A Compendium on Nonlinear Ordinary Differential Equations

P. L. Sachdev

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A Wiley-Interscience Publication
JOHN WILEY & SONS, INC.
New York • Chichester • Brisbane • Toronto • Singapore • Weinheim

This text is printed on acid-free paper.

Library of Congress Cataloging in Publication Data:

Sachdev, P. L.

A compendium on nonlinear ordinary differential equations / P.L. Sachdev.

p. cm.

"A Wiley-Interscience publication."

Includes bibliographical references (p. –).

ISBN 0-471-53134-0 (cloth : acid-free paper)

1. Differential equations. I. Title.

QA372.S145 1997

515'.352—dc20 96-10204

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PREFACE

I could not have accomplished the present work if I had not received help and assistance from friends, colleagues, and students, to each of whom I am deeply indebted. Dr. Neelam Gupta accompanied me in my peregrinations to many university libraries around New Jersey and New York. My friend and co-worker Dr. Varughese Philip agreed to undertake the very difficult task of checking through thousands of equations, and accomplished it with cheer and patience. I was fortunate to have Dr. Eric Lord to assist at the crucial stage of organization of the material; his rich experience in mathematical physics was of immense value in devising a sensible scheme for ordering the equations. My student Dr. B. Mayil Vaganan came to my rescue whenever I found the physical and mental effort overpowering. Mr. M. Renugopal put the entire material of the compendium in LaTeX with great care. Others who assisted me include Dr. D. Palaniappan, Dr. K. R. C. Nair, Mr. Ch. Srinivasa Rao, and Dr. R. Sarathy, and my secretary, Mrs. Sandhya. I also wish to thank Professor V. G. Tikekar for his cooperation.

My wife, Rita, patiently listened to my grumblings when I encountered hurdles of various kinds, and bore the major share of the responsibility of bringing up our two sons, Deepak and Anurag.

I would wish to record my sincere thanks to successive editors at John Wiley—Maria Taylor, Kate Roach, Elizabeth Murphy, Maria Allegra, and Jessica Downey—for their patience and cooperation during the long period the present work was in progress.

I am much obliged to the authorities of the Indian Institute of Science, in particular the Curriculum Development Cell, Ministry of Human Resources Development, Government of India, for facilities and financial assistance.

P. L. SACHDEV

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September 1996

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INTRODUCTION

I do not recollect the mystical moment when the thought to prepare this compendium captured my imagination. It was not unnatural to conceive of it after I had completed my book *Nonlinear Ordinary Differential Equations and Their Applications*, since published by Marcel Dekker (1991), but I was only vaguely aware of the task ahead, and the enormity of the effort that would be demanded of me. As I plodded on, thumbing through literally hundreds of volumes of journals, hunting out useful, interesting, known and not-so-well-known equations, I realized that the volume I had envisioned as modest in size would grow and that it could never be exhaustive. However, the search continued and the material piled up. It took a relentless effort of five years to bring this work to its present stage of completion. I ransacked mathematics sections of many libraries: the Courant Institute, NYU, Rutgers, New Jersey Institute of Technology, St. Andrews (UK), TIFR and IIT, Bombay, and the Indian Institute of Science, Bangalore. Almost all journals in applied mathematics, physics, and engineering that deal with nonlinear phenomenon were browsed through. That explains the large size of the biblography and, of course, of the compendium itself. Yet it does not seem possible to exhaust all the equations, since new ones get added to the literature almost every day. The present collection should, nevertheless, meet the needs of a large majority of scientists, engineers, and applied mathematicians.

I, like several generations of engineers, scientists, and mathematicians, had often consulted the classic collection of E. Kamke (1959) to ascertain whether a nonlinear ordinary differential equation that I encountered in my research just might be there. Despite the fact that this work is in German, it has been immensely popular outside Germany. This book, I believe, has been my principal inspiration. Although it concerns mostly linear ordinary differential equations, there are a small number (266) of nonlinear equations. The corresponding collection in English by Murphy (1960) does not go much beyond Kamke.

