Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Springer-Verlag
Berlin Heidelberg New York

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Springer-Verlag
Berlin Heidelberg New York 1983

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AMS Subject Classifications (1980): 30FXX, 30F40, 32G15, 51M10, 14H35

ISBN 3-540-11975-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-11975-2 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Main entry under title: Kleinian groups and related topics. (Lecture notes in mathematics; 971) Papers presented at the Workshop on Kleinian Groups and Related Topics held in Oaxtepec, Mexico, Aug. 10–14, 1981, during the Second Coloquio de Matemáticas held by the Mathematics Dept. of the Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional. 1. Kleinian groups–Congresses. 2. Riemann surfaces–Congresses. 3. Geometry, Hyperbolic–Congresses. 4. Geometry, Algebraic–Congresses. I. Gallo, D. M. (Daniel M.), 1944-. II. Porter, R. M. (R. Michael), 1952-. III. Workshop on Kleinian Groups and Related Topics (1981: Oaxtepec, Mexico) IV. Series: Lecture notes in mathematics (Springer-Verlag) ; 971. QA3.L28 no. 971 [QA331] 510s [515'.223] 82-19656
ISBN 0-387-11975-2 (U.S.)

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

INTRODUCTION

The Workshop on Kleinian Groups and Related Topics was held in Oaxtepec, Mexico, from August 10 to 14, 1981, during the second Coloquio de Matemáticas held by the Mathematics Department of the Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional.

The theory of Kleinian groups has undergone a vast diversification in the last two decades, particularly with its more recent applications to 3-manifold theory. The object of this conference was to provide a stimulus to its development and to make recent progress more accesible to researchers in Mexico. The keynote speaker was B. Maskit.

The contributions to this volume were provided by participants at the Workshop as well as others who responded to a call for papers. In accordance with an agreement made at the time of the conference, all articles contained here have been refereed.

The editors wish to express their appreciation to the Consejo Nacional de Ciencia y Tecnología, the Secretaría de Educación Pública, the Instituto Politécnico Nacional, and the Centro de Investigación y de Estudios Avanzados for their support of this event. In addition we thank all those who had a share in the production of this book, including the Workshop participants, the referees, and Springer-Verlag.

Daniel M. Gallo

R. Michael Porter

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LIFTING SURFACE GROUPS TO SL(2,ℂ)*

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It is an elementary consequence of the uniformization theorem that the fundamental group $\pi_1 S$ of a closed orientable surface S of genus $g \geq 2$ has a faithful representation as a Fuchsian group G. That the totality of such representations forms a connected subset of the real Möbius group, $\text{Möb}_{\mathbb{R}}$, was first shown by Fricke. Since $\text{Möb}_{\mathbb{R}}$ is canonically isomorphic to $PSL(2,\mathbb{R})$, it is natural and computationally relevant to investigate whether a given representation may be lifted to $SL(2,\mathbb{R})$. Fricke showed that it is indeed possible and further that the lift depends continuously on the surface S viewed as a point in Teichmüller space. It is an immediate consequence of the definitions that given a Möbius transformation γ, there are two choices of lifts $\hat{\gamma}$ of γ to $SL(2,\mathbb{R})$. Since G is a surface group of rank $2g$ there are $2g$ choices of the signs of the generators $\hat{\gamma}_i$. Further, the commutator relation

$$\prod_{i \text{ odd}} [\gamma_i, \gamma_{i+1}] = \text{id}$$

lifts to the relation

$$(1) \qquad R = \prod_{i \text{ odd}} [\hat{\gamma}_i, \hat{\gamma}_{i+1}] = \pm \text{Id}$$

where the γ_i are a standard set of generators for G. Fricke's argument shows that there is a constant choice of sign possible in Equation 1. Siegel [5] was first to raise the question as to whether the sign was always positive. Both Bers [3] and Abikoff [2, p. 18] claimed that the result was true. Bers' unpublished proof consists

* This research has been partially supported by the National Science Foundation.

of studying the side identifications of regular 4g-gons in the hyperbolic plane.

