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Edited by A. Dold and B. Eckmann

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Ryszard Jajte

Strong Limit Theorems in Non-Commutative Probability

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continuation on page 153

PREFACE

Recently many authors have extended a series of fundamental pointwise convergence theorems in the theory of probability and ergodic theory to the von Neumann algebra context. They have provided some new tools for mathematical physics and at the same time created interesting techniques in the theory of operator algebras. The main purpose of these notes is to present a self-contained exposition of some ideas and results from this area. We shall confine ourselves to the case of von Neumann algebras and shall not touch on the problems concerning C*-algebras. One of the reasons for this is that we are trying to keep the book on a relatively elementary level. The material presented here has been chosen in such a way that only very little knowledge of the theory of operator algebras is needed for reading it. On the other hand, the von Neumann algebras are very natural non-commutative generalizations of L_∞-algebras, and their rich structure gives the possibilities to obtain the limit theorems in their "almost sure" versions. In a von Neumann algebra one can introduce the "almost uniform" convergence which, in the classical commutative case of the algebra L_∞, is equivalent (via Egoroff's theorem) to the almost sure convergence. This type of convergence will be fundamental for the whole book.

Recently, C. Lance proved a non-commutative version of the individual ergodic theorem for *-automorphisms of a von Neumann algebra. From the point of view of applications in quantum dynamics this result is of great importance. Chapter 2 is devoted to the discussion of some results of this kind and their generalizations. In particular, we prove some "individual" ergodic theorems for normal positive maps of a von Neumann algebra, the non-commutative versions of Kingman's subadditive ergodic theorems for *-automorphisms, a random ergodic theorem and a non-commutative local ergodic theorem for quantum dynamical semigroups. Chapter 3 is devoted to the theory of martingales in von Neumann algebras. Conditional expectations in von Neumann algebras and martingale convergence theorems are important in particular in the theory of measurement in quantum mechanics. The non-commutative martingale convergence theorems of N. Dang-Ngoc and M. S. Goldstein will be proved. Chapter 4 deals with the strong laws of large numbers in the context of von Neumann algebras. Among others, Batty's results will be presented. Chapter 1 has a preparatory character. In it we shall discuss some properties of the almost uniform convergence in von

Neumann algebras.

These notes do not cover, of course, all the results concerning the almost uniform convergence in von Neumann algebras. In particular, we do not discuss the ergodic theorems for weights, only for states. This book is written mainly for a reader familiar with the theory of probability but may be of some interest for those mathematicians and physicists who are interested in some techniques of operator algebras and their applications. As we mentioned, our main goal is to present some ideas which lead us from the classical results well-known in the probability theory to their non-commutative versions, and, consequently to the applications in quantum field theory. In this sense our book is "homogeneous". Most of the results presented here have been obtained recently (Lance 1976-78, Kümmerer 1978, Dang-Ngoc 1978, 1982, Yeadon 1975-1980, Watanabe 1979, Goldstein 1981, and others).

Some theorems will be proved for states and some of them only for traces. Since a state, in general, is not subadditive on the lattice of projections, the techniques for non-tracial states are, as a rule, quite different from those used for traces, and are also much more difficult. It is worth noting here that very often the arguments needed for traces are similar to the classical ones; but in some cases, a new approach is necessary.

The prerequisites for reading this book are the fundamentals of functional analysis and probability. The elements of the theory of von Neumann algebras are collected in the Appendix. We refer also to the Appendix for the terminology and notations used in the book. The Appendix is almost self-contained and can also be read separately, before studying the main chapters.

These notes were prepared during my stay at the University of Tennessee in Knoxville and at the Center for Stochastic Processes at the University of North Carolina at Chapel Hill. I am deeply indebted to all my colleagues from both these institutions for the warm hospitality they extended to me.

Many thanks are due to Ms. Cindi Blair and Mrs. Margaret Ravitch for their patient and skillful typing of the first draft of the manuscript. I sincerely wish to thank Mrs. Ruth Bahr who took great care in the typing of the final version.