Since Kamke's book, research in physics, mathematics, and engineering has spawned such a large number of new and interesting equations that a compendium on the subject has long been overdue. (This gap has been widened enormously by the appearance of that ubiquitous phenomenon called chaos.) I decided that the equations should be dealt with in the following way: If an equation can be solved quickly in a closed form, the steps to arrive at the solution should be given sparsely; otherwise, a summary of the asympototics, stability, existence, or numerical results may be appended to each equation. The contributions of various authors to the same equation are generally not combined; instead, their individual results are enunciated at the same location in the compendium. Equations in an abstract setting; stiff, delay, or stochastic equations; and functional or differential-difference equations have not been included. A large majority of equations in the compendium have arisen from physical models directly or through transformations such as the similarity reduction; initial and boundary conditions have been included wherever these have been imposed. Since the compendium is pretty large as it is, there is no scope for including notes for various classes of equations, as found in Kamke's book. However, the author's book *Nonlinear Ordinary Differential Equations and Their Applications* will be found useful for elements of qualitative analysis of nonlinear ordinary differential equations.

Categorization of such a large number of equations posed some difficulty; the book of Kamke again provided useful clues. A detailed rationale for the classification of equations is given in Section 1.1. However, the basic principle adopted was to list the equations in order of increasing complexity, as they would appear to the user: One should be able to look for an equation in the same manner as one would look for a word in the dictionary. For the convenience of the reader, the equations belonging to each order—second, third, fourth, fifth, and higher—have been divided into a large number of subclasses; equations in each subclass bear a subtitle and have been ordered as explained below. The scheme given here may not be perfect but should be adequate.

There are equations, such as those of Lorenz, Van der Pol, and Painlevé, to which entire shelves of literature have been devoted. We have given only a few recent results concerning these equations. On the other hand, the present compendium includes many less known equations that the reader will find interesting, curious, or useful.

The present volume may be helpful in many ways. It should be a standard reference in the sense that Kamke's book has been for (almost precisely) five decades: An engineer or scientist can look up an equation and may find a ready-made analysis with a reference thereto or a set of similar examples of known behavior. The compendium provides a wealth of examples for teaching and exposition. Moreover, since the equations have been drawn from diverse fields, the book should cross-fertilize thinking and analysis in unrelated fields.

1.1 Instructions to the User

The equations have been ordered like the words in a dictionary; once the user has familiarized himself or herself with the system of organization that we have employed, any equation in the book can be located rapidly. The user will then find concise information about explicit solutions and the method of solution, when these can be found, and indications of the nature and behavior of solutions where explicit solutions may not be found. Also, references to the source material are given. Equations of related type will be found in close proximity.

The basis of our ordering scheme is the following hierarchy of functions of a single variable $f(x)$:

Polynomials

Rational functions (P/Q where P and Q are polynomials)

Functions involving fractional powers [e.g., $x^{3/2}, (1+x^2)^{1/2}$, etc.]

Functions involving unspecified powers (e.g., x^p, where p is not necessarily an integer)

Trigonometical functions

Functions involving exp

Functions involving ln

Functions involving a modulus (e.g., $|x|$, etc.)

Unspecified functions

Polynomials $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ are ordered according to a dictionary ordering of the "word" $a_n a_{n-1} \cdots a_0$. The "alphabet" of coefficients has the following order:

(a) Zero

(b) Positive numbers, in increasing order

(c) Negative numbers, in decreasing order

(d) Unspecified constants

Having established the ordering scheme for functions of a single variable, consider the problem of ordering functions of two variables, $f(x, y)$. We regard them as functions of y, with parameters that can be numbers or functions of x. Thus we append to the alphabet of coefficients of a polynomical in y, coefficients that are

(e) Functions of x

Similarly, functions $f(x, y, y')$ are ordered by regarding them as functions of y', with parameters that can be numbers, functions of x, or

(f) Functions of x and y

Proceeding in this way, we arrive at a principle for *ordering* a list of differential equations. To render it completely rigorous would involve more detailed considerations (e.g., does $\sin^2 x$ precede or follow $\sin x^2$?). That was not our aim. Our aim was only the pragmatic one of devising a scheme to facilitate looking up a differential equation.