Here we prove the result by showing the following

THEOREM. *If* $\hat{G}$ *is a lift of the surface group* G *to* SL(2,R) *and the genus of* G *satisfies* $g \geq 2$, *then for any choice* $\gamma_1, \ldots, \gamma_{2g}$ *of standard generators of* G, *we have*

$$\prod_{i \text{ odd}} [\hat{\gamma}_i, \hat{\gamma}_{i+1}] = \text{Id}.$$

That this proof appear in print is in response to a question raised by Irwin Kra and the members of a student seminar at Stony Brook.

The proof of the theorem is an immediate consequence of the following three lemmas and the techniques used in the study of augmented Teichmüller spaces (see Abikoff [1] and [2] and Harvey [4]).

If H is a Fuchsian group representing a thrice punctured sphere then G is a free group on two generators but is usually presented as $< \gamma_1, \gamma_2, \gamma_3 : \gamma_1\gamma_2\gamma_3 = \text{id} >$. It is easy to see that by appropriate choice of sign of tr $\hat{\gamma}_3$, the relation may be chosen to lift to

$$\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3 = \text{Id}. \tag{2}$$

LEMMA 1. *If* H *is lifted to* SL(2,R) *so that the traces* tr $\hat{\gamma}_1$ *and* tr $\hat{\gamma}_2$ *have the same sign and the relation* (2) *is valid, then* tr $\hat{\gamma}_3$ *has negative sign.*

LEMMA 2. *If*

$$\hat{\theta} = \prod_{\substack{i \text{ odd} \\ i < 2g-1}} [\hat{\gamma}_i, \hat{\gamma}_{i+1}],$$

then tr $\hat{\theta}$ *and* tr R *have opposite sign.*

LEMMA 3. $\operatorname{tr}\hat{\theta} < 0$.

Lemmas 2 and 3 immediately give the desired result.

1. Proof of Lemma 1.

We normalize $\hat{\gamma}_1$ and $\hat{\gamma}_2$ as follows:

$$\hat{\gamma}_1 = \begin{bmatrix} \pm 1 & \pm 1 \\ 0 & \pm 1 \end{bmatrix} \quad \text{and} \quad \hat{\gamma}_2 = \begin{bmatrix} \pm 1 & 0 \\ \pm\alpha & \pm 1 \end{bmatrix} .$$

Since γ_3^{-1} is parabolic, $\operatorname{tr}\gamma_3^{-1} = \pm 2$, but $\operatorname{tr}\gamma_3^{-1} = 2 + \alpha$. The choice of 2 for $\operatorname{tr}\gamma_3^{-1}$ yields $\alpha = 0$ which is impossible since H is a free group of rank 2.

2. Proof of Lemma 2.

Since $R = \pm \mathrm{Id}$ we have

$$\hat{\theta}\,\hat{\gamma}_{2g-1} = \pm\hat{\gamma}_{2g}\hat{\gamma}_{2g-1}\hat{\gamma}_{2g}^{-1} . \tag{3}$$

Using Fricke's result and the connectedness of the augmented Teichmüller space, we see that a consistent choice of sign may be made for any discrete faithful representation of $\pi_1 S$ in $SL(2,\mathbb{R})$ and that the choice must persist as θ becomes parabolic. We may then assume, by conjugation, that

$$\hat{\theta} = \begin{bmatrix} \pm 1 & \pm 1 \\ 0 & \pm 1 \end{bmatrix} \quad \text{and} \quad \hat{\gamma}_{2g-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

Since trace is a conjugation invariant, Equation 3 implies

$$\operatorname{tr}\hat{\theta}\hat{\gamma}_{2g-1} = \pm\operatorname{tr}\hat{\gamma}_{2g-1}$$

or

$$\pm(a + c + d) = \pm(a + d).$$

If both choices of sign are identical, then $c = 0$. It follows that the Möbius transformation $\gamma_{2g-1} : z \mapsto (az+b)/(cz+d)$ fixes infinity. Since θ also fixes infinity, well-known properties of Fuchsian groups show that γ_{2g-1} and θ commute, but group theory says that they don't.

3. *Proof of Lemma 3.*

In the augmented Teichmüller space a thrice punctured sphere is a limiting deformation of a torus with one hole. Two of the punctures come from pinching one curve; the corresponding matrices are then conjugate in the limit group, hence have traces of the same sign. It follows from Lemma 1 that the sign of the matrix representing the remaining puncture, which came from the border curve, is negative.