Chapel Hill, October 1984 R. Jajte

CONTENTS

INTRODUCTION 1

CHAPTER 1. ALMOST UNIFORM CONVERGENCE IN VON NEUMANN ALGEBRAS

1.1 Preliminaries 4

1.2 Various kinds of 'almost sure' convergence in von Neumann algebras. 5

1.3 Non-commutative version of Egoroff's theorem 8

1.4 Notes and remarks 9

CHAPTER 2. ERGODIC THEOREMS

2.1 Preliminaries .11

2.2 Almost uniform ergodic theorems 12

2.3 Non-commutative subadditive ergodic theorems. 32

2.4 Random ergodic theorem. 47

2.5 Local ergodic theorem for quantum dynamical semigroups. 52

2.6 Notes and remarks 55

CHAPTER 3. CONVERGENCE OF CONDITIONAL EXPECTATIONS AND MARTINGALES IN VON NEUMANN ALGEBRAS

3.1 Conditional expectations 57

3.2 Almost uniform convergence of conditional expectations and martingales 64

3.3 Notes and remarks 80

CHAPTER 4. STRONG LAWS OF LARGE NUMBERS IN VON NEUMANN ALGEBRAS

4.1 Preliminaries .81

4.2 Independence .81

4.3 Almost complete convergence in von Neumann algebras . .82

4.4 Strong limit theorem for orthogonal sequences 83

4.5 A non-commutative extension of Glivenko-Cantelli's theorem 87

4.6 Generalized Kolmogorov's inequality for traces and some of its consequences. 90

4.7 Strong laws of large numbers for traces 91

4.8 Convergence rate in the strong law of large numbers . 100

4.9 Notes and remarks 105

APPENDIX 106
1. Generalities, commutant, bicommutant theorem, A-invariant subspaces 106
2. Positive linear functionals, GNS-representation, pure states and irreducible representation. 107
3. Partial isometries, polar decomposition, equivalence of projectors 111
4. Locally convex topologies on B(H) 113
5. Hilbert-Schmidt and trace-class operators, predual of a von Neumann algebra. 115
6. Normal positive linear functionals, traces, normal positive linear mappings, Kadison's inequality, singular positive linear functionals, metrization of the strong topology in the unit ball of a von Neumann algebra. . . 119
7. Measure topology, non-commutative L_p-spaces 127
8. The algebra $L_\infty(\Omega, F, \mu, A)$ 132
9. Commutative von Neumann algebras 132
10. Tomita-Takesaki modular theory. 135
11. Tomita-Takesaki modular theory, continuation 136

BIBLIOGRAPHY 139

INDEX 151

INTRODUCTION

Contemporary scientific literature offers ample evidence that the algebraic methods which have revolutionized pure mathematics are now in the process of having a similar impact in the physical sciences. The algebraic approach to statistical mechanics and quantum field theory is an example of this new orientation.

Gerard G. Emch

Non-commutative probability theory has as its motivation the mathematical foundations of quantum mechanics. On the other hand it can be treated as a natural generalization of the classical probability theory. Let us begin with some comparisons. In the classical mechanics to any physical system of point particles there corresponds a differentiable manifold $\mathcal{U}$. The states of the system are represented by the points of $\mathcal{U}$, and the physical quantities (observables) are described then by (measurable) functions over the manifold $\mathcal{U}$. In quantum mechanics, to any physical system there corresponds a Hilbert space H. For systems with a finite number of degrees of freedom the (pure) states are given by vectors (rays) in H, or, more generally, mixed states, by positive trace class operators (density operators). The observables are represented by the self-adjoint operators acting on H. For particle systems with an infinite number of degrees of freedom, one identifies the states of the system with the (mathematical) states over a suitable algebra A of operators. In most of the situations A can be taken as a von Neumann algebra of operators acting on a separable complex Hilbert space. The algebra of all bounded linear operators acting on H is the von Neumann algebra, so both cases are covered by the general algebraic set up. Moreover, the classical situation leads to the commutative von Neumann algebra $L_\infty(\mathcal{U}, B_\mathcal{U},\mu)$ of bounded measurable functions over a measure space $(\mathcal{U}, B_\mathcal{U},\mu)$. The unbounded observables (measurable functions) are then "affiliated" with $L_\infty(\mathcal{U}, B_\mathcal{U},\mu)$ in a natural way. The measure μ (after its unique extension to the integral $f \to \int_\mathcal{U} f(d\mu)$ is a normal faithful state on L_∞. Thus the classical case can be treated also in frames of (commutative) von Neumann algebras. Conversely, every commutative von Neumann algebra is isomorphic to some L_∞, so that every commutative case is in fact the classical one.

