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Polyharmonic Boundary Value Problems

Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains

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Dedicated to our wives Chiara, Brigitte and Barbara.

The cover figure displays the solution of $\Delta^2 u = f$ in a rectangle with homogeneous Dirichlet boundary condition for a nonnegative function f with its support concentrated near a point on the left hand side. The dark part shows the region where $u < 0$.

Preface

Linear elliptic equations arise in several models describing various phenomena in the applied sciences, the most famous being the second order stationary heat equation or, equivalently, the membrane equation. For this intensively well-studied linear problem there are two main lines of results. The first line consists of existence and regularity results. Usually the solution exists and "gains two orders of differentiation" with respect to the source term. The second line contains comparison type results, namely the property that a positive source term implies that the solution is positive under suitable side constraints such as homogeneous Dirichlet boundary conditions. This property is often also called positivity preserving or, simply, maximum principle. These kinds of results hold for general second order elliptic problems, see the books by Gilbarg-Trudinger [198] and Protter-Weinberger [347]. For linear higher order elliptic problems the existence and regularity type results remain, as one may say, in their full generality whereas comparison type results may fail. Here and in the sequel "higher order" means order at least four.

Most interesting models, however, are nonlinear. By now, the theory of second order elliptic problems is quite well developed for semilinear, quasilinear and even for some fully nonlinear problems. If one looks closely at the tools being used in the proofs, then one finds that many results benefit in some way from the positivity preserving property. Techniques based on Harnack's inequality, De Giorgi-Nash-Moser's iteration, viscosity solutions etc., all use suitable versions of a maximum principle. This is a crucial distinction from higher order problems for which there is no obvious positivity preserving property. A further crucial tool related to the maximum principle and intensively used for second order problems is the truncation method, introduced by Stampacchia. This method is helpful in regularity theory, in properties of first order Sobolev spaces and in several geometric arguments, such as the moving planes technique which proves symmetry of solutions by reflection. Also the truncation (or reflection) method fails for higher order problems. For instance, the modulus of a function belonging to a second order Sobolev space may not belong to the same space. The failure of maximum principles and of truncation methods, one could say, are the main reasons why the theory of nonlinear higher order elliptic equations is by far less developed than the theory of analogous second order equations. On the other hand, in view of many applications and increasing interest especially in the last twenty years, one should try to develop new tools suitable for higher order problems involving polyharmonic operators.

The simple example of the two functions $x \mapsto \pm|x|^2$ shows that already for the biharmonic operator the standard maximum principle fails. Nevertheless, taking also boundary conditions into account could yield comparison or positivity preserving properties and indeed, in certain special situations, such behaviour can be observed. It is one goal of the present exposition to describe situations where positivity preserving properties hold true or fail, respectively, and to explain how we have tackled the main difficulties related to the lack of a general comparison principle. In the present book we also show that in many higher order problems positivity preserving "almost" occurs. By this we mean that the solution to a problem inherits the sign of the data, except for some small contribution. By the experience from the present work, we hope that suitable techniques may be developed in order to obtain results quite analogous to the second order situation. Many recent higher order results give support to this hope.

A further goal of the present book is to collect some of those problems, where the authors were particularly involved, and to explain by which new methods one can replace second order techniques. In particular, to overcome the failure of the maximum principle and of the truncation method several ad hoc ideas will be introduced.

Let us now explain in some detail the subjects we address within this book.

Linear Higher Order Elliptic Problems

The polyharmonic operator $(-\Delta)^m$ is the prototype of an elliptic operator L of order $2m$, but with respect to linear questions, much more general operators can be considered. A general theory for boundary value problems for linear elliptic operators L of order $2m$ was developed by Agmon-Douglis-Nirenberg [4–6, 148]. Although the material is quite technical, it turns out that the Schauder theory as well as the L^p-theory can be developed to a large extent analogously to second order equations. The only exception are maximum modulus estimates which, for linear higher order problems, are much more restrictive than for second order problems. We provide a summary of the main results which hopefully will prove to be sufficiently wide to be useful for anybody who needs to refer to linear estimates or existence results.