If S_g is a surface of genus g, we dissect it as shown in Figure 1

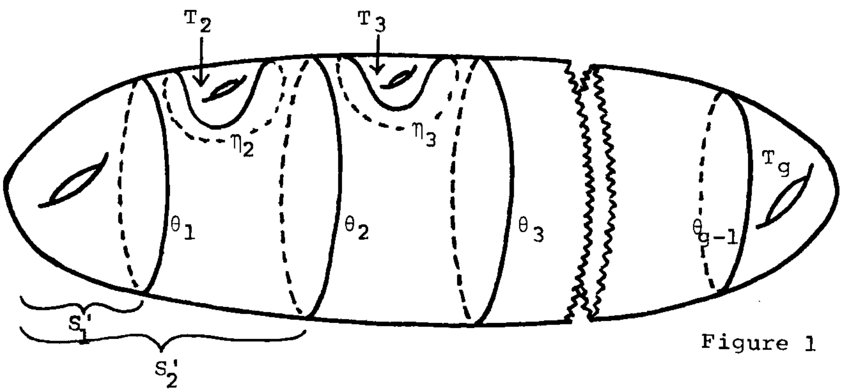

Figure 1

and use induction to show that $\mathrm{tr}\,\hat{\theta} = \mathrm{tr}\,\hat{\theta}_{g-1} < 0$. For $g = 2$, it was done above. For $g > 2$, S_g is the union of a surface S'_{g-2} of genus $g-2$ with one border curve θ_{g-2}, a pants and two tori T_{g-1} and T_g each missing a disk. S_g is also the union of S'_{g-1} and T_g. As inductive hypothesis we take $\mathrm{tr}\,\hat{\theta}_{g-2} < 0$. Since η_{g-1} borders T_{g-1}, $\mathrm{tr}\,\hat{\eta}_{g-1} < 0$. By Lemma 2, it follows that $\mathrm{tr}\,\hat{\theta}_{g-1} < 0$.

4. *Some Remarks.*

After proving this result, we learned that several other proofs exist. These other proofs use heavy machinery which is either topological, algebraic or cohomological in nature.

Andrew Haas has observed that this proof may be used to show that $\operatorname{tr}\hat{\theta} < 0$ whenever θ is a simple dividing loop on a compact surface.

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NEC GROUPS AND KLEIN SURFACES

EMILIO BUJALANCE

1. *Introduction*

In the last few years several results on non-Euclidean crystallographic groups and on applications of these groups to the study of Klein surfaces have been obtained. This paper consists mainly of a brief survey of these results.

In Section 2 NEC groups are introduced and their main properties are given. In Section 3 we study the NEC normal subgroups of NEC groups. In Section 4 we show the existing relations between NEC groups and Klein surfaces as well as between normal NEC subgroups and automorphism groups of Klein surfaces. Finally in Section 5 we study the automorphism groups of Klein surfaces.

2. NEC *groups*

Let us consider transformations of the Riemann sphere $\mathbb{C}^+$ of the following forms:

i) $w(z) = \dfrac{az+b}{cz+d}$, $z \in \mathbb{C}^+$: $ad-bc = 1$, $a,b,c,d \in \mathbb{R}$;

ii) $w(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$, $z \in \mathbb{C}^+$: $ad-bc = -1$, $a,b,c,d \in \mathbb{R}$.

These transformations form a group G which maps the upper half plane $\operatorname{Im} z > 0$, which we denote by D, into itself.

The transformations of the form i) preserve the orientation of D and form a subgroup of G of index two (*the hyperbolic group*); the transformations of the form ii) reverse the orientation of D.

The transformations of the hyperbolic group are of three types:

1) *hyperbolic*: if $|a+d| > 2$ with two fixed points on the real axis,

2) *elliptic*: if $|a+d| < 2$ with one fixed point in D, and

3) *parabolic*: if $|a+d| = 2$ with one fixed point of multiplicity two on the real axis.

The transformations of the form ii) are either *glide-reflections* (if $a+d \neq 0$) with two fixed points on the real axis, or *reflections* (if $a+d=0$) with a circle of fixed points.

