Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

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Optimal Control of Nonsmooth Distributed Parameter Systems

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INTRODUCTION

Starting with the pioneering work of Lions and Stampacchia [73], much attention is paid in the literature to the investigation of nonlinear partial differential equations involving nondifferentiable and even discontinuous terms. This includes the case of variational inequalities and free boundary problems and the main motivation of their interest is given by the many models arising in such a form with an important area of applications. See the monographs of Kinderlehrer and Stampacchia [62], Friedman [47], Ockendon and Elliott [45] where a large number of examples from various domains are discussed.

A natural direction of development of the theory is the study of related control problems and the first papers along these lines belong to Lions [71], Yvon [144], Mignot [74]. This may be also viewed as a continuation of the classical analysis of control systems governed by linear partial differential equations and we quote the wellknown book of Lions [68] in this respect.

The present work may be inscribed as a contribution to the general effort of research in nonsmooth optimization problems associated with nonlinear partial differential equations.

More precisely, the main aim of these notes is to examine distributed control problems governed by nonlinear evolution equations (parabolic or hyperbolic), in the absence of differentiability properties. In this setting, a special emphasis is given to nonlinear hyperbolic problems which are less discussed in the literature.

In order to obtain a more complete image of the area of research we have mentioned and to show other possible applications of the methods, several sections deal with problems which don't enter strictly in the announced subject: nonlinear delay differential equations, boundary control, elliptic problems and optimal design.

The material of the book is divided into three parts, after the type of the nonlinear term which occurs in the state system: semilinear and quasilinear problems, variational inequalities, free boundary problems. Among the different topics underlined throughout the work, we refer to: existence of optimal pairs, first order necessary conditions, general and efficient approximation procedures. Each time when this is possible, we indicate applications to regularity or bang-bang results for the optimal control. Examples clarifying and motivating the theory are included in every chapter.

An important role in the conception of the work is played by the methods: the adapted penalization, V.Barbu [13], the unstable systems control theory, J.L.Lions [72], the variational inequality approach [129]. From a technical point of view, in order to make these notes more useful, we have tried to use different types of arguments in different problems, of course in certain limits.

Chapter I, which introduces some fundamental notions, notations and results, is kept at a minimum length. In particular, for the existence, uniqueness and regularity theory of the various equations which appear in the text, we generally prefer to quote appropriately the scientific literature. A precious auxiliary in this respect, containing a large amount of the needed results, are the monographs of V.Barbu [12], H. Brezis [20].

The book is based mainly on the results obtained by the author during several years, but it also gives a survey of the existing literature in this area of research, via numerous comments, references, comparisons. However, as the subject is still under active development, there is no attempt to be comprehensive in any sense. We mention only the contributions of Barbu [17], Lions [72], Pawlow [90], Neittaanmaki and Haslinger [59], which are closer to our topic.

In the elaboration of this work, we received constant support and encouragement during the debates in the specialised seminars of the University of Iasi, leaded by prof.dr. V.Barbu and of the University of Bucharest, leaded by prof.dr. A.Halanay. To them and to all the other participants in the seminars, we express our gratitude. We also acknowledge with thanks the financial support of the Institute of Mathematics of the Romanian Academy of Sciences, which was decisive in the preparation of the present work.

Bucharest, June 1990 **Dan Tiba**

LIST OF SYMBOLS

R^N - the finite dimensional Euclidean space,

Ω - bounded domain in R^N,

mesΩ - the Lebesgue measure of Ω,

$\Gamma = \partial\Omega$ - the boundary of Ω,

$Q = \Omega \times]0,T[$,

$\Sigma = \partial\Omega \times [0,T]$,

$\mathcal{D}(\Omega)$, $C_0^\infty(\Omega)$ - the space of indefinitely differentiable functions, with compact support in Ω,

$\mathcal{D}'(\Omega)$ - the space of distributions on Ω,

$L^p(\Omega)$, $1 \leq p \leq \infty$ - the space of real functions, p-integrable in Ω (with the usual modification when $p = \infty$),

$L^p(\Omega;X)$ - the space of p-integrable functions in the sense of Bochner, with values in the Banach space X,

$C^m(\bar{\Omega};X)$ - the space of m times continuously differentiable functions on $\bar{\Omega}$, with values in X,

$W^{k,p}(\Omega)$ - the Sobolev space of real p-integrable functions, with distributional derivatives up to order k, p - integrable in Ω,

$BV(0,T;X)$ - the space of bounded variation functions on $[0,T]$, with values in X,