For further convenience, the work has been divided into sections, each containing equations of a general type. For example:

2.1 $y'' + f(y) = 0$

2.2 $y'' + g(x)h(y) = 0$

2.3 $y'' + f(x, y) = 0$

The rule at this level of organization is that *separable functions* $g(x)h(y)$ precede inseparable functions $f(x, y)$. The scheme is less formidable than might appear from its description. Further clarification can be obtained, if necessary, simply by looking through the pages to see how it looks in practice.

Ambiguities can arise because, to a certain extent, the form of a nonlinear equation is dependent on how one chooses to write it (for example, $xy'' + 2y' + xe^y = 0$ and $y'' + 2y'/x + e^y = 0$ would occupy different positions in our scheme of ordering). In general, we have chosen to leave each equation in the form in which it appeared in the source literature. In a few instances we have deviated from this principle when it would have led to the separation of an equation from the class of equations to which it naturally belongs. The user is advised to multiply or divide by an obvious factor if an equation is not found where expected, and to try again.

No linear equations, and no first order equations, have been dealt with. The major parts of the book have been labeled 2, 3, 4, 5, 6, N to denote second order, third order, ..., sixth and higher orders, and Nth order. At the end of each of these parts are collected *systems* of simultaneous equations, of the appropriate order.

The systems of each order have been grouped according to the highest order of differentiation that occurs (the independent variable is always either x or t, with differentiation denoted by a prime or a dot, respectively). Within each group, the ordering scheme is exactly like that for single equations, except that the dependent variable and its derivatives (y, y', etc.) are now vector quantities. To facilitate the use of this scheme to look up a system of equations, the systems have been written with all terms involving *derivatives* of the dependent variables on the left-hand side and all undifferentiated terms on the right.

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SECOND ORDER EQUATIONS

2.1 $y'' + f(y) = 0,\ f(y)$ polynomial

1. $$\boldsymbol{y'' + \mu y + y^2 - C = 0},$$

where μ and C are constants.

Put

$$\begin{aligned} y &= au(\theta) + u_1, u_1 = \left(\frac{1}{2}\right)\left\{(\mu^2 + 4C)^{1/2} - \mu\right\}, \theta = rx, \\ r &= (\mu + 2u_1)^{1/2}, a = \left(\frac{1}{2}\right) r^2 \end{aligned}$$

to obtain the canonical form of the given DE as

$$u'' + u + (1/2)u^2 = 0. \tag{1}$$

A first integral of (1) is

$$u'^2 = E - u^2 - (1/3)u^3, \tag{2}$$

where E is an integration constant related to the energy of the oscillator. The phase diagram of (2) is drawn. The closed orbit solutions, filling the region inside the loop of the separatrix, namely,

$$y_s = 3\,\mathrm{sech}^2(\theta/2) - 2, \quad E = 4/3, \tag{3}$$

are easily drawn. These closed (periodic) solutions tend asymptotically to the separatrix. Goldshtik, Hussain, and Shtern (1991)

2. $$\boldsymbol{y'' + 4y(y-1) - A = 0},$$

where A is a constant.