If, on D, we introduce the Riemannian metric $ds = |dz|y^{-1}$ $(z = x+iy)$, then D becomes a model of the hyperbolic plane and G its group of isometries.

A *non-Euclidean crystallographic* (NEC) group is a discrete subgroup Γ of G with compact quotient space D/Γ. An NEC group containing only orientation-preserving transformations is a *Fuchsian group*.

In order to derive presentations for NEC groups, Wilkie [17] proved the following result:

Let Γ be an NEC group. Then there is a fundamental region P for Γ which is a polygon in D whose perimeter, described counterclockwise, is one of the following:

(1) $\xi_1\xi_1'\cdots\xi_\tau\xi_\tau' \quad \varepsilon_1\gamma_{10}\cdots\gamma_{1s_1}\varepsilon_1' \cdots \varepsilon_k\gamma_{k0}\cdots\gamma_{ks_k}\varepsilon_k' \quad \alpha_1\beta_1\alpha_1'\beta_1'\cdots\alpha_g\beta_g\alpha_g'\beta_g'$

(2) $\xi_1\xi_1'\cdots\xi_\tau\xi_\tau' \quad \varepsilon_1\gamma_{10}\cdots\gamma_{1s_1}\varepsilon_1' \cdots \varepsilon_k\gamma_{k0}\cdots\gamma_{ks_k}\varepsilon_k' \quad \alpha_1\alpha_1^*\cdots\alpha_g\alpha_g^*$

In these symbols, each letter denotes an oriented side of the polygon. The apostrophe means that the corresponding sides $\xi\xi', \alpha\alpha', \beta\beta', \varepsilon\varepsilon'$ of the polygon are identified by generators of the group which preserve orientation, and the asterisk means that the corresponding sides $\alpha\alpha^*$ of the polygon are identified by generators of the group which reverse orientation. As a consequence, if we identify corresponding points on the related edges of the polygon, we obtain from it a surface with boundary. In the case (1), the surface will be a sphere with k disks removed and g handles added. In the case (2) the surface will be a sphere with k disks removed and g cross-caps added.

Moreover, we have the following properties: a) the stabilizer in Γ of the vertex M_i common to the sides ξ_i, ξ_i' is a cyclic group of rotations of order m_i (x_i will denote a generator of the group); b) the stabilizer of the vertex N_{ij} common to the sides $\gamma_{i(j-1)}$, γ_{ij} is a dihedral group of order $2n_{ij}$, with generators $c_{i(j-1)}$, c_{ij}, that are reflections in $\gamma_{i(j-1)}$, γ_{ij} respectively; c) the stabilizer of an inner point of the side γ_{ij} joining N_{ij} and $N_{i(j+1)}$ is a group isomorphic to $\mathbb{Z}_2$, generated by c_{ij}; d) the stabilizer of the vertex $N_{i(s_i+1)}$ common to the sides γ_{is_i} and ε_i' is a group isomorphic to $\mathbb{Z}_2$, generated by c_{is_i}. No other points of P are fixed points for Γ.

Macbeath [9] associated to the NEC groups of type (1) the *NEC signature:*

$$(*) \quad (g\,;+\,;[m_1,\ldots,m_\tau]\,;\,\{(n_{11}\ldots n_{1s_1}),\ldots,(n_{k1}\ldots n_{ks_k})\})$$

and to the groups of type (2) the signature

$$(**) \quad (g\,;-\,;[m_1,\ldots,m_\tau]\,;\,\{(n_{11}\ldots n_{1s_1}),\ldots,(n_{k1}\ldots n_{ks_k})\})$$

These signatures characterize the algebraic structure of the groups.

