Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

970

Twistor Geometry and Non-Linear Systems

Proceedings, Primorsko, 1980

Edited by H. D. Doebner and T. D. Palev

Springer-Verlag
Berlin Heidelberg New York

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

970

Twistor Geometry and Non-Linear Systems

Review Lectures given at
the 4th Bulgarian Summer
School on Mathematical Problems of
Quantum Field Theory
Held at Primorsko, Bulgaria, September 1980

Edited by H. D. Doebner and T. D. Palev

Springer-Verlag
Berlin Heidelberg New York 1982

Editors

Heinz-Dietrich Doebner
Institut für Theoretische Physik, Technische Universität Clausthal
3392 Clausthal-Zellerfeld, Federal Republic of Germany

Tchavdar D. Palev
Institute of Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences
1184 Sofia, Bulgaria

ISBN 3-540-11972-8 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-11972-8 Springer-Verlag New York Heidelberg Berlin

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically those of translation, reprinting, re-use of illustrations, broadcasting, reproduction by photocopying machine or similar means, and storage in data banks. Under § 54 of the German Copyright Law where copies are made for other than private use, a fee is payable to "Verwertungsgesellschaft Wort", Munich.

© by Springer-Verlag Berlin Heidelberg 1982
Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

PREFACE

The mathematical structure and the physical application of twistor geometry together with special properties of solution varieties of non-linear PDO and their quantisation have been an active and fruitful field of research in mathematical physics during the last years and no doubt this situation will prevail in the next future. The twistor approach emerged directly from a description of physical systems in Minkowski space with non-linear dynamics in general. Examples are field theory, including non-abelian gauge fields, and general relativity with the Einstein equations. The approach relates physical problems and complex manifold theory, algebraic topology and sheaf theory thus providing one example where more theoretical parts of mathematics are applicable to fundamental and practical physical problems, yielding fruitful results presumably not obtainable otherwise. The non-linearity of the physical system in question reflects itself in the twistor geometry. Part of this relation has to be explored yet; infinite-dimensional Lie-algebras will be useful there and the singularity structure as well as the dynamical symmetries are of special interest. Furthermore the quantisation of such systems will rely also on complex manifold techniques.

Review lectures covering authoritatively part of the above programme were given at the Fourth Bulgarian Summer School on Elementary Particles and High Energy Physics: "Mathematical Problems in Quantum Field Theory" held in Primorsko in September 1980. The lectures are collected and edited in an updated version in this volume. Twistor geometry and its application to certain non-linear physical systems were treated. Some reviews present a detailed account of the formalisms, others show its applicability and relevance to physical systems.

The material is organized as follows:

Part I : Twistor Geometry

with theoretical lectures by S.G. GINDIKIN on integral geometry and YU.I. MANIN on analytic sheaf cohomology, including also side-views to gauge theories and with lectures on applications by Z. PERJES treating particle theory and by N.J. HITCHIN on the Einstein equations.

Part II : Non-Linear Systems

with a theoretical lecture by A.A. KIRILLOV on infinite dimensional Lie-groups and with more applied lectures by A.S. SCHWARZ on a construction of solutions of non-linear equations, by A.K. POGREBKOV and M.C. POLIVANOV, A.V. MELNIKOV, M.A. SEMENOV-TIANSHANSKY on singularities and group theoretical properties, by A.V. MIKHAILOV on the inverse scattering method and by P.A. NIKOLOV and I.T. TODOROV on relativistic particle dynamics.

Considered as proceedings of the IV. Bulgarian School the volume contains only part of the lectures and seminars presented there. The editors agree with the general editorial requirements that a lecture notes volume should be homogeneous. So it was necessary to center the material around the main topics of the school. It was not possible to include either contributions on supermathematics or quantisation methods or the lectures with a strong bias towards physics. The same holds for papers having already been published in the form of a review or having the character of a research announcement.

The responsibility for the final preparation of the manuscripts for printing was in the hand of one of the editors (H.D.D.). We received all the manuscripts in English, if these manuscripts were translations, the orginal version was not at our disposal; whenever possible though, unclear parts of the translations were corrected.

ACKNOWLEDGEMENTS

The IV. Bulgarian School on Elementary Particles and High Energy Physics in 1980 was organized by the Institute for Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences. It was sponsored be the Bulgarian Academy of Sciences, the Central Committee of the Dimitrov Communist Youth League, the Bulgarian Physical Society and the Bulgarian Mathematical Society. Dr. B.L. MARKOWSKY was acting secretary of the school.