$M(0,T;X)$ - the space of X-valued measures on $]0,T[$,

X^* - the dual of the space X,

$S(x,a)$ - the ball with centre x and radius a, in a given metric space,

$(.,.)_{X \times X^*}$ - the pairing between X and X^*,

$(.,.)_H$ - the scalar product in the Hilbert space H,

$|\cdot|_X$ - the norm in the normed space X,

$|\cdot|$ - the Euclidean norm in R^N,

dom(A) - the domain of the mapping A,

R(A) - the range of the mapping A,

A^* - the adjoint of the linear operator A,

$\text{dist}(v,M)_X$ - the distance between $v \in X$ and the set $M \subset X$, in the metric of X,

int M - the interior of the set M,

$\bar{M}$ - the closure of the set M,

MxN - the cartesian product of the sets M and N,

$[m,n]$ - an ordered pair in M x N,

cl f - the lower semicontinuous closure of f,

∂f - the subdifferential of the convex function f,

$\partial K = [-\partial_x K, \partial_y K]$ - the subdifferential of the saddle function K,

Df - the Clarke generalized gradient of f,

grad f - the gradient of the differentiable function f on R^N,

∇f - the Gateaux differential of f in normed spaces,

y', $\dot{y}$, y_t, y_x, dy/dt, $\partial y/\partial t$, $\partial y/\partial x$ - different notations of derivatives,

$\partial/\partial n$ - the outward normal derivative to Γ.

Note: sometimes we use other notations, which will be specified in the text.

I. ELEMENTS OF NONLINEAR ANALYSIS

This introductory chapter contains some prerequisites in the theory of monotone operators, convex analysis, generalized gradients, Sobolev spaces and nonlinear differential equations, collected for easier reference .

For the sake of brevity, many of the results are indicated without proof and others are omitted. The material presented here is standard and, with minor exceptions, may be found in the wellknown monographs of V.Barbu [12], [13], V.Barbu and Th.Precupanu [14], H.Brezis [20], [21], I.Ekeland and R.Temam [42], J.L.Lions [69], R.T.Rockafellar [99], D.Kinderlehrer and G.Stampacchia [62], K.Yosida [145].

1. Function spaces and compactness principles

We denote $R =]-\infty,+\infty[$, R^N, $N \in \{1,2,...\}$, the finite dimensional Euclidean spaces. Let Ω be a measurable subset in R^N and X be a Banach space with norm $|\cdot|_X$. By $L^p(\Omega;X)$ we mean the space of equivalence classes, modulo the equality a.e., of functions strongly measurable in Ω, with values in X and with the norm p - integrable, $1 < p < \infty$. $L^p(\Omega;X)$ is a Banach space with respect to the norm

$$|u|^p_{L^p(\Omega;X)} = \int_\Omega |u(x)|^p_X dx .$$

For $p = \infty$, $L^\infty(\Omega;X)$ is the space of equivalence classes of functions, modulo the equality a.e., measurable from Ω to X and essentially bounded in Ω. It also is a Banach space with the norm

$$|u|_{L^\infty(\Omega;X)} = \operatorname*{esssup}_{x\in\Omega} |u(x)|_X .$$

Let Ω be a bounded domain in R^N. A remarkable subspace of $L^\infty(\Omega;X)$ is $C(\bar{\Omega};X)$, the Banach space of functions, continuous on $\bar{\Omega}$, with values in X, endowed with the topology of the uniform convergence. Two situations appear usually in the text:

- $\Omega =]0,T[\subset R$ and we use the notations $L^p(0,T;X)$, $L^\infty(0,T;X)$, $C(0,T;X)$;
- $X = R$ and we use the notations $L^p(\Omega)$, $L^\infty(\Omega)$, $C(\bar{\Omega})$.

The spaces $L^p(\Omega;X)$ have wellknown properties [45], [125]. We state only the Egorov theorem, which will be of frequent use:

Theorem 1.1. Assume that $f_n \to f$ strongly in $L^p(\Omega)$, $1 \le p \le \infty$. Then for every $\varepsilon > 0$ there is a measurable subset $\Omega_\varepsilon \subset \Omega$, $mes(\Omega - \Omega_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on Ω_ε.

We consider the space $\mathcal{D}(\Omega)$ (sometimes also denoted by $C_0^\infty(\Omega)$) of infinitely differentiable functions with compact support in Ω and its dual - the space of distributions $\mathcal{D}'(\Omega)$ to be known (see [145]). If $u \in L^p(\Omega)$, $1 \le p \le \infty$, then the functional

$$h \to \int_\Omega u(x)h(x)dx$$

defined on $\mathcal{D}(\Omega)$, is a distribution on Ω, denoted u, and called distribution of type function. Let k be a natural number. The Sobolev space $W^{k,p}(\Omega)$ is the space of all the distributions $u \in \mathcal{D}'(\Omega)$ of type function, such that all the distributional derivatives, $D^\alpha u$, up to order k, of u, are distributions of type function and belong to $L^p(\Omega)$. $W^{k,p}(\Omega)$ is a Banach space with the norm

$$|u|^p_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega |D^\alpha u(x)|^p dx,$$

where α is a multiindex, $|\alpha|$ is its length and D^α is the distributional derivative of order α.