In this book we shall be concerned with some limit theorems describing the asymptotic behavior of sequences of observables treated as elements of a von Neumann algebra A (or affiliated with A). More exactly, we shall treat the observables as the elements of the "kinematical structure" described by a von Neumann algebra A with a faithful normal state ϕ. For an observable ξ from A, the number $\phi(\xi)$ represents the expected value of ξ - the only measurable quantity of the theory - when the system is in the state ϕ. In particular, for a self-adjoint ξ with spectral representation $\xi = \int_{-\infty}^{\infty} \lambda e(d\lambda)$, and Borel Z on the real line, the number $\phi(e(Z))$ is the probability that ξ takes its values from Z when the system is in the state ϕ.

For finite quantum systems the dynamics is determined by specification of a self-adjoint operator H (the Hamiltonian operator) which is usually a function of the particle position and momentum, e.g.,

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m} + v(q_1, q_2, \ldots, q_n),$$

where p_i and q_i (momentum and position operators) satisfy the canonical commutation relations

$$p_i p_j - p_j p_i = 0 = q_i q_j - q_j q_i$$

and

$$p_i q_j - q_j p_i = - i h \delta_{ij}$$

Then the evolution of the system in time is given by the one-parameter group of unitary operators $U_t = \exp\{itH\}$ and is described either by changes of the observables

$$A \to A_t = U_t A U_t^* \quad \text{(Heisenberg picture)}$$

or by changes of the states

$$\phi \to \phi_t = \exp(-itH)\phi \quad \text{(Schrödinger picture).}$$

The well-known theorem of Wigner [132] [135] says that any *-automorphism of B(*H*) is given by a unitary operator, so we can say that the dynamics of the system is always described by a one-parameter group of *-automorphisms of the algebra of (bounded) observables, and any *-automorphism of the algebra of observables describes some dynamics with the Hamiltonian operator equal to the infinitesimal generator of the unitary group U_t (via Stone theorem). Thus the most natural way to describe the dynamics of an infinite system is to determine it by a continuous one-parameter group (α_t) of *-automorphisms of the underlying von Neumann algebra (describing the "kinematical structure" of the system). Then the motion of the system is represented by the equations for the evolution in time of the expected values $t \to \phi(\alpha^t \xi)$ (in the Heisenberg picture) or by $t \to \tilde{\alpha}^t \phi(\xi)$ (in the Schrödinger picture), where $\tilde{\alpha}^t$ is the (pre) dual of α^t.

A common approach to understanding the dynamical behavior of physical systems is through 'time averages' of the form

$$(*) \qquad s_N = \frac{1}{N} \sum_{t=0}^{N-1} \alpha^t \xi$$

(and their expected values). In particular we are interested in the existence of the limit of s_N as $N \to \infty$ in a sense as strong as possible. In the sequel we shall discuss in details the conditions for the 'almost sure' (almost uniform) convergence of the sums

$$x_N = \frac{1}{N} \sum_{t=0}^{N-1} \xi_t$$

where ξ_t are the operators from a von Neumann algebra A (or affiliated with A). In particular we shall be concerned with the ergodic averages (*), where α is a *-automorphism of A.

We shall start with an analysis of the notion of almost uniform convergence in von Neumann algebras.