We wish to express our gratitude to the Academicians Prof. CHRISTOV and Prof. TODOROV for their encouragement. We thank especially Prof. STIG I. ANDERSSON (Clausthal) for his help in preparing the material and we acknowledge the assistance of Dipl.Math. UTE GEHRINGER.

H.D. Doebner

T.D. Palev

此为试读,需要完整PDF请访问: www.ertongbook.com

TABLE OF CONTENTS

I. TWISTOR GEOMETRY

INTEGRAL GEOMETRY AND TWISTORS

S.G. Gindikin

The aim of these lectures is to give a notion of a new development of the Penrose idea how to realize the four-dimensional manifolds with selfdual metrics as manifolds of curves [1] . These constructions of Penrose happened to be in close connection with those ones which arose in the recent years in integral geometry [4] . The integral geometry hints a natural generalization of the selfdual metrics problem and it seems to us that considerations in this more broader framework is more prolific. The new results presented here (II, III and Appendix) are obtained in collaboration with J.N. Bernstein [2,3] .

Introduction. The twistor realization of the flat space time (from Plücker-Klein to Minkowski-Penrose)

The fundamental idea by Penrose is that points of four-dimensional space-time (either Minkowsky or Euclid) may be considered as complex lines in an auxiliary three-dimensional space (named twistor space). The role of the complex geometry is crucial in the investigation of the real manifold. The twistor programme which had already produced a series of interesting results consists in a systematic interpretation of space time data in terms of three-dimensional twistor space data. The idea is that the three-dimensional data arising must be simpler than their four-dimensional prototypes. The return way is connected with "extra" relations which appear because the dimension of the space grows. Thus, massless equations or Yang-Mills equations correspond to different variants of the Cauchy-Riemann equations (see e.g. [5,6]).

The Plücker coordinates. The geometric idea of Penrose is the first cousin of the fundamental ideas of geometricians in the second half of the 19th century. First we must recall how in mathematics, perhaps for the first time, appeared a four-dimensional manifold. The geometry is in debt to Julius Plücker for a number of wonderful discoveries. He terminated the glorious history of invention of various coordinates in projective space introducing homogeneous coordinates that serve all points of the space at the same time. Recall that to a point of three-dimensional space $\mathbb{P}^3 = \mathbb{RP}^3$ the sets of four numbers $(x_0, x_1, x_2, x_3) \neq (0,0,0,0)$ are assigned so that (x_0, x_1, x_2, x_3) and $(\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3)$ correspond to the same point. On the subset $\{x_0 \neq 0\}$ the unhomogeneous coordinates $X_j = \frac{x_j}{x_0}$, where $j = 1, 2, 3$, maybe introduced, the plane

$\{x_0=0\}$ being considered as the one at infinity.The discovery of Plücker has made the Poncelet-Gergonne duality absolutely clear, since it be - comes evident that planes form the dual copy of the projective space with homogeneous coordinates $(\xi^0, \xi^1, \xi^2, \xi^3)$ where $\xi^0 x_0 + \xi^1 x_1 + \xi^2 x_2 + \xi^3 x_3 = 0$ is the equation of the plane. In the last memoire by

Plücker entitled "New geometry of the space based on the consideration of a line as an element of the space" , its posthumous issue made in 1868-69 by Klein and Clebsch, the space is introduced whose elements (points!)are lines in their projective space $\mathbb{P}^3$ .It is clear that the dimension of this set equals four: almost all lines in homogeneous coordinates can be expressed as $X_1 = \alpha_1 X_3 + \beta_1$, $X_2 = \alpha_2 X_3 + \beta_2$ so $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ may be considered as local coordinates.But Plücker seeks for coordinates that will do for all lines in $\mathbb{P}^3$. For this, as it was in the case of homogeneous coordinates in $\mathbb{P}^3$,he introduces "extra" coordinates.He defines a line by a pair of different points $x = (x_0, x_1, x_2, x_3)$ and $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ where (x_j) is a set of its homogeneous coordinates and so he constitutes expressions that do not depend on the choice of points on the line:

$$p_{ij} = x_i \tilde{x}_j - x_j \tilde{x}_i \tag{1}$$

It is clear that $p_{ii} = 0$, $p_{ij} = -p_{ji}$ and one may confine to the six numbers $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$ that we will call the Plücker co- ordinates of the line.Since the points were defined by homogeneous co- ordinates, the sets $\{p_{ij}\}$ and $\{\lambda p_{ij}\}$ correspond to the same line.If all p_{ij} are zero, then x and $\tilde{x}$ are proportional i.e.points coincide and that was forbidden.Thus, it is natural to con- sider the non-zero set of six numbers $\{p_{ij}\}$ up to a multiple as homogeneous coordinates of a point in the five-dimensional projective space $\mathbb{P}^5$.