The completion of $\mathcal{D}(\Omega)$ in the topology of $W^{k,p}(\Omega)$ is denoted by $\overset{\circ}{W}{}^{k,p}(\Omega)$ and its dual is denoted by $W^{-k,q}(\Omega)$, $p^{-1}+q^{-1} = 1$. In the case p=2, the following notations are used $W^{k,2}(\Omega) = H^k(\Omega)$, $\overset{\circ}{W}{}^{k,2}(\Omega) = \overset{\circ}{H}{}^k(\Omega)$, $W^{-k,2}(\Omega) = H^{-k}(\Omega)$. The spaces $H^k(\Omega)$ are Hilbert spaces with the scalar product

$$(u,v)_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha u . D^\alpha v dx.$$

Analogously, we define the spaces of vectorial distributions. $\mathcal{D}'(0,T;X)$ is the space of linear, continuous operators from $\mathcal{D}(]0,T[)$ to the Banach space X. $W^{k,p}(0,T;X)$ is the space of vectorial distributions $u \in \mathcal{D}'(0,T;X)$ with the distributional derivatives up to order k in $L^p(0,T;X)$. The elements of $W^{k,p}(0,T;X)$ are absolutely continuous functions together with their distributional derivatives, up to the order k - 1.

In the applications to parabolic problems, we write shortly $W^{2,1,p}(Q)$, $Q =]0,T[\times\Omega$, $p \ge 1$, for the space $L^p(0,T;W^{2,p}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega))$.

Take $\Omega = R^N$. By means of the Fourier transform

$$F : f \to Ff = (2\pi)^{-N/2} \int_{R^N} \exp(-ix.\xi) f(x)dx$$

we obtain the equivalent definition

$$H^m(R^N) = \left\{ u \in L^2(R^N); (1 + |\xi|^2)^{m/2} Fu \in L^2(R^N) \right\}.$$

In this way, one can define $H^s(R^N)$ for every $s \in R$, which remains a Hilbert space by the scalar product:

$$(u,v)_{H^s(R^N)} = ((1 + |\xi|^2)^{s/2} Fu, (1 + |\xi|^2)^{s/2} Fv)_{L^2(R^N)}.$$

If the boundary Γ of Ω is sufficiently regular, for instance if it is a C^∞ manifold, we can define the spaces $H^s(\Gamma)$ by local charts.

Proceeding by interpolation, one obtains the spaces $H^s(\Omega)$, $s \in R$. The norm in $H^s(\Omega)$

is equivalent with

$$|u|_{H^s(\Omega)} = \inf |v|_{H^s(R^N)},$$

where v is choosen such that its restriction to Ω equals u a.e. .

For the functions from $H^s(\Omega)$, $s > 0$, one may define the trace on Γ and the normal derivatives $\partial^j u/\partial n^j$ up to the order [s-1/2] (the greatest positive integer majorized by s-1/2). This is the so called trace theorem and the proof may be found in [70].

Theorem 1.2. The mapping

$$u \to \{\partial^j u/\partial n^j;\ j = 0,1,...,[s-1/2]\}$$

from $\mathcal{D}(\Omega)$ to $\mathcal{D}(\Gamma)^{[s-1/2]+1}$ may be extended to a linear, continuous operator from $H^s(\Omega)$ onto $\prod H^{s-j-1/2}(\Gamma)$, $j = 0,1,...,[s-1/2]$.

In particular, the space $H^k_o(\Omega)$ coincides with the kernel of the trace operator.

If Ω is a bounded domain then the embedding $H^{s_1}(\Omega) \subseteq H^{s_2}(\Omega)$, $s_1 > s_2$, is compact (the Rellich theorem). Other types of embeddings are given by the Sobolev theorem:

Theorem 1.3.

i) For $q^{-1} > p^{-1} - kN^{-1}$ and $1 \le p,q \le \infty$, $k \ge 1$, then $W^{k,p}(\Omega) \subset L^q(\Omega)$. Moreover, if $p,q < \infty$, the embedding is compact.

ii) For $m < k - Np^{-1}$ we have $W^{k,p}(\Omega) \subset C^m(\bar{\Omega})$.

We continue with several other specific compactness criterions, in function spaces:

Theorem 1.4. Arzela-Ascoli

A sequence $\{x_n\}$ is relatively compact in $C(\bar{\Omega};X)$ iff:

a) it is equibounded and equicontinuous,

b) $\{x_n(z)\}$ is relatively compact in X for all $z \in \Omega$.