CHAPTER 1

ALMOST UNIFORM CONVERGENCE IN VON NEUMANN ALGEBRAS

1.1 Preliminaries

Throughout the book we constantly use the terminology of operator algebras. For this terminology we refer to the Appendix. In the sequel A will denote a von Neumann algebra acting in a complex Hilbert space H; we denote by A' the commutant of A. ϕ will be a state on A. A_+ will stand for the cone of positive elements of A; Proj A will denote the set of all orthogonal projections in A. For $p \in$ Proj A, always $p^\perp = 1-p$. We shall write 1 for the identity operator in A. For a Borel subset Z of the real line and a self-adjoint operator x, we denote by $e_Z(x)$ the spectral projection of x corresponding to Z. For $x \in A$ we put $|x|^2 = x^*x$. In the next sections we introduce and discuss in detail the notion of the almost uniform convergence in von Neumann algebras. Let us begin with some comparisons. For a probability space (Ω,F,μ), let $\mathbb{L}_\infty(\Omega,F,\mu)$ be the algebra (of equivalent classes) of all complex-valued F-measurable and essentially bounded functions on Ω. It can be treated as a commutative von Neumann algebra acting in $\mathbb{L}_2(\Omega,F,\mu)$ if we identify the functions $g \in \mathbb{L}_\infty$ with the multiplication operators a_g: $f \to fg$, for $f \in \mathbb{L}_2$. The algebra $A = \mathbb{L}_\infty(\Omega,F,\mu)$ has a faithful normal tracial state τ_μ (given by $\tau_\mu(f) = \int_\Omega f d\mu$). By Ergoroff's theorem the μ-almost sure convergence of a sequence (f_n) from A is equivalent to its almost uniform convergence. This fact makes it possible to express the almost sure convergence purely in terms of the algebra A, without any reference to the base space Ω. Namely, we may restate the almost sure convergence by means of the $\mathbb{L}_\infty$-norm, state τ_μ and the characteristic functions (of "large" sets). This suggests the following definition.

1.1.1. DEFINITION. *Let A be a von Neumann algebra with a faithful normal state ϕ. We say that a sequence (x_n) of elements of A converges almost uniformly to an element $x \in A$ if, for each $\varepsilon > 0$, there is a projection $p \in A$ with $\phi(1-p) < \varepsilon$ and such that $\|(x_n-x)p\| \to 0$ as* $n \to \infty$.

1.1.2. It is worth noting here that, in fact, the above definition does not depend on the choice of ϕ; namely, the almost uniform convergence just defined is equivalent to the following two conditions:

(*) in any strong neighbourhood of the identity in A, there is a projection p such that $||(x_n - x)p|| \to 0$ as $n \to \infty$.

(**) for every faithful normal state ϕ on A and $\varepsilon > 0$, there exists a projection $p \in A$ with $\phi(1-p) < \varepsilon$ such that $||(x_n - x)p|| \to 0$.

This follows immediately from the fact that if ϕ is a normal faithful state then the strong topology in the unit ball S in A can be metrized by the formula $\text{dist}(x,y) = \phi[(x-y)^*(x-y)]^{1/2}$ (see Appendix).

1.1.3 THEOREM. *Let A be a von Neumann algebra with a faithful normal state ϕ. For bounded sequences of operators (x_n) from A, the almost uniform convergence implies the strong (σ-strong) convergence of (x_n).*

Proof. Let $x_n \to 0$ almost uniformly. In our case, the GNS representation of A associated with ϕ is faithful and normal so, without any loss of generality, we can assume that A acts in its GNS representation space H_ϕ in a standard way. In particular, we have $\phi(x) = (x\xi,\xi)$ for $x \in A$, where ξ is a cyclic and separating vector in H_ϕ. Let $\varepsilon > 0$ be given. Assume that $||x|| \leq 1$. There is a projection $p \in A$ with $\phi(1-p) < \varepsilon$ and $||x_n p|| \to 0$. Let $y \in A'$ (commutant of A). Then, denoting by $||\cdot||_\phi$ the norm in H_ϕ we have

$$||x_n y\xi||_\phi \leq ||x_n p y\xi||_\phi + ||x_n(1-p)y\xi||_\phi .$$

But $||x_n p y\xi||_\phi \leq ||x_n p||\cdot||y\xi||_\phi < \varepsilon$ for n large enough, and

$$\begin{aligned} ||x_n(1-p)y\xi||_\phi &= ||yx_n(1-p)\xi||_\phi \leq ||yx_n||\ ||(1-p)\xi||_\phi \\ &= ||yx_n||[\phi(1-p)]^{1/2} \leq ||yx_n||\varepsilon^{1/2}, \end{aligned}$$

which shows that $||x_n y\xi||_\phi \to 0$ for all $y \in A'$. Since the set of vectors $\{y\xi,\ y \in A'\}$ is dense in H_ϕ and (x_n) is uniformly bounded, it implies the strong (σ-strong) convergence of x_n to zero. □