The main properties of higher – at least second – order Sobolev spaces will be recalled. Since more orders of differentiation are involved, several different equivalent norms are available in these spaces. A crucial role in the choice of the norm is played by the regularity of the boundary. For the second order Dirichlet problem for the Poisson equation a nonsmooth boundary leads to technical difficulties but, due to the maximum principle, there is an inherent stability so that, when approximating nonsmooth domains by smooth domains, one recovers most of the features for domains with smooth boundary, see [46]. For Neumann boundary conditions the situation is more complicated in domains with rather wild boundaries, although even for polygonal boundaries they do not show spectacular changes. For higher order boundary value problems some peculiar phenomena occur. For instance, the so-called Babuška and Sapondžyan paradoxes [28, 358] forces one to be very careful in the choice of the norm in second order Sobolev spaces since some boundary value problems strongly depend on the regularity of the boundary. This phenomenon and its consequences will be studied in some detail.

Positivity in Higher Order Elliptic Problems

As long as existence and regularity results are concerned, the theory of linear higher order problems is already quite well developed as explained above. This is no longer true as soon as qualitative properties of the solution related to the source term are investigated. For instance, if we consider the clamped plate equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

the "simplest question" seems to find out whether the positivity of the datum implies the positivity of the solution, Or, physically speaking,

does upwards pushing of a clamped plate yield upwards bending?

Equivalently, one may ask whether the corresponding Green function G is positive. In some special cases, the answer is "yes", while it is "no" in general. However, in numerical experiments, it appears very difficult to display the negative part and heuristically, one feels that the negative part of G – if present at all – is small in a suitable sense compared with the "dominating" positive part. We discuss not only the cases where one has positive Green functions and develop a perturbation theory of positivity, but we shall also discuss systematically under which conditions one may expect the negative part of the Green function to be small. We expect such smallness results to have some impact on future developments in the theory of non-linear higher order elliptic boundary value problems.

Boundary Conditions

For second order elliptic equations one usually extensively studies the case of Dirichlet boundary conditions because other boundary conditions do not exhibit too different behaviours. For the biharmonic equation $\Delta^2 u = f$ in a bounded domain of $\mathbb{R}^n$ it is not at all obvious which boundary condition would serve as a role model. Then a good approach is to focus on some boundary conditions that describe physically relevant situations. We consider a simplified energy functional and derive its Euler-Lagrange equation including the corresponding natural boundary conditions. We start with the linearised model for the beam. From a physical point of view, as long as the fourth order planar equation is considered, the most interesting seem to be not only the Dirichlet boundary conditions but also the Navier or

Steklov boundary conditions. The Dirichlet conditions correspond to the clamped plate model whereas Navier and Steklov conditions correspond to the hinged plate model, either by neglecting or considering the contribution of the curvature of the boundary. Each one of these boundary conditions requires the unknown function to vanish on the boundary, the difference being on the second boundary condition. These three boundary conditions have their own features and none of them may be thought to play the model role. We discuss all of them and emphasise their own peculiarities with respect to the comparison principles, to their variational formulation and to solvability of related nonlinear problems.

Eigenvalue Problems

For second order problems, such as the Dirichlet problem for the Laplace operator, one has not only the existence of infinitely many eigenvalues but also the simplicity and the one sign property of the first eigenfunction. For the biharmonic Dirichlet problem, this property is true in a ball but it is false in general. Again, a crucial role is played by the sign of the corresponding Green function. Concerning the isoperimetric properties of the first eigenvalue of the Dirichlet-Laplacian, the Faber-Krahn [162, 254, 255] result states that, among domains having the same finite volume it attains its minimum when the domain is a ball. A similar result was conjectured to hold for the biharmonic operator under homogeneous Dirichlet boundary conditions by Lord Rayleigh [351] in 1894. Although this statement has been proved only in domains of dimensions $n = 2, 3$, it is the common feeling that it should be true in any dimension. The minimisation of the first Steklov eigenvalue appears to be less obvious. And, indeed, we will see that a Faber-Krahn type result does not hold in this case.

Semilinear Equations

Among nonlinear problems for higher order elliptic equations one may just mention models for thin elastic plates, stationary surface diffusion flow, the Paneitz-Branson equation and the Willmore equation as frequently studied. In membrane biophysics the Willmore equation is also known as Helfrich model [228]. Moreover, several results concerning semilinear equations with power type nonlinear sources are also extremely useful in order to understand interesting phenomena in functional analysis such as the failure of compactness in the critical Sobolev embedding and in related inequalities.