Theorem 1.5. Helly - Foias [68]

Let $V \subset U$ compactly, be Banach spaces. If $\{p_n\}$ is bounded in $L^\infty(0,T;V)$ and $\{(p_n)_t\}$ is bounded in $L^1(0,T;U)$ then, on a subsequence again denoted p_n, we have

$$p_n(t) \to p(t), \quad t \in [0,T]$$

strongly in U and $p \in BV(0,T;U)$.

Theorem 1.6. Dunford-Pettis

A sequence $\{f_n\}$ is weakly relatively compact in $L^1(\Omega;X)$ iff the integrals of $|f_n|_X$ are uniformly absolutely continuous on Ω.

Theorem 1.7. Lions [71]

Let B_o, B and B_1 be three Banach spaces such that $B_o \subset B \subset B_1$ and the injection $B_o \subset B$ is compact. Then, for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that:

$$|u-w|_B \le \varepsilon(|u|_{B_o} + |w|_{B_o}) + C_\varepsilon |u-w|_{B_1}$$

for all $u, w \in B_o$.

Theorem 1.8. Aubin [2]

In the hypotheses of Theorem 1.7, let $\mathcal{V}$ be a bounded subset in $L^{p_o}(0,T;B_o)$ with

$$|dv/dt|_{L^{p_1}(0,T;B_1)} \leq C, \quad v \in \mathcal{V},$$

where $1 < p_o, p_1 < \infty$. Then $\mathcal{V}$ is relatively compact in $L^{p_o}(0,T;B)$.

We close this section with a variant of the Gronwall lemma, due to Brezis [20].

Proposition 1.9. Let $m \in L^1(0,T)$, $m \geq 0$ a.e. $[0,T]$ and a be a positive constant. Let $\Phi \in C(0,T)$ satisfy the inequality

$$1/2\Phi^2(t) \leq 1/2a^2 + \int_0^t m(s)\Phi(s)ds, \quad t \in [0,T].$$

Then, we have:

$$|\Phi(t)| \leq a + \int_0^t m(s)ds, \quad t \in [0,T].$$

2. Monotone operators

Consider two sets, X and Y, and X x Y their cartesian product. A subset $A \subset X \times Y$ is called a multivalued operator defined on X with values in Y. We have:

$$Ax = \{y \in Y;\ [x,y] \in A\},\ x \in X;$$

$$\mathrm{dom}(A) = \{x \in X;\ Ax \neq \phi\} \subset X;$$

$$R(A) = \bigcup_{x \in X} Ax \subset Y, \text{ the range of } A;$$

$$A^{-1} = \{[y,x];\ [x,y] \in A\} \subset Y \times X.$$

Let X be a Hilbert space. A (multivalued) operator $A : X \to X$ is called monotone if

$$(x_1 - x_2, y_1 - y_2)_X \geq 0 \quad \forall [x_i, y_i] \in A,\ i=1,2.$$

If the inequality is strict for $x_1 \neq x_2$, $x_1, x_2 \in \mathrm{dom}(A)$, then A is strictly monotone. If, moreover, the following relation holds:

$$(x_1 - x_2, y_1 - y_2)_X \geq \alpha |x_1 - x_2|_X^2, \quad \alpha > 0, \forall [x_i, y_i] \in A, \quad i=1,2$$

then A is called strongly monotone.

The operator $A : X \to X$ is ω- monotone if $A + \omega I$ is monotone (the definition is useful for $\omega > 0$). The monotone operator $A : X \to X$ is called maximal monotone if its graph, as a subset in X x X, is maximal, that is it cannot be strictly included in any other monotone graph from X x X.

Proposition 2.1. Let $A : X \to X$ be a maximal monotone operator. Then:

i) A^{-1} is maximal monotone

ii) For every $x \in dom(A)$, the set Ax is convex and closed in X.

iii) A is demiclosed, that is $x_n \to x$ strongly in X, $y_n \to y$ weakly in X and $y_n \in Ax_n$ imply that $y \in Ax$.

The proof of this statement is based on the definitions and on the continuity of the scalar product. The next two results are fundamental and their proof is less elementary and we quote Barbu [12], Ch. II for a general argument.

A single valued operator $A : X \to X$ with $dom(A) = X$ is called hemicontinuous if for all $x \in X$, $y \in X$ we have $A((1-t)x + ty) \to Ax$, weakly in X, as $t \to 0$.

Proposition 2.2. A hemicontinuous monotone operator is maximal monotone.

Theorem 2.3. The following statements are equivalent:

i) A is maximal monotone in $X \times X$.

ii) A is monotone and $R(I+A) = X$.

iii) $(I + \lambda A)^{-1}$ is a contraction (nonexpansive) mapping on the whole space X, for all $\lambda > 0$.

