separate books, '*Foundations of Discrete Mathematics*' and '*Applied Discrete Structures*'. The first book, the present one, gives the fundamental concepts and techniques of discrete mathematics and a fairly thorough exposure to algebra. The second book is more applications oriented. (See the Epilogue for a detailed preview of '*Applied Discrete Structures*'.) It contains a review of the first book, with the help of which it can be read independently of the first book.

The present book has seven chapters. The first chapter introduces the subject matter. It also contains the list of problems mentioned earlier which are nicknamed for a ready reference in future. The results of the second chapter are elementary. Still, they are treated rather formally, so as to familiarise the reader with the style of presentation to be encountered in the later chapters. The third chapter introduces the process of abstraction, studies two elementary structures on sets and gives the generalities about algebraic structures. The next three chapters deal with specific algebraic structures. The last chapter presents advanced counting techniques based on generating functions and recurrence relations.

Each chapter is divided into four sections. Each section contains exercises followed by 'Notes and Guide to Literature'. The latter are generally intended to direct the interested reader to appropriate references for further reading or to acknowledge credit to the sources from where I have borrowed something. A few historical remarks are also made occasionally. However, such remarks and references are only indicative. No claim is made about thier being complete or most up-to-date.

There are virtually no prerequisites for reading this book. Some facts about power series and differential equations will be referred to in Chapter 7. But that is more by way of relating discrete mathematics with continuous mathematics rather than a strict pre-requisite. Although this book is avowedly written to make a case for discrete mathematics, the idea is definitely not to be little continuous mathematics. A great majority of the readers of this book will have already studied calculus or at least be studying it concurrently. Consequently, wherever possible, I have indulged into comments about continuous mathematics as well. While a mature reader may find them platitudinous, I hope they will help the average reader gain new insights into the nature of continuous mathematics and thereby appreciate discrete mathematics even more.

Like most authors of textbooks, I had to strike a balance between pairs of mutually conflicting virtues such as expanse versus depth, clarity versus brevity and abstract versus concrete. The objectives of including almost all standard topics, of reaching a reasonable depth and of building up the mathematical maturity of the reader so that he can handle abstract concepts soon proved to be somewhat incompatible with each other and threatened to blow the size of the book out of proportion even after it was split into two. As a result, I was forced to make a few unpleasant choices. Some of the standard results had to be relegated to the exercises. It is hoped that with the generous hints given, the reader can work them out. (Answers to most of the exercises are given are the end of the book.) Also the diagrammatic and the numerical illustrations have been kept to a minimum. The emphasis is more on thoughts and less on numerical dexterity. I am of course, not unaware of the pedagogical importance of diagrams and worked-out examples. Had space permitted, I would have loved to include more of them. Another reason is that there already exist books which cater to these aspects very nicely. For the algebraic part we recommend William J. Gilbert's '*Modern Algebra with Applications*' and for the combinatorial part, Alan Tucker's '*Applied Combinatorics*'.

Exercises form an integral part of the book. The results of many of them are used freely in the text. There are virtually no exercises whose sole purpose is to provide numerical drill. Nearly all exercises require some thinking for their solution, the degree and the quality of which obviously vary considerably, Some of the exercises merely ask the reader to supply parts of a proof (occasionally an entire proof). A few require the application of the results proved in the text while a few others are meant to prove some standard results which could not be incorporated in the text. Hints for solution and comments about the significance are given liberally. The degree of difficulty of a problem, especially one where some thought is involved, is always a matter of personal opinion. Many challenging problems look deceptively simple once you know their solutions. It is therefore, very difficult to rank the exercises quantitatively in terms of their difficulty and to give the estimated time for working them out as is done by Donald Knuth in his pioneering volumes of '*The Art of Computer Programming*'. My own experience with such a time scale is, in fact, that when I could not do a problem within the stipulated time, it unnecessarily created an inferiority complex. I have therefore refrained from giving any quantitative assessment of the difficulty of a problem. Still, some qualitative indication seems to be in order. So I have put a star (*) over those problems, which, in my opinion, require a little originality of thought. Unusually demanding exercises are doubly starred (**). These include some standard theorems whose proofs are far from simple and also a few problems which, to my knowledge, are unsolved. These exercises are not really meant to be solved, even by the hlghly gifted student. He is merely expected to appreciate their difficulty. Here I recall with full agreement, a comment by I.N. Herstein in his '*Topics in Algebra*' that the value of a problem is not so much in coming up with the answer as in the ideas and the attempted ideas it forces on the would-be solver.