Multiplying the given DE by y' and integrating once, we have

$$\frac{3}{8}\left(\frac{dy}{dx}\right)^2 = -y^3 + \frac{3}{2}y^2 + 34Ay + B = (y - y_1)(y - y_2)(y_3 - y), \quad \text{say,} \tag{1}$$

where $y_1 < y_2 < y_3$. The general form of the cubic in (1) is drawn. It is evident that the only real solutions of this equation occur when $\left(\dfrac{dy}{dx}\right)^2 \geq 0$; thus the solution is either

$y = y_1$ or a nonlinear oscillation between y_2 and y_3. Two special cases of the general cubic are when $y_2 \to y_1$, giving the solitary wave, and when $y_2 \to y_3$, giving a discontinuity between y_1 and y_3. The solution can be written in terms of a Jacobian elliptic function $\mathrm{cn}(u; \nu)$ as

$$y = y_2 + (y_3 - y_2)\,\mathrm{cn}^2\left(x\left[(2/3)(y_3 - y_1)\right]^{1/2}; \nu\right),$$

where $\nu = (y_3 - y_2)/(y_3 - y_1)$. In the case $A = B = 0$, $\nu = 1$ and the solution becomes $y = (3/2)\,\mathrm{sech}^2 x$, the solitary wave. Johnson (1970)

3. $$\boldsymbol{y'' + 6y^2 + a = 0, \quad y(0) = y(\pi) = 0,}$$

where $a \in R$ is a parameter.

It is shown that for all $k \in N$, there exist values $a_k < \cdots < a_1$ such that for $a < a_k$, there exist k solutions of the given Sturm-Liouville problem. Ruf and Solimini (1986)

4. $$\boldsymbol{y'' - y - (1/4)y^2 = 0, \quad -1 < x < 1, y(-1) = y(1) = 1.}$$

Multiplication by y' and integration leads to

$$y'^2 = y^2 - y_0^2 + (1/6)(y^3 - y_0^3) \tag{1}$$

on using evenness of solution in x, and setting $y(0) = y_0$, as yet unknown. Separation of variables in (1) and integration yields a solution in terms of elliptic functions,

$$y(x) = y_0 - u_1\ \mathrm{sc}^2(\lambda^{-1}6^{-1/2}x; m),$$

where $u_{1,2} = -3(1 + y_0/2) \pm (1/2)[3(2 - y_0)(6 + y_0)]^{1/2}$, $\lambda^{-1} = (1/2)(-u_2)^{1/2}$, and $m = (u_1 - u_2)/(-u_2)$.

In order to satisfy the BC, we require that y_0 satisfies

$$y_0 = 1 + u_1\mathrm{sc}^2(\lambda^{-1}6^{-1/2}; m) \equiv f(y_0). \tag{2}$$

Numerical solution of (2) leads to $y_0 = 0.60850$. Hart (1980)

5. $$\boldsymbol{y'' - y - (1/4)y^2 = 0, \quad -1 < x < 1, y(-1) = 1, \ y(1) = 1.}$$

(a) The given DE arises in the determination of the stationary temperature distribution in a bar whose ends $x = \pm 1$ are kept at the temperature $y = 1$, etc. It is stated from previous results that the given problem has a unique solution such that $0 \le y(x) \le 1$ and the bounds are given by

$$c_1(x) \le y((x) \le c_2(x), b_1(x) \le y(x) \le b_2(x),$$

where

$$\begin{aligned} c_1(x) &= 1 - 0.35(1 - x^2) - 0.05(1 - x^4), \\ c_2(x) &= 1 - 0.35(1 - x^2) - 0.04(1 - x^4), \end{aligned}$$

and

$$\begin{aligned} b_1(x) &= \{\cosh(3/2)^{1/2}x\}/\cosh(3/2)^{1/2}, \\ b_2(x) &= \{\cosh x\}/\{\cosh 1\}. \end{aligned}$$

These bounds are considerably improved upon, by using the maximum principle. Numerical results are also given.