Thus, the set of lines happened to be naturally embedded in $\mathbb{P}^5$. But, since it depends on four parameters only, the numbers p_{ij} must satisfy one more equation.It is not difficult to discover:

$$p_{01} p_{23} - p_{02} p_{13} + p_{03} p_{12} = 0 \tag{2}$$

It is also not difficult to verify that there are no other relations, i.e.from any set $\{p_{ij}\}$ satisfying (2) points x and $\tilde{x}$ satisfying (1) may be recovered.From the geometrical viewpoint (2) defines a second order surface in $\mathbb{P}^5$. If we pass to coordinates

$p_{01} = u_0 - u_3$, $p_{23} = u_0 + u_3$, $p_{02} = u_4 - u_1$, $p_{13} = u_4 + u_1$; $p_{03} = u_2 - u_5$, $p_{12} = u_2 + u_5$
then (2) changes into

$$u_0^2 + u_1^2 + u_2^2 - u_3^2 - u_4^2 - u_5^2 = 0 \qquad (2')$$

Thus, the set of lines in the three-dimensional projective space $\mathbb{P}^3$ is embedded as a second order surface ("quadric") (2) - (2') in the five-dimensional projective space $\mathbb{P}^5$. This discovery of Plücker played the principal role in the moulding of mathematical ideology. It established the isomorphism of two completely different geometric structures: that of manifolds of lines in and of quadrics in $\mathbb{P}^5$. In the years that followed the best geometricians Sophus Lie, Felix Klein, Eli Cartan lovingly collected such isomorphisms. Later interests shifted to the general judgement of manifolds, when they dealt with coordinates ignoring the geometric nature of points.

The interpretation of quadrics with other signatures.

When, almost 50 years later in the relativistic theory the four-fold of Minkowski appeared it was the time when everyone's thing was the four-dimensional generations and perhaps nobody did seriously compare the four-fold of Minkowski with the four-fold of Plücker. It is wonderful that between them there is a deep connection, as was later discovered, which was for a long time hidden in the general theory of homogeneous symmetric spaces.It took 50 years more before Penrose made this connection a constant source of analytical results.

One of the probable ways from Plücker geometry to Minkowski geometry starts from a quite naive question. Since to lines in $\mathbb{P}^5$ corresponds the quadric of signature (3,3) in $\mathbb{P}^5$, maybe points of quadrics with other signatures do admit geometric interpretation? This question stirred very respectable mathematicians. Sophus Lie discovered that on the set of spheres in the three-dimensional space homogeneous coordinates can be naturally introduced so that they "fill" the quadric of signature (4,2) in $\mathbb{P}^5$ (the geometry of Lie spheres). Felix Klein introduced in four-dimensional space quite refined coordinates which he called hexaspherical and these coordinates "filled" the quadric of signature (5,1) in $\mathbb{P}^5$.

There is a more straightforward way to answer this question, quite in the spirit of 19th century geometry. Taking into account that all real quadrics in $\mathbb{P}^5$ are real forms of the same complex quadric in $\mathbb{CP}^5$, we must first complexify the problem. Namely, consider (complex) lines in $\mathbb{CP}^3$ and introduce the Plücker coordinates which identify the set of lines with the quadric (2) in $\mathbb{CP}^5$, where p_{ij} are supposed to be complex. To investigate various real forms of the quadric (2) we must intersect it by different five-dimensional real planes $\mathbb{R}P^5$ whose complex span coincides with $\mathbb{C}P^5$. If we just consider all u_j in (2) to be real, then we obtain the already considered quadric of real lines in $\mathbb{R}P^3$. But we may consider u_0, u_1, u_2, u_5 to be real and $u_3 = i v_3$, $u_4 = i v_4$ to be purely imaginary or only $u_3 = i v_3$ purely imaginary, while the other coordinates are real. We obtain real surfaces, respectively

$$u_0^2 + u_1^2 + u_2^2 + v_3^2 + v_4^2 - u_5^2 = 0 \qquad (S)$$

$$u_0^2 + u_1^2 + u_2^2 + v_3^2 - u_4^2 - u_5^2 = 0 \qquad (M)$$

They are elipsoid and hyperboloid of one sheet, respectively. Since these real lines belong to the complex quadric the complex lines correspond to the points of the quadric. It is natural to try to find out which complex lines correspond to points of surfaces S and M .