The material is so arranged that it can be covered in two courses of one semester each. (See the 'Suggested course coverage' for more delails about this.) In fact, it is hoped that the present book, along with '*Applied Discrete Structures*' would eventually be used as texts for a sequence of four one-semester courses in discrete mathematics. Such a sequence, coupled with the traditional sequence in continuous mathematics, would provide a college student with a solid mathematical background, regardless of what he specialises in later on.

It was mostly by accident that I was prompted to write this book. My own mathematical training did not include discrete mathematics and like many others in my position I thought of it as something which is possibly useful but inherently trivial. Then I happened to read C.L. Liu's fine book '*Introduction to Combinatorial Mathematics*' and used it as a text for a course to students of computer science. In doing so, I realised that the mathematics in it was important enough to deserve a place in the mainstream of mathematics. Other books which have influenced me profoundly are those of Knuth and Herstein, mentioned above. The latter's influence extends not only to the coverage of algebra in this book, but also to my style in general. I am deeply indebted to these three authors.

My own expertise in discrete structures being limited to algebra, I frequently had to consult others on many points. I am grateful to many colleagues of mine who helped me by informal discussions and by pointing out appropriate references. I must especially mention prof. G.A. Patwardhan and Prof. M.N. Vartak. If despite their best help, any errors have occurred, I am entirely responsible.

Financial support for the preparation of the manuscript of this book was given by the Curriculum Development Cell at the Indian Institute of Technology, Bombay and is hereby thankfully acknowledged. I also thank Mr. Parameswaran for his sincerity and patience in typing the manuscript. The printers too have done a neat job. The rather large number of misprints that still remain is due mostly to my own negligence in proof-reading and is a sad testimony to the observation that the author himself is generally a poor proof-reader of his own work. A few misprints have been corrected in the 'Errata'.

The introduction of discrete mathemetics in the core programme is still in the experimental stage and not yet fully implemented. I shall, therefore, be most interested in the views, comments and suggestions, not only from mathematicians who may be teaching it but from others as well. I hope that through such a dialogue a standard syllabus of discrete mathematics will evolve in near future. The twentieth century is coming to an end and the talk of orienting ourselves for the twentyfirst century has become a fashion of the day. Let us hope that before the dawn of the twentyfirst century, the undergraduate mathematics curriculum is freed, at least partially, from the tight grip of the nineteenth century physics.

Bombay, K.D. JOSHI
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Contents

One

Introduction and Preliminaries

This chapter is meant to give an idea of the kind of things that would be discussed in this book and also to review the prerequisites needed for their understanding. The spirit of discrete mathematics, especially in comparison to the continuous mathematics is discussed in Section 1. A rather long list of problems is given in Section 2. The comments about them in Section 3 guide the reader to the chapters in which the machinery needed to solve the problems will be developed. Section 4 gives a warm-up in logic.

1. What is Discrete Mathematics?

Discrete mathematics began at least as early as man (or perhaps some other animals) learnt to count. The fundamental idea behind counting is to establish a one-to-one correspondence (or a bijection, as it is technically called) between two sets of objects and even today this continues to be one of the most widely used devices in discrete mathematics. Nearly all the mathematics that we pick up in early school comes under discrete mathematics. This includes, the addition, multiplication and other arithmetical operations we do with integers and rational numbers. The fairly interesting topics of permutations and combinations and related problems in probability are an important part of discrete mathematics.

Thus 'discrete mathematics' is far from a new innovation in the history of mathematics. Perhaps the only new thing about it is its name. The dictionary meaning of the word 'discrete' is 'separate and distinct, unrelated, made up of distinct parts'. But this only remotely reflects the way the word is used in mathematics. In fact, the best way to understand the spirit of discrete mathematics is by camparing it with non-discrete mathematics! The latter is more popularly known as the **continuous or continuum mathematics.** Let us, therefore, take a look at what continuous mathematics is and how it came into being.