One further motivation to study nonlinear higher order elliptic reaction-diffusion type equations like

$$(-\Delta)^m u = f(u) \tag{$*$}$$

in bounded domains is to understand whether the results available in the simplest case $m = 1$ can also be proved for any m, or whether the results for $m = 1$ are special, in particular as far as positivity and the use of maximum principles are concerned. The differential equation $(*)$ is complemented with suitable boundary conditions. As already mentioned above, if $m = n = 2$, equation $(*)$ may be considered as a nonlinear plate equation for plates subject to nonlinear feedback forces, one may think e.g. of suspension bridges. In this case, $(*)$ may also be interpreted as a reaction-diffusion equation, where the diffusion operator Δ^2 refers to (linearised) surface diffusion.

The first part of Chapter 7 is devoted to the proof of symmetry results for positive solutions to $(*)$ in the ball under Dirichlet boundary conditions. As already mentioned, truncation and reflection methods do not apply to higher order problems so that a suitable generalisation of the moving planes technique is needed here.

Equation $(*)$ deserves a particular attention when $f(u)$ has a power-type behaviour. In this case, a crucial role is played by the critical power $s = (n+2m)/(n-2m)$ which corresponds to the critical (Sobolev) exponent which appears whenever $n > 2m$. Indeed, subcritical problems in bounded domains enjoy compactness properties as a consequence of the Rellich-Kondrachov embedding theorem. But compactness is lacking when the critical growth is attained and by means of Pohožaev-type identities, this gives rise to many interesting phenomena. The existence theory can be developed similarly to the second order case $m = 1$ while it becomes immediately quite difficult to prove positivity or nonexistence of certain solutions. Nonexistence phenomena are related to so-called critical dimensions introduced by Pucci-Serrin [348, 349]. They formulated an interesting conjecture concerning these critical dimensions. We give a proof of a relaxed form of it in Chapter 7. We also give a functional analytic interpretation of these nonexistence results, which is reflected in the possibility of adding L^2–remainder terms in Sobolev inequalities with critical exponent and optimal constants. Moreover, the influence of topological and geometrical properties of Ω on the solvability of the equation is investigated. Also applications to conformal geometry, such as the Paneitz-Branson equation, involve the critical Sobolev exponent since the corresponding semilinear equation enjoys a conformal covariance property. In this context a key role is played by a fourth order curvature invariant, the so-called Q-curvature. Our book does not aim at giving an overview of this rapidly developing subject. For this purpose we refer to the monographs of Chang [89] and Druet-Hebey-Robert [149]. We want to put a spot on some special aspects of such kind of equations. First, we consider the question whether in suitable domains in euclidean space it is possible to change the euclidean background metric conformally into a metric which has strictly positive constant Q-curvature, while at the same time, certain geometric quantities vanish on the boundary. Secondly, we study a phenomenon of nonuniqueness of complete metrics in hyperbolic space, all being conformal to the Poincaré-metric and all having the same constant Q-curvature. This result is in strict contrast with the corresponding problem involving constant negative scalar curvature.

We conclude the discussion of semilinear elliptic problems with some observations on fourth order problems with supercritical growth. Corresponding second order results heavily rely on the use of maximum principles and constructions of many refined auxiliary functions having some sub- or supersolution property. Such techniques are not available at all for the fourth order problems. In symmetric situations, however, they could be replaced by different tools so that many of the results being well established for second order equations do indeed carry over to the fourth order ones.

A Dirichlet Problem for Willmore Surfaces of Revolution

A frame invariant modeling of elastic deformations of surfaces like thin plates or biological membranes gives rise to variational integrals involving curvature and area terms. A special case is the Willmore functional