1.2. Various kinds of 'almost sure' convergence in von Neumann algebras

In Definition 1.1.1 we introduced the concept of the almost uniform convergence which generalizes to the von Neumann algebra context the notion of the almost sure convergence. One can consider the other noncommutative versions of this notion.

Let A be as before a von Neumann algebra with a faithful normal state ϕ. Let us write four conditions for x_n and x in A.

(i) for any $\varepsilon > 0$, there is a projection p in A with $\phi(1-p) < \varepsilon$

and a positive integer N such that $||(x_n - x)p|| < \varepsilon$ for $n \geq N$.

(ii) for any $\varepsilon > 0$, there is a projection $p \in A$ with $\phi(1-p) < \varepsilon$, such that $||(x_n - x)p|| \to 0$ as $n \to \infty$.

(iii) for any $\varepsilon > 0$, there is a sequence of projections (p_n) in A increasing to 1 (in the strong topology) such that $||(x_n - x)p_n|| < \varepsilon$ for $n = 1,2,\ldots$.

(iv) for any non-zero projection p in A there is a non-zero projection $q \in A$ such that $q \leq p$ and $||(x_n - x)q|| \to 0$ as $n \to \infty$.

Of course, the condition (ii) means the almost uniform (a.u.) convergence of x_n to x. If the condition (i) or (iii) or (iv) is satisfied, then (x_n) is said to converge to x closely on large sets (c.l.s.) or nearly everywhere (n.e.) or quasi uniformly (q.u.), respectively.

Evidently, in the case of a commutative von Neumann algebra $\mathbb{L}_\infty(\Omega, F, \mu)$, all four conditions just formulated are equivalent to the μ-almost sure convergence.

1.2.1. THEOREM. *Let A be a von Neumann algebra with a faithful normal state ϕ. For any bounded sequence (x_n) in A, all four conditions (i) through (iv) are equivalent.*

Proof: We assume that $x = 0$ and $||x_n|| \leq 1$ for $n = 1,2,\ldots$. Let $p \in \mathrm{Proj}\, A$, $y \in A$ and $\phi(p|y|^2p) < \varepsilon^4 < 1$. Then, putting $q = pe_{[p,\varepsilon^2]}\{p|y|^2p\}$, we have that $q \leq p$, $\phi(p-q) \leq \varepsilon$ and $||yq|| < \varepsilon$. Indeed, clearly, $q \leq p$. Moreover, $\phi(p-q) \leq \varepsilon^2\phi(p|y|^2p) < \varepsilon$, and $||yq||^2 = ||q|y|^2q|| = ||qp|y|^2p|| < \varepsilon^2$.

Let us also notice that, for $y \in A$ with $||y|| \leq 1$ and $q,r \in \mathrm{Proj}\, A$, if $||q^\perp r|| < \alpha$ and $||yq|| < \beta$, then $||yr|| < \alpha + \beta$. To prove this it is sufficient to estimate $||yr\xi|| \leq ||yq^\perp r\xi|| + ||yqr\xi||$.

From the facts just proved it follows that (i) implies the following condition

(*) for each $\varepsilon > 0$ and $q \in \mathrm{Proj}\, A$, there is a projection $r \in A$ such that $r \leq q$, $\phi(q-r) < \varepsilon$ and $||x_n r|| < \varepsilon$ for large enough.