<u>The interpretation of real quadrics in terms of complex lines</u> (case S). We have

$$p_{01} = u_0 - i v_3 , \quad p_{23} = u_0 + i v_3 , \quad p_{02} = i v_4 - u_1 ,$$

$$p_{13} = i v_4 + u_1 , \quad p_{03} = u_2 - u_5 , \quad p_{12} = u_2 + u_5 .$$

Thus, to points of S correspond the Plücker coordinates that satisfy

$$p_{23} = \overline{p_{01}} , \quad p_{13} = \overline{p_{02}} , \quad \operatorname{Im} p_{03} = \operatorname{Im} p_{12} \qquad (3)$$

and by these conditions points of S are completely defined. Then, if the line with such Plücker coordinates goes through the point $z = (z_0, z_1, z_2, z_3)$ we easily verify that we may assume that the other point is $\tilde{z} = (-\bar{z}_3, \bar{z}_2, -\bar{z}_1, \bar{z}_0)$. Thus, to points of a real quadric S correspond complex lines in $\mathbb{C}P^3$ that join points (z_0, z_1, z_2, z_3) and $(-\bar{z}_3, \bar{z}_2, -\bar{z}_1, \bar{z}_0)$

What is remarkable in these lines? Through each point $z \in \mathbb{C}P^3$ goes exactly one line of this kind. As a result the space $\mathbb{C}P^3$ splits into the union of non-intersecting lines. This is well-known the mathematics of fibration of $\mathbb{C}P^3$ over the sphere S^4 with complex lines as fibers. If we intersect this fibration with real projective space $\mathbb{R}P^3$ we obtain the fibration $\mathbb{R}P^3$ with real lines that join points (x_0, x_1, x_2, x_3) and $(-x_3, x_2, -x_1, x_0)$ as fibers. In terms of elementary geometry we have obtained the splitting of three-dimensional space into mutually skew lines.

<u>The interpretation of a hyperboloid as a family of lines.</u>

In case M we have

此为试读,需要完整PDF请访问: www.ertongbook.com

$$p_{23} = \overline{p_{01}}\ , \quad Im\ p_{13} = Im\ p_{02} = Im\ p_{03} = Im\ p_{12} = 0 \quad (4)$$

This situation is somewhat more complicated. Let first $p_{03} \neq 0$, for simplicity. Since the coordinates are homogeneous, we may assume that $p_{03} = 1$ and pick points on the corresponding line with coordinates $z_0 = \tilde{z}_3 = 1$, $z_3 = \tilde{z}_0 = 0$. These points are unique. It follows from (4) that $z = (1,a,c,0)$ and $\tilde{z} = (0,\bar{c},b,1)$, where a and b are real.

What is remarkable in these lines? It is quite straightforward that all points that belong to them, i.e. $w = \lambda z + \mu \tilde{z}$ where $\lambda, \mu \in \mathbb{C}$ satisfy

$$Im\ (w_1 \bar{w}_0 + w_2 \bar{w}_3) = 0 \qquad (5)$$

and if we remove the restriction $p_{03} \neq 0$ we will see that there are no other lines such that all their points satisfy (5). Thus, if N stands for the real surface of dimension 5 defined by (5) then all lines that belong to N are exactly those lines whose Plücker coordinates satisfy (4), hence, the lines corresponding to the points of the real surface M. Note that N contains the whole projective space $\mathbb{RP}^3$.

Generally speaking, the family of complex lines that depends on four real parameters fills, as follows from the computation of dimensionalities, the domain in $\mathbb{CP}^3$. Therefore we may expect that the surface N has the following specific property: it contains a 4-parameter family of lines. This result has a real analogue. There are a lot of non-flat surfaces, but only on a hyperboloid with one sheet there are two different families of line elements (recall that from the projective viewpoint the hyperboloids with one sheet and hyperbolic paraboloids are equivalent).