Let Ω be a bounded, measurable set in R^N and A be a maximal monotone operator in the Hilbert space X. It is possible to define $\tilde{A}$ on $L^2(\Omega;X)$ by $v \in \tilde{A}u$ iff $v(x) \in Au(x)$ a.e.Ω. The operator $\tilde{A}$ is maximal monotone in $L^2(\Omega;X)$. The monotonicity of $\tilde{A}$ is obvious. For the maximality, we use Thm. 2.3. The equation

$$v(x) + Av(x) \ni f(x), \quad x \in \Omega$$

with $f \in L^2(\Omega;X)$, has a unique solution $v(x) \in X$, a.e. $x \in \Omega$. As $(I+A)^{-1}$ is nonexpansive, we see that $v \in L^2(\Omega;X)$ and it is the solution of the equation $v + \tilde{A}v \ni f$.

Now, assume that Ω is a bounded domain in R^N, with smooth boundary. We consider the nonlinear differential operator:

$$Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \dots, D^m u),$$

where $A_\alpha(x,\xi)$ are real functions defined on $\Omega \times R^K$, satisfying the conditions:

1) A_α are measurable in x and continuous in ξ,

2) $|A_\alpha(x,\xi)| \leq C(|\xi|^{p-1} + g(x))$

with $g \in L^q(\Omega)$, $p>1$, $q^{-1} + p^{-1} = 1$.

3) $\sum_{|\alpha| \leq m} (A_\alpha(x,\xi) - A_\alpha(x,\eta))(\xi_\alpha - \eta_\alpha) \geq 0$

for all $\xi, \eta \in R^K$ and a.e. $x \in \Omega$.

Here we denote by K the length of the vector with components u and all its derivatives up to the order m.

To the operator $A : W_0^{m,p}(\Omega) \to W^{-m,q}(\Omega)$, we associate its realization in $L^2(\Omega)$ by:

$$A_{L^2(\Omega)} u = Au, \quad u \in dom(A_{L^2(\Omega)}),$$
$$dom(A_{L^2(\Omega)}) = \{u \in W_0^{m,p}(\Omega);\ Au \in L^2(\Omega)\}.$$

This is a maximal monotone operator in $L^2(\Omega)$ if we assume one more condition:

$$4)(Au,u)_{W_0^{m,p}(\Omega) \times W^{-m,q}(\Omega)} \geq \alpha |u|^p_{W_0^{m,p}(\Omega)} + C, \quad \alpha > 0,$$

for all $u \in W_0^{m,p}(\Omega)$.

We recall that the operator $A : X \to X$ is coercive iff

$$\lim_{|u|_X \to \infty} (Au,u)_X / |u|_X = +\infty.$$

Theorem 2.4. A coercive, maximal monotone operator is surjective.

Proof

Let $x^* \in X$ be arbitrary fixed. By Thm 2.3, for every $\lambda > 0$, there is $x_\lambda \in X$ such that

(*) $$\lambda x_\lambda + Ax_\lambda = x^*,$$

where $A \subset X \times X$ is the given maximal monotone coercive operator. By shifting the domain of A, we may assume that $0 \in$ dom A and multiplying (*) by x_λ, we get

$$\lambda |x_\lambda|_X \leq |A0|_X + |x^*|_X$$

by the monotonicity of A. Then, again (*), shows that $\{Ax_\lambda\}$ is bounded in X and the coercivity assumption yields that $\{x_\lambda\}$ is bounded in X. If we pass to a subsequence, we may assume that $x_\lambda \to x$ weakly in X and $Ax_\lambda \to x^*$ strongly in X. The demiclosedness of A gives $Ax = x^*$.

Remark 2.5. The monotonicity property may be defined in Banach spaces too, by replacing the inner product with the pairing $(\cdot,\cdot)_{X \times X^*}$. All the above properties remain true in the general setting of Banach spaces.

The nonlinear differential operator A, introduced above, is called the generalized divergence operator or the Leray-Lions operator. If we consider it as acting between the spaces $W_0^{m,p}(\Omega)$ and $W^{-m,q}(\Omega)$ we remark that it is monotone and hemicontinuous, under hypotheses 1)-3). Therefore, in this setting, it is maximal monotone without condition 4).

We say that an operator $A : X \to X$ is locally bounded in $x_0 \in X$ if there is a neighbourhood V of x_0 in X, such that $A(V) = \bigcup_{x \in V} Ax$ is a bounded subset of X.

Theorem 2.6. Any monotone operator $A : X \to X$ is locally bounded on the interior of dom(A).

Proof

By shifting dom(A), we may suppose that $0 \in \text{intdom}(A)$. If A is not locally bounded at 0, we shall derive a contradiction. Thus, consider $\{x_n\} \subset \text{dom}(A)$, $x_n^* \in Ax_n$, such that $x_n \to 0$ and $|x_n^*|_X \to \infty$ as $n \to \infty$.