Indeed, let $0 < \varepsilon_n \to 0$. By (i) we can find a sequence $(r_n) \subset \mathrm{Proj}\, A$ with $\phi(r_n^\perp) < \varepsilon_n$ and a sequence of positive integers $m(n)$ such that $||x_m r_n|| < \varepsilon_n$ for $m > m(n)$. Let $q \in \mathrm{Proj}\, A$ be given. Then, by the normality of Φ, $\phi(qr_n^\perp q) \to 0$ and we can fix n_0 such that $\varepsilon_{n_0} < \varepsilon$ and $\phi(qr_{n_0}^\perp q) < \varepsilon^4$. Putting $r = qe_{qr_{n_0}^\perp q}[0,\varepsilon^2)$, we have $r \leq q$, $\phi(q-r) < \varepsilon$ and $||r_{n_0}^\perp r|| < \varepsilon$. Moreover, $||x_m r|| < 2\varepsilon$ for $m > m(n_0)$.

To prove the implication (i) $\to$ (ii), let us fix some $\varepsilon > 0$ and assume that (i) holds. By (*), we find a sequence $(p_n) \subset \operatorname{Proj} A$ such that $1 = p_1 \geq p_2 \geq \ldots$, $\phi(p_n - p_{n+1}) < 2^{-n}\varepsilon$ and $\|x_m p_n\| < \varepsilon$ for $m > m(n)$. Put $p = \inf_k p_k$. Then $\phi(p^\perp) = \sum_n \phi(p_n - p_{n+1}) < \varepsilon$, and $\|x_m p\| \leq \|x_m p_{n_o}\| < \varepsilon$ for $m > m(n_o)$. This means that $x_m \to 0$ almost uniformly.

By an easy modification of the above proof we can show the implication (i) → (iv). Namely, for a given $0 \neq p \in \operatorname{Proj} A$ and $\varepsilon > 0$ we find a sequence of projections $p = p_1 \geq p_2 \geq \ldots$ with $\|x_m p_n\| < \varepsilon$ for $m > m(n)$ and $\phi(p_n - p_{n+1}) < 2^{-(n+1)}\phi(p)$. Then it is enough to put $q = \inf_k p_k$. The implication (ii) → (i) is trivial.

Suppose now that (iii) holds so that, for $\varepsilon > 0$, there exist projections p_n in A with $p_n \uparrow 1$ and $\|x_n p_n\| < \varepsilon$ for all n. Then $\phi(p_m) \geq 1 - \varepsilon$ for $n \geq m$ which means that (i) holds so that (iii) implies (i).

It remains to prove the implication (iv) → (iii). Let $\varepsilon > 0$, $0 < \varepsilon_k < \varepsilon_{k+1} \to \varepsilon$ and $0 < \delta_k \to 0$. To show that (iii) holds it is enough to find an increasing sequence $(q_k) \subset \operatorname{Proj} A$ and an increasing sequence of positive integers such that $\phi(q_k^\perp) < \delta_k$ and $\|x_n q_k\| < \varepsilon_k$ for $n \geq n_k$ (then we can put $p_1 = \cdots = p_{n_1} = 0$, $p_{n_1+1} = \ldots = p_{n_2} = q_1$, etc. Thus it is enough to show that if $\varepsilon < \varepsilon'$, $\delta > 0$ and $\|x_n p\| < \varepsilon$ for $n > \ell$, where $p \in \operatorname{Proj} A$, then there exist $q \in \operatorname{Proj} A$ and $\ell' > \ell$ such that $q \geq p$, $\phi(q^\perp) < \delta$ and $\|x_n q\| < \varepsilon'$ for $n > \ell'$. Let $(p_t, t \in T)$ be a maximal family of mutually orthogonal projections in A such that $p_t \leq p^\perp$ and $\|x_n p_t\| \to 0$ as $n \to \infty$ for every $t \in T$. This family is at most countable (because there is a faithful normal state ϕ on A; comp. Appendix). Since ϕ is normal and faithful, from (iv) it follows that there exists a sequence (p_k) of mutually orthogonal projections in A such that $\sum_{k=1}^{\infty} p_k = p^\perp$ and $\|x_n p_k\| \to 0$ as $n \to \infty$ for $k = 1, 2, \ldots$.
Taking N large enough we obtain $\phi(p^\perp - \sum_{k=1}^{N} p_k) < \delta$ and, consequently, $\phi(q^\perp) < \delta$ for $q = p + \sum_{k=1}^{N} p_k$. Moreover, $\|x_n q\| < \varepsilon'$ for n sufficiently large. The proof is completed. □