Make a summing up. We began with the quadric of real lines in $\mathbb{P}^3$ then passed over to the quadric Q of complex lines in $\mathbb{CP}^3$. Among the real surfaces of second order which belong to this complex surface there are not only surface of real lines but also two other types of surfaces; one, that corresponds to a fibration of $\mathbb{CP}^3$ by complex lines as fibers, the other corresponds to five-dimensional real surfaces in $\mathbb{CP}^3$ that have a family of complex lines that depend on four real parameters. This example explicitly shows the phenomenon that is the product of martyrdom of 19th century geometricians.

First, purely real geometric data often admit interpretation in terms of complex data.

Second, if we complexify a real problem and then try to see which real problems led to this complex one, we often find new meaningful geometrical facts.

The metric in the manifold of lines. The relation between the smart realization of four-dimensional real quadrics S and M and the four-dimensional space time is still not quite clear. The discovery of this relation supposes the introduction of a metric on the surfaces S and M. It turns out that there is a wonderful invariant way to introduce a metric (more precisely, a metric up to conformal equivalence) using the above interpretation of quadrics.

Let us begin with the complex quadric of lines $Q \subset \mathbb{C}P^5$. Lines in a three-dimensional space sometimes intersect. How does it become manifest in the Plücker coordinates? We see that if $\{p_{ij}\}$ and $\{p_{ij}'\}$ are the Plücker coordinates of two lines, then they intersect if

$$p_{01}p_{23}' - p_{02}p_{13}' + p_{03}p_{12}' + p_{23}p_{01}' - p_{13}p_{02}' + p_{12}p_{03}' = 0 \qquad (6)$$

To avoid determinants of the fourth order, let us deduce (6) under the simplifying assumption (which was once accepted). Let $p_{03} \neq 0$ and $p_{03}' \neq 0$. Then we may assume that $p_{03} = p_{03}' = 1$ and the lines join points $(1, \alpha_1, \alpha_2, 0)$ and $(0, \beta_1, \beta_2, 1)$ respectively $(1, \alpha_1', \alpha_2', 0)$ and $(0, \beta_1', \beta_2', 1)$ (essentially we have passed from homogeneous coordinates to non-homogeneous ones). The points of the line p are defined by the equations

$$z_1 = \alpha_1 z_0 + \beta_1 z_3 \quad , \quad z_2 = \alpha_2 z_0 + \beta_2 z_3$$

and similarly for the line p'. The lines intersect if there is a common solution (z_0, z_1, z_2, z_3) of this system of four equations i.e. if

$$\begin{aligned} z_0(\alpha_1 - \alpha_1') + z_3(\beta_1 - \beta_1') &= 0 \\ z_0(\alpha_2 - \alpha_2') + z_3(\beta_2 - \beta_2') &= 0 \end{aligned} \qquad (7)$$

Thus, the lines intersect if

$$\varrho(\alpha, \beta, \alpha', \beta') = (\alpha_1 - \alpha_1')(\beta_2 - \beta_2') - (\beta_1 - \beta_1')\cdot(\alpha_2 - \alpha_2') \qquad (8)$$

vanishes. When the modern mathematician looks at a quadratic expression he feels an irresistible desire to claim it as distance. So do we. It goes without saying that on Q the distance ϱ is complex. But the principal fact is that it vanishes if the lines intersect. Moreover, by this condition the distance ϱ is defined uniquely up to a conformal transformation.

Let us assign to each point $p \in Q$ the isotropy cone $V_p \subset Q$, i.e. the set of points p' which are at zero distance from p (lines p and p' intersect). Then V_p coincides with the intersection of the quadric Q and the tangent plane to Q at the point p.

Metrics on surfaces S and M. Let us restrict ρ to the quadric S. We confine again with points such that $p_{03} = 1$. Then (3) implies that $\beta_1 = \bar{\alpha}_2$, $\beta_2 = -\bar{\alpha}_1$ and as coordinates on S a pair of complex number (α_1, α_2) will do, and then

$$\rho_S(\alpha, \alpha') = |\alpha_1 - \alpha_1'|^2 + |\alpha_2 - \alpha_2'|^2 \qquad (9)$$

This distance is non-negative definite and non-degenerate. It agrees with the fact that lines which correspond to points of S do not intersect. We have obtained the usual Euclidean distance on the four-dimensional real sphere.