We take r positive and sufficiently small such that $S(0,r) \subset \text{dom}(A)$. We show that there exist $y \in S(0,r)$ and a subsequence $n_k \to \infty$ such that

(+) $$(x_{n_k} - y, x_{n_k}^*)_X \to -\infty.$$

By contradiction, we assume that for every $u \in S(0,r)$ there is $C_u > -\infty$ such that

$$(x_n - u, x_n^*)_X \geq C_u, \quad \forall\, n \in N.$$

If $E_k = \{u \in S(0,r);\ (x_n - u,\ x_n^*)_X \geq -k,\ \forall\ n \in N\}$, then $S(0,r) = \cup E_k$ and a category argument gives that there are $\varepsilon > 0$, $k_o \in N$, $y \in S(0,r)$ such that $S(y,\varepsilon) \subset E_{k_o}$. We get

$$(2x_n + y - u, x_n^*)_X \geq C_{-y} - k_o \quad \forall\, n \in N,\ \forall\, u \in S(y,\varepsilon).$$

Let n_o be such that $|x_n|_X \leq \frac{\varepsilon}{4}$ for $n \geq n_o$. Then $u = 2x_n + y - v \in S(y,\varepsilon)$ for $n \geq n_o$ and $|v|_X \leq \frac{\varepsilon}{2}$. In other words, it yields

$$(v, x_n^*)_X \geq C_{-y} - k_o, \quad \forall\, n \geq n_o, \quad \forall\, |v|_X \leq \frac{\varepsilon}{2},$$

contrary to $|x_n^*|_X \to \infty$. This shows that (+) is true and the monotonicity of A implies that

$$(x_{n_k} - y, Ay)_X \to -\infty, \text{ as } n_k \to \infty.$$

But $\{x_{n_k}\}$ is bounded and this final contradiction concludes the proof.

The next result is a generalization of Thm 2.4:

Theorem 2.7. Let A be a maximal monotone operator in X. Then A is surjective iff A^{-1} is locally bounded.

Proof

The "only if" part is a direct consequence of Thm 2.6. For the "if" part, we prove that R(A) is simultaneously a closed and open subset of X.

Let $x_o^* \in R(A) = \text{dom}(A^{-1})$ and let $\{x_n^*\} \subset R(A)$ be such that $x_n^* \to x_o^*$ as $n \to \infty$. For $x_n \in A^{-1}x_n^*$, we have $|x_n|_X \leq M$ be the hypothesis and we may assume tht $x_n \to x_o$ weakly in X.

We have

$$(x_n^* - x^*, x_n - x)_X \geq 0$$

for all $[x^*,x] \in A^{-1}$ and passing to the limit $n \to \infty$, we see that

$$(x_0^* - x^*, x_0 - x)_X \geq 0$$

for all $[x,x^*] \in A$, so $[x_0,x_0^*] \in A$ and $x_0^* \in R(A) = \overline{R(A)}$.

To show that $R(A)$ is open in X, we take $[x_0,y_0] \in A$ and $\rho > 0$ such that A^{-1} is bounded on the subset $\{ y \in R(A);\ |y - y_0|_X \leq \rho \}$. Since A is maximal monotone, the equation

$$Ax_\varepsilon + \varepsilon x_\varepsilon - \varepsilon x_0 \ni y$$

has a solution $x_\varepsilon \in \mathrm{dom}(A)$, for any $\varepsilon > 0$. We have

$$(x_\varepsilon - x_0, y_\varepsilon - y_0)_X \geq 0, \quad y_\varepsilon = y - \varepsilon x_\varepsilon + \varepsilon x_0 .$$

We take $y \in X$ with $|y - y_0|_X \leq \rho/2$ and the above inequality implies that

$$\varepsilon |x_\varepsilon - x_0|_X \leq |y - y_0|_X \leq \rho/2, \quad \forall \varepsilon > 0.$$

Then $|y_\varepsilon - y_0| \leq \rho$ and the boundedness properties of A yields that $\{x_\varepsilon\}$ is bounded in X. Since $y_\varepsilon \to y$ strongly in X and $x_\varepsilon \to x$ weakly on a subsequence in X, we get $[x,y] \in A$ in virtue of the maximal monotonicity of A and the set $\{ y \in X;\ |y - y_0|_X \leq \rho/2 \} \subset R(A)$ which ends the proof.