(b) The given DE is autonomous. Its general solution is easily found to be

$$y(x) = (x - x_0)\ln(x - x_0) + \alpha(x - x_0),$$

where x_0 and α are arbitrary constants. Anderson and Arthurs (1982), Kruskal and Clarkson (1992)

6. $$\boldsymbol{y'' - y - y^2 = 0.}$$

The given DE has $(y', y) = (0, 0)$ as a saddle point, while $(y', y) = (-1, 0)$ is a center. Writing $y = -1 + \epsilon z$, we find that $z'' + z = \epsilon z^2$, $z(0) = 1$, $z'(0) = 0$. Putting $wx = \theta$, $w^{-2} = 1 - \epsilon\eta(\epsilon)$, we have

$$z'' + z = \epsilon\{\eta z + (1 - \epsilon\eta)z^2\}, \quad ' \equiv \frac{d}{d\theta}. \tag{1}$$

An equivalent integral equation form of (1) is

$$\begin{aligned} z(\theta) &= \cos\theta + \epsilon\int_0^\theta \sin(\theta - \tau)[\eta z + (1 - \epsilon\eta)z^2]d\tau, \\ z'(\theta) &= -\sin\theta + \epsilon\int_0^\theta \cos(\theta - \tau)[\eta z + (1 - \epsilon\eta)z^2]d\tau. \end{aligned} \tag{2}$$

We know that for ϵ small enough the solutions of (2) are periodic. Expanding z and η in powers of ϵ and applying the periodicity condition, we find that $\eta(\epsilon) = 0.\epsilon - (5/6)\epsilon + \epsilon^2$, with $T = 2\pi/w, T_0 = 2\pi, T_1 = 0, T_2 = 5\pi/6$. Verhulst (1990), p.143

7. $$\boldsymbol{y'' - (3/2)y^2 = 0, \quad y(1) = 1,\ y'(1) = 1.}$$

Writing the given DE as $y'y'' - (3/2)y^2y' = 0$, integrating twice, and using the IC, we have $y = 4(3 - x)^{-2}$. Reddick (1949), p.192

8. $$\boldsymbol{y'' - 2y^2 + y = 0, \quad y(0) = 1,\ y'(0) = 0.}$$

Upper and lower bounds of the solution are found. Exact solution is $y = \sec x$. Eliason (1972)

9. $$\boldsymbol{y'' - 6y^2 = 0.}$$

Put $p(y) = y'$, $p\dfrac{dp}{dy} = 6y^2$, $y' = \pm(4y^3 - C_1)^{1/2}$, $y = P(x + C_2)$; P is the Weierstrass P-function with invariants $g_2 = 0, g_3 = C_1$; C_1 and C_2 are arbitrary constants. Kamke (1959), p.542; Murphy (1960), p.380

10. $$\boldsymbol{y'' - 6y^2 + (3/2)y_0^2 = 0, \quad y(x_0) = y_0,\ y'(x_0) = 0.}$$

The solution is easily found to be

$$y = y_0 + (3/2)y_0\tan^2[(3y_0/2)^{1/2}(x - x_0)].$$

Bartashevich (1973)

11. $$\boldsymbol{y'' - 6y^2 + 4y = 0.}$$

Multiply by y' and integrate:

$$y'^2 - 4y^3 + 4y^2 + C = 0, \quad x = \int \frac{dy}{\{4y^3 - 4y^2 - C\}^{1/2}} + C_1;$$

the solution is obtained in terms of elliptic functions. C and C_1 are arbitrary constants. For $C = 0$,

$$y = \frac{1}{\sin^2(x + C_1)}.$$

Kamke (1959), p.543

12. $$\boldsymbol{y'' - k^2 y^2 = 0,}$$

where k is a constant.

Putting $y' = p$, we have $y'' = p\dfrac{dp}{dy}$; we can integrate the resulting DE twice $\left(\text{using } p = \dfrac{dy}{dx}\right)$ to obtain

$$\int \frac{dy}{\{C_1 + (2/3)k^2 y^3\}^{1/2}} + C_2 = \pm x,$$

where C_1 and C_2 are arbitrary constants. The quadrature may be evaluated explicitly. Martin and Reissner (1958), p.77

13. $$\boldsymbol{\epsilon y'' - y^2 = 0, \quad \epsilon \ll 1, \quad y(0) = \alpha, \quad y(1) = \beta.}$$

Two boundary layer solutions are constructed. They are

$$y^i = \alpha/\{1 + (\alpha/6)^{1/2}\xi\}^2 + \cdots, \xi = x\epsilon^{-1/2}$$

and

$$y^I = \beta/\{1 + (\beta/6)^{1/2}\zeta\}^2 + \cdots, \zeta = (1 - x)/\epsilon^{1/2}$$

near $x = 0$ and $x = 1$, respectively. Nayfeh (1985), p.347

14. $$\boldsymbol{y'' + k^2 y^2 = 0,}$$

where k is a constant.