Now restrict ρ to the hyperboloid M assuming again $p_{03} = 1$. Let $M_0 \subset M$ be the set of points such that $p_{03} = 1$. Then (4) implies that α_1 and β_2 are real, while $\bar{\beta}_1 = \alpha_2$. Let us make the substitution

$$\alpha_1 = t - x_1, \quad \beta_2 = t + x_1, \quad \beta_1 = x_2 + i x_3$$

where t and x_j are real, which makes (8) into

$$\rho_M(t, x; t', x') = (t-t')^2 + (x_1 - x_1')^2 - (x_2 - x_2')^2 - (x_3 - x_3')^2$$

This is exactly the Minkowsky metric. The intersection of the cone V_p with M_0 is the light cone C_p with the vertix p.

Thus the distance naturally arising from geometry induces on S the conformal Euclidean metric and on M the conformal Minkowsky metric.

We have obtained that M_0 with the metric ρ_M is the Minkowsky space. To the points of M_0 correspond those lines on the surface $N \subset \mathbb{C}P^3$ which do not intersect the line ℓ of the form $z_0 = z_3 = 0$. The manifold M is the conformal compactification of the Minkowski space. It is obtained from M_0 by adding the light cone at infinity.

The space Q maybe considered as the complexified compactified Minkowsky space.

Automorphism groups. On the space $\mathbb{C}P^3$ the group $SL(4, \mathbb{C})$ acts by projective transformations. Since lines go into lines under these transformations the action removes to the quadric of lines Q. Further, since under projective transformations skew lines go into skew ones on Q we obtain conformal automorphisms of the metric ρ. In particular, the cones V_p will transform into each other.

Similarly, if we consider the subgroup of projective transformations that preserve the surface N , i.e. $SU(2,2) \subset SL(4, \mathbb{C})$, we obtain automorphisms of M , which are conformal with respect to g_M ,i.e. the conformal transformations of the Minkowsky space. Finally, if we consider projective transformations which preserve also the line ℓ we obtain the affine automorphisms of the Minkowsky space M_0 .

The interpretation of twistors. Now let us show how to recover naturally N and $\mathbb{C}\mathbb{P}^3$ from M and Q .On M , consider light cones C_p , where $p \in M$. To their points correspond lines $p' \subset N$ that intersect the line p . It is easier to verify that to points which belong to the same generator of C_p correspond lines in N which intersect at the same point. The result is a correspondence between the generators of the light cones (light lines) on compactified Minkowsky space M and points of the surface N . It means that N may be considered as the abstract set of light lines.
But in applications the structure of N connected with its embedding in $\mathbb{C}\mathbb{P}^3$ (called CR structure) is also important.

On Q , the points of $\mathbb{C}\mathbb{P}^3$ correspond to the following data. The cones V_p have two-dimensional generators. Under the notations of (8) we have

$$\pi_1 : \quad (\alpha_2 - \alpha_2') = \lambda(\alpha_1 - \alpha_1') , \quad (\beta_2 - \beta_2') = \lambda(\beta_1 - \beta_1')$$

$$\pi_2 : \quad (\beta_1 - \beta_1') = \mu(\alpha_1 - \alpha_1') , \quad (\beta_2 - \beta_2') = \mu(\alpha_2 - \alpha_2')$$

where $p = (\alpha, \beta)$

To generators of the family π_1 correspond those lines that go through fixed points of the initial space $\mathbb{C}\mathbb{P}^3$, and to a generator of π_2 there correspond families of lines that lie in the fixed planes of the space $\mathbb{C}\mathbb{P}^3$. Thus, if we start from complexified Minkowsky space Q with a conformal structure defined by a system of isotropic cones V_p , then twistor space $\mathbb{C}\mathbb{P}^3$ is a manifold of generators of V_p of the type π_1 ("complex light planes").

Analytical applications of twistor theory are based on the fact that N (thus differing from M) has complex tangent directions. It enables us to consider different variants of tangent Cauchy-Riemann equation on N . In some way the integral of a solution by lines in N gives on M a solution of different equations of mathematical physics.

1. The interpretation of selfdual metrics in terms of manifolds of curves

In the introduction we dealt with the flat space-time. The following idea is due to Penrose and it aims at interpreting curved space-time as a manifold of complex curves on a three-dimensional complex manifold. The conformal class of the metric must be defined by the requirement that the distance between points corresponding to intersecting curves is zero.