By Thm.2.3 one may define the mappings:

$$J_\lambda : X \to X, \quad J_\lambda x = (I + \lambda A)^{-1} x, \quad \lambda > 0,$$

$$A_\lambda : X \to X, \quad A_\lambda x = \lambda^{-1}(I - J_\lambda)x, \quad \lambda > 0.$$

They are called the resolvent, respectively the Yosida approximation of the maximal monotone operator A. The following properties are valid:

Theorem 2.8. Let $A : X \to X$ be maximal monotone. Then:

i) $\lim_{\lambda \to 0} J_\lambda x = \mathrm{Proj}_{\overline{\mathrm{dom}(A)}}(x), \quad x \in X;$

ii) A_λ is maximal monotone and Lipschitzian of constant $1/\lambda$ on X.

iii) For every $x \in \mathrm{dom}(A)$, we have:

$$|A_\lambda x|_X \leq |A^0 x|_X, \qquad A_\lambda x \to A^0 x, \ \lambda \to 0$$

where $A^0 x = \mathrm{Proj}_{Ax} 0$,

iv) $A_\lambda x \in A J_\lambda x, \quad x \in X.$

Since the proof is quite lengthy we omit it and we refer to Brezis [21].

Now, we are able to state an important refinement of Proposition 2.1, iii):

Theorem 2.9. If $\lambda_n \to 0$, $x_n \to x$ weakly in X, $A_{\lambda_n} x_n \to y$ weakly in X and moreover

$$\limsup_{n,m}(x_n - x_m, A_{\lambda_n}x_n - A_{\lambda_m}x_m)_X \leq 0,$$

then $[x,y] \in A$ and

$$\lim_{n,m\to\infty} (x_n - x_m, A_{\lambda_n}x_n - A_{\lambda_m}x_m)_X = 0.$$

Proof

We have:

$$(x_n - x_m, A_{\lambda_n}x_n - A_{\lambda_m}x_m)_X = (J_{\lambda_n}x_n - J_{\lambda_m}x_m, AJ_{\lambda_n}x_n - AJ_{\lambda_m}x_m)_X + $$
$$+ (\lambda_n A_{\lambda_n}x_n - \lambda_m A_{\lambda_m}x_m, A_{\lambda_n}x_n - A_{\lambda_m}x_m)_X .$$

Denoting $J_{\lambda_n}x_n = \tilde{x}_n$, since $\{A_{\lambda_n}x_n\}$ is bounded in X, we get that $\tilde{x}_n \to x$ weakly in X and

$$\lim(\tilde{x}_n - \tilde{x}_m, A\tilde{x}_n - A\tilde{x}_m)_X = 0$$

by the above identity and the monotonicity of A.

We may assume that on a subsequence n_k, $(\tilde{x}_{n_k}, A\tilde{x}_{n_k})_X \to \mu$. Then

$$0 = \lim_{n_l\to\infty} [\lim_{n_k\to\infty} (\tilde{x}_{n_l} - \tilde{x}_{n_k}, A\tilde{x}_{n_l} - A\tilde{x}_{n_k})_X] = 2\mu - 2(x,y)_X$$

so $\mu = (x,y) = \lim_{n\to\infty} (x_n, Ax_n)_X$. For every $[a,b] \in A$, we have

$$(\tilde{x}_n - a, A\tilde{x}_n - b)_X \geq 0$$

and, passing to the limit, we see that

$$(x - a, y - b)_X \geq 0 \quad \forall [a,b] \in A,$$

i.e. $y \in Ax$.

Generally, it is possible that the sum of two maximal monotone operators, A + B, is not maximal monotone since, for instance, its domain may be void.

Theorem 2.10. Let A and B be maximal monotone operators in X x X such that $\operatorname{int}\operatorname{dom}(A) \cap \operatorname{dom}(B) \neq \emptyset$. Then A + B is maximal monotone in X x X.

Proof

By P 2.2 and Thm 2.3, the equation

(x) $\qquad x + A_\mu x + B_\lambda x = y$

has a unique solution denoted x_λ^μ for any $y \in X$. Here $\lambda > 0$, $\mu > 0$, A_μ and B_λ are the Yosida approximations of A, B and they are hemicontinuous. First, we keep the index μ fixed and we denote shortly x_λ, A instead of x_λ^μ, A_μ.

Without loss of generality, we may assume that $0 \in \mathrm{int}(\mathrm{dom}(A)) \cap \mathrm{dom}(B)$, $0 \in A0$, $0 \in B0$. Then (x) shows that $\{x_\lambda\}$ is bounded in X.

By Thm 2.6, there are $\rho > 0$, $M > 0$ such that

$$|x^*|_X \le M, \quad \forall x^* \in \bigcup_{|x|_X \le \rho} Ax .$$

We define $y_\lambda = \frac{\rho}{2}(y - B_\lambda x_\lambda - x_\lambda)/|y - B_\lambda x_\lambda - x_\lambda|_X$ and we notice that $|y_\lambda|_X \le \frac{\rho}{2}$, so $|Ay_\lambda|_X \le M$.