1.2.2. THEOREM. *If ϕ is a tracial faithful normal state (i.e. A is a finite von Neumann algebra) then all four conditions are equivalent.)*

Proof. The proof of (ii) → (i), (iii) → (i) and (iv) → (iii) is the same as in Theorem 1.2.1, so it remains to prove the implications (i) → (iv) and (iv) → (ii). Let (i) hold, and let $0 \neq p \in \operatorname{Proj} A$. Put $\varepsilon_k = 2^{-k-1}\phi(p) > 0$. We find $q_k \in \operatorname{Proj} A$ such that $\phi(q_k^\perp) < \varepsilon_k$,

$||x_n q_k|| < \varepsilon_k$ for $n \geq n(k)$. Put $q = p \wedge \bigwedge_{k=1}^{\infty} q_k$. Obviously, $||x_n q|| \to 0$ as $n \to \infty$, and $q \leq p$. Moreover $\phi(q^{\perp}) \leq \phi(p^{\perp}) + \sum_k \phi(q_k^{\perp}) < \phi(p^{\perp}) + \phi(p/2) <$ < 1. i.e. $q \neq 0$. Thus the implication (i) $\to$ (iv) is proved. Assume now that (iv) holds. Then there exists a sequence (p_k) of mutually orthogonal projections in A such that $\sum_{k=1}^{\infty} p_k = 1$ and $||x_n p_k|| \to 0$ as $n \to \infty$ for $k=1,2,\ldots$. Taking N large enough we obtain $\phi(\sum_{k=1}^{N} p_k) \geq 1-\varepsilon$ and $||x_n \sum_{k=1}^{N} p_k|| \to 0$ as $n \to \infty$. The proof is completed. □

1.2.3 Let us assume that ϕ is a trace (finite or semifinite). Consider the *-algebra $\tilde{A}$ of operators measurable with respect to (A,ϕ) in the sense of Segal-Nelson (see Appendix). The almost uniform convergence (nearly everywhere convergence etc.) can also be considered for sequences in $\tilde{A}$, in particular for sequences (x_n) in $\mathbb{L}_1(A,\phi)$. It is easy to modify the Definition 1.1.1 in a suitable way. Namely, a sequence (x_n) in $\tilde{A}$ is said to be convergent almost uniformly to x if, for each $\varepsilon > 0$, there is a projection $p \in A$ such that $\phi(p^{\perp}) < \varepsilon$, $(x_n - x)p \in A$ for $n > n_0$ and $||(x_n - x)p|| \to 0$ as $n \to \infty$. A similar modification of the conditions (i), (iii) and (iv) we leave to the reader. In the sequel we shall also use the bilateral convergence.

1.2.4 DEFINITION. *A sequence* (x_n) *in* A *(in* $\tilde{A}$ *if* ϕ *is a trace) is said to be bilaterally almost uniformly convergent to* $x \in A(\tilde{A})$ *if for each* $\varepsilon > 0$ *there exists a projection* $p \in A$ *such that* $\phi(1-p) < \varepsilon$ *and* $||p(x_n - x)p|| \to 0$ *as* $n \to \infty$.

We omit the formulation of natural versions of the conditions (i), (iii) and (iv). Theorem 1.2.1 and Theorem 1.2.2 hold also for the bilateral version of the conditions (i)-(iv). We leave the proof to the reader.

1.3. Non-commutative version of Egoroff's theorem.

We start with the following proposition

1.3.1. PROPOSITION *Let* A *be a von Neumann algebra acting in a Hilbert space* H. *If a sequence* (x_i) *in* A *converges strongly to* x_o *then, for every* $\varepsilon > 0$, *there exists a sequence* $(p_i) \subset$ *Proj* A *such that* $p_i \to 1$ *strongly and* $||(x_i - x_0)p_i|| < \varepsilon$ *for* $i=1,2,\ldots$.