Writing $y' = p, y'' = p\dfrac{dp}{dy}$, we can integrate the resulting DE twice to obtain

$$\int \frac{dy}{[C_1 - (2/3)k^2 y^3]^{1/2}} + C_2 = \pm x,$$

where C_1 and C_2 are arbitrary constants. The quadrature may be evaluated explicitly. Martin and Reissner (1958), p.77

15. $$\epsilon y'' + y^2 - 1 = 0.$$

Multiplying by y' and integrating, we get

$$(1/2)\epsilon y'^2 + (1/3)y^3 - y = E,$$

where E is a constant. The solution of this equation is $y = a\zeta(x/\delta) + b$, where $\zeta(x)$ satisfies the equation

$$\zeta'^2 = 4(1-m^2)\zeta - 4(1-2m^2)\zeta^2 - 4m^2\zeta^3.$$

The coefficients a, b, and δ are related to m and ϵ according to

$$\begin{aligned} b &= (1-2m^2)/(m^4 - m^2 + 1)^{1/2}, \quad a = 3m^2/(m^4 - m^2 + 1)^{1/2}, \\ \delta &= [2\epsilon(m^4 - m^2 + 1)^{1/2}]^{1/2}, \quad b^3 - 3b - 3E = 0. \end{aligned}$$

More explicitly,

$$\begin{aligned} y &= [3m^2/(m^4 - m^2 + 1)^{1/2}]\ \mathrm{cn}^2[(x - x_1)/\{2\epsilon(m^4 - m^2 + 1)^{1/2}\}^{1/2}; m] \\ &\quad -(2m^2 - 1)/(m^4 - m^2 + 1)^{1/2}. \end{aligned}$$

Note that the period of $\mathrm{cn}(x)$ is $4K(m)$. The exact solution is related to asymptotic solutions, using a variational approach. Kath, Knessl, and Matkowsky (1987)

16. $$y'' - my^2 + n = 0, \quad y(0) = 0,\ y(1/2) = 1,\ y'(1/2) = 0,$$

where $m \in R$ is given. The constant n forms the eigenvalue.

The given problem occurs in viscous flow between parallel plates, and is solved in terms of elliptic functions by Tang (1967). Bespalova (1984)

17. $$y'' + y + \epsilon y^2 = 0,$$

$y(0) = A,\ y'(0) = 0$. Here ϵ is small.

A uniform perturbation solution to $O(\epsilon^3)$ is

$$\begin{aligned} y(\theta, \epsilon) &= A\cos\theta + \epsilon(A^2/6)(-3 + 2\cos\theta + \cos 2\theta) \\ &\quad +\epsilon^2(A^3/3)[-1 + (29/48)\cos\theta + (1/3)\cos 2\theta + (1/16)\cos 3\theta] + O(\epsilon^3), \end{aligned}$$

where $\theta = wx$ and $w(\epsilon) = 1 - \epsilon^2(5A^2/12) + O(\epsilon^3)$. Mickens (1981), p.39

18. $$y'' + y - \alpha y^2 = 0, \quad \alpha > 0.$$

The method of harmonic balance gives the approximate solution for small α as

$$y = c + a\cos wx,$$

where $w^2 = 1 - 2\alpha c$, $c = 1/(2\alpha) - \{[1/(2\alpha)\}(1 - 2\alpha^2 a^2)^{1/2}$, implying that the frequency-amplitude relation is

$$w = (1 - 2\alpha^2 a^2)^{1/4},\ a < 1/(2\alpha)^{1/2}.$$

Jordan and Smith (1977), p.120

19. $$y'' - \epsilon y^2 + y - \alpha = 0,$$

where α and ϵ are positive constants.