$$\int_{\Gamma} \mathrm{H}^2\, d\omega,$$

which up to a boundary term is conformally invariant. Here H denotes the mean curvature of the surface Γ in $\mathbb{R}^3$. Critical points of this functional are called Willmore surfaces, the corresponding Euler-Lagrange equation is the so-called Willmore equation. It is quasilinear, of fourth order and elliptic. While a number of beautiful results have been recently found for closed surfaces, see e.g. [35, 156, 263–265, 372], only little is known so far about boundary value problems since the difficulties mentioned earlier being typical for fourth order problems due to a lack of maximum principles add here to the difficulty that the ellipticity of the equation is not uniform. The latter reflects the geometric nature of the equation and gives rise e.g. to the problem that minimising sequences for the Willmore functional are in general not bounded in the Sobolev space H^2. In this book we confine ourselves to a very special situation, namely the Dirichlet problem for symmetric Willmore surfaces of revolution. Here, by means of some refined geometric constructions, we succeed in considering minimising sequences of the Willmore functional subject to Dirichlet boundary conditions and with suitable additional C^1-properties thereby gaining weak H^2- and strong C^1-compactness. We expect the theory of boundary value problems for Willmore surfaces to develop rapidly and consider this chapter as one contribution to outline directions of possible future research in quasilinear geometric fourth order equations.

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Contents

Preface ... vii

Acknowledgements ... xiii

1 Models of Higher Order ... 1
1.1 Classical Problems from Elasticity ... 1
1.1.1 The Static Loading of a Slender Beam ... 2
1.1.2 The Kirchhoff-Love Model for a Thin Plate ... 5
1.1.3 Decomposition into Second Order Systems ... 8
1.2 The Boggio-Hadamard Conjecture for a Clamped Plate ... 9
1.3 The First Eigenvalue ... 12
1.3.1 The Dirichlet Eigenvalue Problem ... 13
1.3.2 An Eigenvalue Problem for a Buckled Plate ... 14
1.3.3 A Steklov Eigenvalue Problem ... 15
1.4 Paradoxes for the Hinged Plate ... 16
1.4.1 Sapondžyan's Paradox by Concave Corners ... 17
1.4.2 The Babuška Paradox ... 17
1.5 Paneitz-Branson Type Equations ... 18
1.6 Critical Growth Polyharmonic Model Problems ... 21
1.7 Qualitative Properties of Solutions to Semilinear Problems ... 22
1.8 Willmore Surfaces ... 23

2 Linear Problems ... 27
2.1 Polyharmonic Operators ... 27
2.2 Higher Order Sobolev Spaces ... 29
2.2.1 Definitions and Basic Properties ... 29
2.2.2 Embedding Theorems ... 32
2.3 Boundary Conditions ... 33
2.4 Hilbert Space Theory ... 36
2.4.1 Normal Boundary Conditions and Green's Formula ... 36
2.4.2 Homogeneous Boundary Value Problems ... 38
2.4.3 Inhomogeneous Boundary Value Problems ... 42

2.5 Regularity Results and A Priori Estimates 44
2.5.1 Schauder Theory 44
2.5.2 L^p-Theory 46
2.5.3 The Miranda-Agmon Maximum Modulus Estimates 48
2.6 Green's Function and Boggio's Formula 50
2.7 The Space $H^2 \cap H_0^1$ and the Sapondžyan-Babuška Paradoxes 52
2.8 Bibliographical Notes 59

3 Eigenvalue Problems 61
3.1 Dirichlet Eigenvalues 62
3.1.1 A Generalised Kreĭn-Rutman Result 62
3.1.2 Decomposition with Respect to Dual Cones 63
3.1.3 Positivity of the First Eigenfunction 68
3.1.4 Symmetrisation and Talenti's Comparison Principle 71
3.1.5 The Rayleigh Conjecture for the Clamped Plate 73
3.2 Buckling Load of a Clamped Plate 77
3.3 Steklov Eigenvalues 82
3.3.1 The Steklov Spectrum 83
3.3.2 Minimisation of the First Eigenvalue 90
3.4 Bibliographical Notes 96

4 Kernel Estimates 99
4.1 Consequences of Boggio's Formula 99
4.2 Kernel Estimates in the Ball 101
4.2.1 Direct Green Function Estimates 101
4.2.2 A 3-G-Type Theorem 110
4.3 Estimates for the Steklov Problem 116
4.4 General Properties of the Green Functions 122
4.4.1 Regularity of the Biharmonic Green Function 122
4.4.2 Preliminary Estimates for the Green Function 123
4.5 Uniform Green Functions Estimates in $C^{4,\gamma}$-Families of Domains 125
4.5.1 Uniform Global Estimates Without Boundary Terms 126
4.5.2 Uniform Global Estimates Including Boundary Terms 135
4.5.3 Convergence of the Green Function in Domain Approximations 141
4.6 Weighted Estimates for the Dirichlet Problem 142
4.7 Bibliographical Notes 146