Wilkie [17] proves that an NEC group with signature (*) has the presentation given by generators:

x_i : $i = 1,\ldots,\tau$	elliptic transformations
a_j, b_j : $j = 1,\ldots,g$	hyperbolic transformations
c_{ij} : $i = 1,\ldots,k$ $j = 0,\ldots,s_i$	reflections
e_i : $i = 1,\ldots,k$	hyperbolic transformations

and relations:

$$x_i^{m_i} = 1 : \ i = 1,\ldots,\tau$$

$$c_{is_i} = e_i^{-1}c_{i0}e_i : \ i = 1,\ldots,k$$

$$c_{i(j-2)}^2 = c_{ij}^2 = (c_{i(j-1)}c_{ij})^{n_{ij}} = 1 : \ i = 1,\ldots,k\,;\ j = 1,\ldots,s_i$$

$$x_1 \dots x_\tau e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

In a group Γ with signature (**) we have the presentation given by generators:

x_i : $i = 1,\dots,\tau$	elliptic transformations
d_j : $j = 1,\dots,g$	hyperbolic transformations
c_{ij} : $i = 1,\dots,k$; $j = 0,\dots,s_i$	reflections
e_i : $i = 1,\dots,k$	hyperbolic transformations

and relations:

$$x_i^{m_i} = 1 : \ i = 1,\dots,\tau$$

$$c_{is_i} = e_i^{-1} c_{i0} e_i : \ i = 1,\dots,k$$

$$c_{i(j-1)}^2 = c_{ij}^2 = (c_{i(j-1)} c_{ij})^{n_{ij}} = 1 : \ i = 1,\dots,k \ ; \ j = 1,\dots,s_i$$

$$x_1 \dots x_\tau e_1 \dots e_k d_1^2 \dots d_g^2 = 1.$$

From now on, we will denote by $x_i, e_i, c_{ij}, a_i, b_i, d_j$ the above generators associated with an NEC group.

Let c, c' be two period cycles $c = (n_1 \dots n_s)$ and $c' = (n_1' \dots n_s')$. Then c, c' are called *directly equivalent* if one is a cyclic permutation of the other, that is, if $s = s'$ and there is an integer k such that $n_i = n'_{k+i}$, suffixes being read modulo s.

c, c' are called *inversely equivalent* if one is a cyclic permutation of the other reversed, that is, if $s = s'$ and there is an integer k such that $n_i = n'_{k-i}$, where the suffixes are again reduced modulo s.

If Γ is a NEC group of signature

$$(g \, ; \pm ; [m_1 \dots m_\tau] \, ; \ \{(c_1 \dots c_k)\})$$

then Γ is isomorphic to the group Γ' of signature

$$(g' ; \pm ; [m_1' \ldots m_{\tau'}'] ; \{(c_1' \ldots c_{k'}')\})$$

if and only if they have the same sign, $g = g'$, $\tau = \tau'$, $k = k'$, $[m_1', \ldots m_{\tau'}'] = [m_1 \ldots m_\tau]$, and there is a permutation ϕ of $(1,2 \ldots k)$ such that either (a) for each i, c_i' is directly equivalent to $c_{\phi(i)}$, or (b) for each i, c_i' is inversely equivalent to $c_{\phi(i)}$, in the case of the sign "+" ; and there is a permutation of $(1 \ldots k)$ such that for each i, c_i' is either directly or inversely equivalent to $c_{\phi(i)}$ in the case of the sign "-" (Macbeath [9] and Wilkie [17]).

In the same way as in the case of Fuchsian groups, in the NEC groups the measures of the fundamental regions will allow us to relate the NEC signatures of the groups with those of the subgroups. Similarly these measures give some conditions for the achievement of the NEC signatures. We will call the *measure of a fundamental region of an NEC group* Γ the area of any fundamental region associated to the group, and we will denote it by $|\Gamma|$. The measure of the fundamental region of an NEC group with signature (*) is (Singerman [16])

$$|\Gamma| = \pi\left(2\tau + 2k + 4g - 4 + \sum_{i=1}^{k} s_i - \sum_{i=1}^{k}\sum_{j=1}^{s_i} \frac{1}{n_{ij}} - \sum_{i=1}^{\tau} \frac{2}{m_i}\right) = \pi R$$

and the measure of the fundamental region of an NEC group Γ with signature (**) is

$$|\Gamma| = \pi\left(2\tau + 2k + 2g - 4 + \sum_{i=1}^{k} s_i - \sum_{i=1}^{k}\sum_{j=1}^{s_i} \frac{1}{n_{ij}} - \sum_{i=1}^{\tau} \frac{2}{m_i}\right) = \pi R'$$

If the expressions R and R' are > 0, then there are NEC groups of signatures (*) and (**) respectively (Zieschang [18]).