This is the relativistic equation for the central orbit of a planet, where $y = 1/r$, and r, x are polar coordinates of the planet in the plane of its motion. The term ϵy is the Einstein correction; ϵ and α are positive constants with ϵ very small. The equilibrium point $y = \{1 + (1 - 4\epsilon\alpha)^{1/2}\}/2$ is a center according to linear approximation. Jordan and Smith (1977), p.58

20. $$y'' + w^2 y - \epsilon y^2 = 0, \quad y(0) = 0,\ y(X) = L,$$

where w, ϵ, X, and L are constants.

Introducing $\lambda = y'(0)$ and putting $\zeta = wy/L$, we get a first integral of the given system as

$$\frac{d\zeta}{dx} = \pm w\ \alpha^{1/2} P(\zeta)^{1/2}, \tag{1}$$

where $\alpha = 2\epsilon\lambda/(3w^3), P(\zeta) = \zeta^3 - (1/\alpha)\zeta^2 + 1/\alpha$; λ is yet an unknown constant. Equation (1) may be solved subject to $\zeta(0) = 0, \zeta(X) = wL/\lambda$. Assume that $\alpha < 2/3(3)^{1/2}$, so that $P(\zeta) = 0$ has three real roots, ζ_1, ζ_2, and ζ_3 ($\zeta_1 > \zeta_2 > \zeta_3$). For $\lambda > 0$, the general solution can be expressed in terms of elliptic integrals as $wx = \alpha^{-1/2}\gamma^{-1}F(\phi/\beta)$, where $F(\phi/\beta)$ is the elliptic integral, $\gamma = (1/2)(\zeta_1 - \zeta_3)^{1/2}, \sin^2\beta = (\zeta_2 - \zeta_3)/(\zeta_1 - \zeta_3)$, and $\cos^2\phi = (\zeta_1 - \zeta_2)(\zeta_1 - \zeta_3)/\{(\zeta_2 - \zeta_3)(\zeta_1 - \zeta)\}$.

Similar results may be obtained when $\lambda < 0$. Explicit approximate solution is obtained when α is small. Numerical results are depicted. Becket (1980)

21. $$y'' + y - k(1 + \epsilon y^2) = 0,$$

where $\epsilon \ll 1$ and k are parameters.

The given DE is the orbital equation of a planet about the sun. The perturbation solution with the initial conditions $y(0) = k(e+1), y'(0) = 0$, where e is the eccentricity of the unperturbed orbit, is

$$y = k(e\cos x + 1) + \epsilon k^3\{ex\sin x + (1/2)e^2 + 1 - \{(e^2/3) + 1\}\cos x - (e^2/6)\cos 2x\} + O(\epsilon^2).$$

Jordan and Smith (1977), p.149

22. $$y'' - \epsilon y^2 + y - \mu = 0,$$

where ϵ (small) and μ are constants.

Writing $y = \mu + a\cos(\theta + \psi)$, $y' = -a\sin(\theta + \psi)$, and following the Lagrange method of averaging, we obtain, by substitution in the original DE, equations for a and ψ as functions of θ:

$$\begin{aligned} \frac{da}{d\theta} &= -\epsilon\{\mu + a\cos(\theta + \psi)\}^2 \sin(\theta + \psi), \\ \frac{d\psi}{d\theta} &= -(\epsilon/a)\{\mu + a\cos(\theta + \psi)\}^2 \cos(\theta + \psi). \end{aligned} \tag{1}$$

Averaging the RHSs of (1) with respect to θ over 0 to 2π gives

$$\frac{da}{d\theta} = 0, \quad \frac{d\psi}{d\theta} = -\epsilon\mu. \tag{2}$$