5 Positivity and Lower Order Perturbations 147
5.1 A Positivity Result for Dirichlet Problems in the Ball 148
5.2 The Role of Positive Boundary Data 153
5.2.1 The Highest Order Dirichlet Datum 153
5.2.2 Also Nonzero Lower Order Boundary Terms 157
5.3 Local Maximum Principles for Higher Order Differential Inequalities 165

5.4 Steklov Boundary Conditions 167
5.4.1 Positivity Preserving 168
5.4.2 Positivity of the Operators Involved in the Steklov Problem 175
5.4.3 Relation Between Hilbert and Schauder Setting 178
5.5 Bibliographical Notes 185

6 Dominance of Positivity in Linear Equations 187
6.1 Highest Order Perturbations in Two Dimensions 188
6.1.1 Domain Perturbations 190
6.1.2 Perturbations of the Principal Part 193
6.2 Small Negative Part of Biharmonic Green's Functions in Two Dimensions 197
6.2.1 The Biharmonic Green Function on the Limaçons de Pascal 197
6.2.2 Filling Smooth Domains with Perturbed Limaçons 201
6.3 Regions of Positivity in Arbitrary Domains in Higher Dimensions ... 208
6.3.1 The Biharmonic Operator 210
6.3.2 Extensions to Polyharmonic Operators 215
6.4 Small Negative Part of Biharmonic Green's Functions in Higher Dimensions 216
6.4.1 Bounds for the Negative Part 217
6.4.2 A Blow-Up Procedure 218
6.5 Domain Perturbations in Higher Dimensions 224
6.6 Bibliographical Notes 226

7 Semilinear Problems 227
7.1 A Gidas-Ni-Nirenberg Type Symmetry Result 229
7.1.1 Green Function Inequalities 231
7.1.2 The Moving Plane Argument 234
7.2 A Brief Overview of Subcritical Problems 238
7.2.1 Regularity for At Most Critical Growth Problems 238
7.2.2 Existence 241
7.2.3 Positivity and Symmetry 243
7.3 The Hilbertian Critical Embedding 244
7.4 The Pohožaev Identity for Critical Growth Problems 254
7.5 Critical Growth Dirichlet Problems 260
7.5.1 Nonexistence Results 260
7.5.2 Existence Results for Linearly Perturbed Equations 263
7.5.3 Nontrivial Solutions Beyond the First Eigenvalue 271
7.6 Critical Growth Navier Problems 281
7.7 Critical Growth Steklov Problems 285
7.8 Optimal Sobolev Inequalities with Remainder Terms 295

7.9 Critical Growth Problems in Geometrically Complicated Domains ..300
7.9.1 Existence Results in Domains with Nontrivial Topology 301
7.9.2 Existence Results in Contractible Domains....302
7.9.3 Energy of Nodal Solutions 304
7.9.4 The Deformation Argument 307
7.9.5 A Struwe-Type Compactness Result 311
7.10 The Conformally Covariant Paneitz Equation in Hyperbolic Space 319
7.10.1 Infinitely Many Complete Radial Conformal Metrics with the Same Q-Curvature 319
7.10.2 Existence and Negative Scalar Curvature 320
7.10.3 Completeness of the Conformal Metric 326
7.11 Fourth Order Equations with Supercritical Terms 335
7.11.1 An Autonomous System 339
7.11.2 Regular Minimal Solutions 347
7.11.3 Characterisation of Singular Solutions 352
7.11.4 Stability of the Minimal Regular Solution 357
7.11.5 Existence and Uniqueness of a Singular Solution 360
7.12 Bibliographical Notes 366

8 Willmore Surfaces of Revolution 371
8.1 An Existence Result 371
8.2 Geometric Background 372
8.2.1 Geometric Quantities for Surfaces of Revolution 372
8.2.2 Surfaces of Revolution as Elastic Curves in the Hyperbolic Half Plane 376
8.3 Minimisation of the Willmore Functional 381
8.3.1 An Upper Bound for the Optimal Energy 382
8.3.2 Monotonicity of the Optimal Energy 383
8.3.3 Properties of Minimising Sequences 387
8.3.4 Attainment of the Minimal Energy 389
8.4 Bibliographical Notes 392

Notations 393

References 397

Author–Index 415

Subject–Index 419

Chapter 1
Models of Higher Order

The goal of this chapter is to explain in some detail which models and equations are considered in this book and to provide some background information and comments on the interplay between the various problems. Our motivation arises on the one hand from equations in continuum mechanics, biophysics or differential geometry and on the other hand from basic questions in the theory of partial differential equations.