The conformal structure on a four-dimensional comples manifold. Penrose [1] considers the complex picture only. The real variant see in [8] . Curved real Minkowsky spaces do not appear in this scheme. Below it is shown that the consideration of a complex situation enlightens a new geometrical meaning of a number of classical geometric notions, e.g. of the operator $*$ and of Riemannian and Weil curvatures.

Let Q be a complex manifold of dimension 4 and $g_{ij}(p)\,dp^i\,dp^j$ the analytical symmetric (not Hermitian!) metric on Q, i.e. $g_p(\xi,\eta) = g_{ij}(p)\,\xi^i\eta^j$ is a non-degenerate bilinear form on the tangent space T_pQ for any $p \in Q$. The conformal class of the metric includes all metrics $\tilde g_{ij}(p) = f(p)\,g_{ij}(p)$ that differ from g by a functional multiple . Let Γ_p be the infinitesimal isotropy cone; it is defined in T_pQ by the equation $g_p(\xi,\xi)=0$ (it is the infinitesimal version of the cone V_p of the introduction). Recall that in the complex case all non-degenerate cones of the second order are linearly equivalent and in the dimension 4 there are two families of two-dimensional flat generators (see (11)). Generators of one family intersect only in the vertex of the cone and through each non-vertex point of the cone there passes a unique generator of each family.

It is of importance that the converse is also true, that is the non-flat cone containing two families of two-dimensional flat generators is automatically quadratic (the equivalent statement is that a non-flat surface in three-dimensional space with two systems of linear generators is a second order surface (cf. page). Note also that the family of flat generators has the canonical structure of a projective line.

Let us call the generators of one family α - planes and the generators of the other β - planes. Let us choose this indexation consistently for all points of Q (it is equivalent to the choice of orientation). Penrose's important observation is that in terms of α - and β - planes it is very convenient to formulate geometrical notions concerned with metrics (clearly in case of real Riemannian or Minkowsky metrics we lack such a possibility). It is especially so with conformal notions.

Selfdual metrics. The α -surface (respectively β - surface) is the two-dimensional submanifold L in Q, such that all its tangent planes are α - planes (respectively β - planes). They are isotropic (light) surfaces with respect to the metric g . Consider the problem of construction of an α - surface with the given tangent α - plane at the fixed point $p \in Q$. We will say that the metric g is α - integrable (selfdual) if this problem is solvable for all initial values (the uniqueness is not difficult to deduce from the Frobenius theorem). The β- integrability (antiselfdualness) is similarly defined. When the metric is α - integrable, the family of α - surfaces depends on three complex parameters and has the natural structure of a complex manifold $\mathcal{T} = \mathcal{T}(Q, g)$. To points $p \in Q$ correspond families of α - surfaces (through p) Curves E_p arise which admit the structure of projective lines, the structure, that is in the set of α - planes of the tangent space $T_p Q$. So to points $p \in Q$ correspond rational curves on $\mathcal{T}$ (four-parameter family).

When do these curves E_p and $E_{\tilde{p}}$ intersect? Clearly they do if on Q there is a α - surface through p and $\tilde{p}$,i.e. when p and $\tilde{p}$ are at zero distance. That means that the metric g is in the desired agreement with the incidence relation of curves E_p . The similar construction is possible for β - integrable metrics.

Proposition. The manifold Q with metric g which is α as well as β - integrable is conformal flat.

This proposition is the reason why a non-flat manifold with α - integrable metric cannot have a real form with real non-conformal flat Minkowsky metric. On such a manifold β - integrability follows from α - integrability. [8] .

Families of curves corresponding to selfdual metrics. Let $\mathcal{T}$ be a complex manifold with several rational curves. We consider a four-parameter family of such curves Q ,and try to introduce conformal structure on Q : consider cones V_p consisting of lines $\tilde{p}$ intersecting p and pass to their images Γ_p in tangent spaces $T_p Q$. The conformal structure on the four-dimensional manifold thus obtained does not as a rule correspond to a conformal metric: cones Γ_p will not be quadratic ones. Let us try to translate the condition of being quadratic in terms of curves in $\mathcal{T}$. As we have already mentioned, Γ_p is quadratic if on Γ_p there are two families of two-dimensional flat generators.

此为试读,需要完整PDF请访问: www.ertongbook.com