Proof. We may assume $||x_i|| \leq 1$ and $x_0 = 0$. Put $y_i = x_i^* x_i$. Then, for every $h \in H$, we may have $||y_i h|| = ||x_i^* x_i h|| \leq ||x_i^*|| \, ||x_i h|| \leq ||x_i h||$, hence (y_i) converges strongly to zero. Put $p_i = e_i([0,\varepsilon^2])$, where $e_i(\cdot)$ denotes the spectral measure of y_i. Then

$$y_i = \int_0^1 u e_i(du) \geq \varepsilon^2 \int_{[\varepsilon^2,1]} \frac{u}{\varepsilon^2} e_i(du) \geq \varepsilon^2(1-p_i),$$

which implies that (p_i) converges strongly to 1. Moreover, we have

$$||x_i p_i||^2 = ||p_i x_i^* x_i p|| \leq ||x_i^* x_i p_i|| = ||y_i p_i|| < \varepsilon^2,$$

which ends the proof. □

1.3.2 THEOREM (Non-commutative Egoroff's theorem). Let A be a von Neumann algebra with a faithful normal state ϕ. Let (x_n) be a sequence in A convergent to x in the strong operator topology. Then, for every projection $p \in A$ and any $\varepsilon > 0$, there exists a projection $q \leq p$ in A and a subsequence (x_{n_k}) of (x_n) such that $\phi(p-q) < \varepsilon$ and $||(x_{n_k}-x)q|| \to 0$ as $k \to \infty$.

Proof. We may assume that $p = 1$ and $x = 0$. By Proposition 1.3.1, there exists a sequence (p_n) of projections in A such that $||x_n p_n|| < \frac{1}{2}$ and $p_n \to 1$ strongly. Choose the index n_1 such that $\phi(1-p_n) < \varepsilon/2$ for $n > n_1$. Put $q_1 = p_{n_1}$. Then $\phi(1-q_1) < \varepsilon/2$, $||x_{n_1} q_1|| < \frac{1}{2}$ and, of course, $x_n q_1 \to 0$ strongly. Putting $y_n^{(1)} = q_1 x_n^* x_n q_1$, we obtain a bounded sequence in $q_1 A q_1$ which converges strongly to zero. Repeating the reasoning for $(y_n^{(1)})$ we find a sequence $(q_n^{(1)})$ of projections in $q_1 A q_1$ such that $q_n^{(1)} \to q_1$ strongly and $||y_n^{(1)} q_n^{(1)}|| < 1/2^2$; we choose an index $n_2 > n_1$ such that $\phi(q_1 - q_n^{(1)}) < \varepsilon/2^2$ and $||y_{n_2}^{(1)} q_2|| < 1/2^2$. Putting $q_2 = q_{n_2}^{(1)}$, we have $q_2 \leq q_1$, $\phi(q_2 - q_1) < \varepsilon/2^2$ and $||y_{n_2}^{(1)} q_2|| < \frac{1}{2^2}$. But $||x_n q_2||^2 = ||q_2 x_n^* x_n q_2|| = ||q_2 q_1 x_n^* x_n q_1 q_2|| = ||q_2 y_n^{(1)} q_2|| \leq$
$\leq ||y_n^{(1)} q_2|| < 1/2^2$ for $n > n_2$. Consequently $||x_m q_2|| < 1/2^2$. By induction we obtain a decreasing sequence (q_n) of projections in A and a sequence of indices $n_1 < n_2 < \ldots$ such that $||x_{n_k} q_k|| < 1/2^k$, $\phi(q_k - q_{k+1}) < \varepsilon/2^k$. Putting $q = \inf_k q$, we get $\phi(1-q) \leq \varepsilon$ and $||x_{n_k} q|| < 1/2^k \to 0$, which completes the proof. □

<u>1.4 Notes and remarks</u>. The 'pointwise' convergence in von Neumann algebras discussed in Chapter 1 was introduced first by I. Segal [117] and has been used systematically in the so called 'noncommutative