3. *Normal subgroups of NEC groups*

In this section we will describe some results that have been obtained for NEC normal subgroups of an NEC group.

If Γ is an NEC group and Γ' is a normal subgroup of Γ with finite index, then Γ' is an NEC group. Conversely, every NEC normal subgroup of an NEC group has finite index.

The orientation in the signatures of the NEC groups is the same for the normal subgroups in the case that the index of the group with respect to the subgroup is odd (Bujalance [2], Hall [6], Hoare, Singerman [7]).

If Γ is a *proper NEC group* (i.e., non Fuchsian), then it has a subgroup Γ^+ of index 2 consisting of the elements which preserve orientation (i.e., $\Gamma^+ = \Gamma \cap G^+$) called the *canonical Fuchsian group* of Γ. In [16] Singerman determines the signature of Γ^+ given that of Γ.

In [2], [3] we have studied the existing relations between the signatures of NEC normal subgroups of an NEC group and the signature of the group, when the index of the group is a prime number $p \neq 2$. As a consequence we have obtained necessary and sufficient criteria on the NEC signatures for the existence of an NEC group with such a signature, which is a normal subgroup of some other NEC group with index different from a power of 2.

We have also established necessary and sufficient conditions for a normal subgroup of odd index of an NEC group to be of a specific signature. These conditions have been obtained using the existence of a certain group of permutations. We have calculated the proper periods of normal NEC subgroups when the index of the subgroup is even.

4. *Compact Klein surfaces*

By a *Klein surface* we shall mean a surface X with or without boundary together with an open covering by a family of sets $\mathcal{U} = \{U_i\}$ with the following properties:

1) For each $U_i \in \mathcal{U}$ there exists a homeomorphism φ_i of U_i onto an open set in the sphere $\mathbb{C}^+$.

2) If $U_i, U_j \in \mathcal{U}$ and $U_i \cap U_j \neq \emptyset$ then $\varphi_i \circ \varphi_j^{-1}$ is an analytic or antianalytic mapping defined on $\varphi_j(U_i \cap U_j)$.

A homeomorphism $f: X \longrightarrow X$ is a called an *automorphism* if $\varphi_i \circ f \circ \varphi_j^{-1}$ is either an analytic or antianalytic mapping in its domain of definition.

If X is orientable and without boundary we will say that it is a *Riemann surface*.

Now let E be the field of all meromorphic functions of a Klein surface X. E is an algebraic function field in one variable over $\mathbb{R}$ and as such it has an algebraic genus. In case X is a Riemann surface, the algebraic genus is equal to the topological genus of X. If X is not a Riemann surface, then the relationship between the topological genus p and the algebraic genus g is given by

$$p = (g-k+1)/2 \quad \text{if } X \text{ is orientable,}$$

$$p = g-k+1 \quad \text{if } X \text{ is non-orientable,}$$

where k denotes the number of boundary components of X.

We will say that an NEC group Γ is a *surface group* if the quotient space $X = D/\Gamma$ is compact and the quotient map $p: D \longrightarrow D/\Gamma$ is unramified. Γ will be called a *bordered surface* if X has boundary.

The following results give the relationship between Klein surfaces and NEC groups:

If Γ is an NEC group, then the quotient space D/Γ has a unique dianalytic structure such that the quotient map $P: D \longrightarrow D/\Gamma$ is a morphism of Klein surfaces (Alling and Greenleaf [1]).

Let X be a compact Klein surface of algebraic genus $g \geqslant 2$. Then X can be represented in the form D/Γ, where Γ is a bordered surface group or a surface group (Singerman [15] and Preston [14]).

We close this section by describing the relation between the automorphism groups of Klein surfaces and the NEC normal subgroups of an NEC group.

In the case of compact non-orientable Klein surfaces without boundary, a necessary and sufficient condition for a finite group G to be a group of automorphisms of a non-orientable Klein surface without boundary of topological genus $p \geqslant 3$ is that there exist a proper NEC group Γ and a homomorphism $\theta: \Gamma \longrightarrow G$ such that the kernel