Using the monotonicity of A, we get

$$0 \le (x_\lambda - y_\lambda, Ax_\lambda - Ay_\lambda)_X = (x_\lambda, Ax_\lambda)_X + (y_\lambda, Ay_\lambda)_X - (y_\lambda, Ax_\lambda)_X - (x_\lambda, Ay_\lambda)_X .$$

Then

$$\frac{\rho}{2}|Ax_\lambda|_X = (y_\lambda, Ax_\lambda)_X \le \text{constant}$$

where we also use that

$$(x_\lambda, Ax_\lambda)_X = -|x_\lambda|_X^2 - (B_\lambda x_\lambda, x_\lambda)_X + (y, x_\lambda)_X \le (y, x_\lambda)_X .$$

Thus, we have shown that $\{x_\lambda\}$, $\{B_\lambda x_\lambda\}$, $\{Ax_\lambda\}$ are bounded subsets of X and we may assume that $x_\lambda \to x_0$, $B_\lambda x_\lambda \to x_1$, $Ax_\lambda \to x_2$ weakly in X, on a subsequence. Again, by the monotonicity of A, we have

$$0 \ge (x_\lambda - x_\varepsilon, x_\lambda - x_\varepsilon)_X + (B_\lambda x_\lambda - B_\varepsilon x_\varepsilon, x_\lambda - x_\varepsilon)_X ,$$

therefore

$$\limsup_{\lambda,\varepsilon \to 0} (B_\lambda x_\lambda - B_\varepsilon x_\varepsilon, x_\lambda - x_\varepsilon)_X \le 0$$

and Thm 2.9 gives that $x_1 \in Bx_0$ and

$$\lim_{\lambda,\varepsilon \to 0} (B_\lambda x_\lambda - B_\varepsilon x_\varepsilon, x_\lambda - x_\varepsilon)_X = 0 .$$

Consequently

$$\limsup_{\lambda,\varepsilon \to 0} (x_\lambda - x_\varepsilon, x_\lambda + Ax_\lambda - x_\varepsilon - Ax_\varepsilon)_X = 0$$

and applying once more Thm 2.9 to the operator I + A, we see that $x_2 = A_\mu x_0$ (we recall that in fact A stands for A_μ up to now).

Assume that $0 \in \mathrm{intdom}(B)$, for the moment. The same argument as above allows to take $\mu \to 0$. Then, we see that the equation

$$x_\lambda + Ax_\lambda + B_\lambda x_\lambda \ni y$$

has a unique solution under the conditions of Thm 2.10. Since $0 \in \mathrm{intdom}(A)$, we can iterate the above proof and we obtain the desired conclusion by Thm 2.3.

Remark 2.11. There is a strong relationship between the maximal monotone operators and the nonlinear contraction semigroups. This idea will be stressed in the last section.

3. Generalized gradients

3.1. The subdifferential of a convex function

A remarkable class of monotone operators is given by the subdifferentials of convex functions.

Let X be a Banach space and $\varphi : X \to [-\infty, +\infty]$ be a convex function. Then, we define $\mathrm{dom}(\varphi) = \{x \in X; \varphi(x) < +\infty\}$ and φ is called proper if $\mathrm{dom}(\varphi) \neq \emptyset$ and $\varphi(x) > -\infty$ for all $x \in X$.

The closure of φ, denoted $\mathrm{cl}\varphi$, is the lower semicontinuous hull of φ:

$$(\mathrm{cl}\varphi)(x) = \liminf_{y \to x} \varphi(y),$$

if $\varphi(x) > -$ for all $x \in X$, and $\mathrm{cl}\varphi = -\infty$ otherwise. The convex function φ is said to be closed if $\varphi = \mathrm{cl}\varphi$. In particular, for a proper, convex function, the closedness is equivalent with the lower semicontinuity.

The subdifferential of the function φ, denoted $\partial\varphi$, is the (possibly multivalued) operator in $X \times X^*$, given by

$$\partial\varphi(x) = \{ w \in X^*; \varphi(x) - \varphi(v) \leq (w, x-v)_{X \times X^*}, \forall v \in X \} .$$

Obviously, x_0 is a minimum point for φ iff $0 \in \partial\varphi(x_0)$.

Theorem 3.1. In Banach spaces, the subdifferential of a lower semicontinuous, proper, convex function φ is a maximal monotone operator.

Proof

We give the argument for the case of reflexive Banach space X. For the general situation, we quote Rockafellar [98]. By the renorming theorem of Asplund [1], we may assume that X and X^* are strictly convex too.

The monotonicity of $\partial\varphi$ is obvious by the definition.

According to the extension of Thm 2.3 to Banach spaces, we have to show that the equation

$$Fx + \partial\varphi(x) \ni x^*$$