In Section 1.1, after providing a few historical and bibliographical facts, we recall the derivation of several linear boundary value problems for the plate equation. In Section 1.8 we come back to this issue of modeling thin elastic plates where the full nonlinear differential geometric expressions are taken into account. As a particular case we concentrate on the Willmore functional, which models the pure bending energy in terms of the squared mean curvature of the elastic surface. The other sections are mainly devoted to outlining the contents of the present book. In Sections 1.2-1.4 we introduce some basic and still partially open questions concerning qualitative properties of solutions of various linear boundary value problems for the linear plate equation and related eigenvalue problems. Particular emphasis is laid on positivity and – more generally – "almost positivity" issues. A significant part of the present book is devoted to semilinear problems involving the biharmonic or polyharmonic operator as principal part. Section 1.5 gives some geometric background and motivation, while in Sections 1.6 and 1.7 semilinear problems are put into a context of contributing to a theory of nonlinear higher order problems.

1.1 Classical Problems from Elasticity

Around 1800 the physicist Chladni was touring Europe and showing, among other things, the nodal line patterns of vibrating plates. Jacob Bernoulli II tried to model these vibrations by the fourth order operator $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ [54]. His model was not accepted, since it is not rotationally symmetric and it failed to reproduce the nodal line patterns of Chladni. The first use of Δ^2 for the modeling of an elastic plate is attributed to a correction of Lagrange of a manuscript by Sophie Germain from 1811.

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For historical details we refer to [79,250,325,398]. For a more elaborate history of the biharmonic problem and the relation with elasticity from an engineering point of view one may consult a survey of Meleshko [300]. This last paper also contains a large bibliography so far as the mechanical engineers are interested. Mathematically interesting questions came up around 1900 when Almansi [8,9], Boggio [62,63] and Hadamard [222,223] addressed existence and positivity questions.

In order to have physically meaningful and mathematically well-posed problems the plate equation $\Delta^2 u = f$ has to be complemented with prescribing a suitable set of boundary data. The most commonly studied boundary value problems for second order elliptic equations are named Dirichlet, Neumann and Robin. These three types appear since they have a physical meaning. For fourth order differential equations such as the plate equation the variety of possible boundary conditions is much larger. We will shortly address some of those that are physically relevant. Most of this book will be focussed on the so-called clamped case which is again referred to by the name of Dirichlet. An early derivation of appropriate boundary conditions can be found in a paper by Friedrichs [173]. See also [58,141]. The following derivation is taken from [388].

1.1.1 *The Static Loading of a Slender Beam*

If $u(x)$ denotes the deviation from the equilibrium of the idealised one-dimensional beam at the point x and $p(x)$ is the density of the lateral load at x, then the elastic energy stored in the bending beam due to the deformation consists of terms that can be described by bending and by stretching. This stretching occurs when the horizontal position of the beam is fixed at both endpoints. Assuming that the elastic force is proportional to the increase of length, the potential energy density for the beam fixed at height 0 at the endpoints a and b would be

$$J_{st}(u) = \int_a^b \left(\sqrt{1+u'(x)^2} - 1 \right) dx.$$

For a string one neglects the bending and, by adding a force density p, one finds

$$J(u) = \int_a^b \left(\sqrt{1+u'(x)^2} - 1 - p(x)u(x) \right) dx.$$

For a thin beam one assumes that the energy density stored by bending the beam is proportional to the square of the curvature:

$$J_{sb}(u) = \int_a^b \frac{u''(x)^2}{\left(1+u'(x)^2\right)^3} \sqrt{1+u'(x)^2}\, dx. \tag{1.1}$$

Formula (1.1) for J_{sb} highlights the curvature and the arclength. A two-dimensional analogue of this functional is the Willmore functional, which is discussed below in