Elementary Mathematical Modeling

Functions and Graphs

Mary Ellen Davis
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ELEMENTARY MATHEMATICAL MODELING
FUNCTIONS AND GRAPHS

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PREFACE

This textbook is for an entry-level college mathematics course at the same academic level as college algebra, but intended for students who are not necessarily preparing for subsequent courses in calculus. Our approach is based on the exploitation of graphing-calculator technology to engage students in concrete modeling applications of mathematics. The mathematical ideas of the course center on functions and their graphs—ranging from linear functions and polynomials to exponential and trigonometric functions—that we hope will seem familiar and friendly to students who complete the course.

BRIEF DESCRIPTION

Specifically, this textbook presents an introduction to mathematical modeling based on the use of elementary functions to describe and explore real-world data and phenomena. It demonstrates graphical, numerical, symbolic, and verbal approaches to the investigation of data, functions, equations, and models. We emphasize interesting applications of elementary mathematics together with the ability to construct useful mathematical models, to analyze them critically, and to communicate quantitative concepts effectively. In short, this is a textbook for

- A graphing technology intensive course that is
- An alternative to the standard college algebra course, and is
- Solidly based on functions, graphs, and data modeling.

RATIONALE FOR A NEW COURSE

The content of the traditional college algebra course is defined largely by the paper-and-pencil skills (mainly symbolic manipulation) that are needed by students whose curricula point them towards a subsequent calculus course. However, many of the students in a typical college algebra course are not really headed for calculus or never make it there. For too many of these students, college algebra consists of revisiting the skills and concepts, either mastered or not, which were "covered" in several previous mathematics courses. This experience leaves students with little enhancement of the quantitative skills they most need for their subsequent studies. It is a missed opportunity for them to begin college with a useful mathematics course that is interesting both to students and to instructors, and which offers a solid chance for progress and success.

There is wide agreement on the need for an alternative new approach to fill this void. Both the NCTM's *Principles and Standards for School Mathematics* and AMATYC's *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus* recommend that mathematics courses teach students to reason mathematically, to model real-world situations, and to make use of appropriate technologies. We offer this as an appropriate textbook for such a course. The evolution of these materials began with a web site that was originally developed (starting in 1996) to support University of Georgia students taking pilot sections of this new course. About two thousand students have now used preliminary versions of the textbook. Many of these students have reacted with enthusiasm belying their typical lack of success in prior mathematical experiences. We hope this apparent success and satisfaction will carry over to the students who use this published textbook.

PURPOSE AND OBJECTIVES

The primary objective of this new course is the development of the quantitative literacy and savvy that college graduates need to function effectively in society and workplace. The course exploits technology and real-world applications to motivate necessary skill development and the ability to reason and communicate mathematically, to use elementary mathematics to solve applied problems, and to make connections between mathematics and the surrounding world.

With a flavor combining functions and graphs with data modeling, the course is based largely on the use of graphing calculator methods in lieu of traditional symbolic manipulations to solve both familiar and nonstandard problems. The focus of the course is "mathematical modeling" and the use of elementary mathematics—numbers and measurement, algebra, geometry, and data exploration—to investigate real-world problems and questions.

As an alternative to the standard college algebra course—though at the same academic level—this course is intended for students who are not necessarily headed for calculus-based curricula, but still need a solid quantitative foundation both for subsequent studies and for life as educated citizens and workers. Graphing technology enables these students to experience the power of mathematics and to enjoy success in solving interesting and significant problems (an experience that they all too rarely enjoy in traditional college algebra courses).

CONTENT AND ORGANIZATION

The book consists of the following chapters:

1. Linear Functions and Models
2. Quadratic Functions and Models
3. Natural Growth Models
4. Exponential and Logarithmic Models
5. Polynomial Models and Linear Systems
6. Trigonometric Models
7. Bounded Growth Models
8. Optimization

Most of these chapters fit a single pattern:

- The first section is a low-key introduction to the type of function to be used as a mathematical model throughout the chapter.
- The next section or two illustrate real-world applications of this new function.
- The final section of the chapter is devoted to data modeling using this type of function.

For example, in Chapter 3:

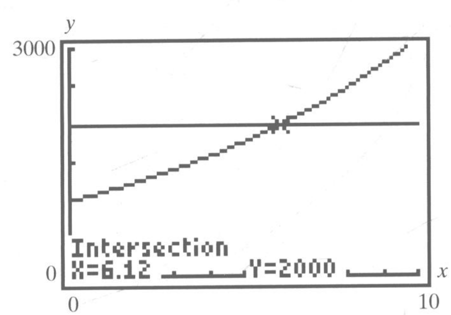
- Section 3.1 begins with the concept of constant-percentage growth and exponential functions of the form

$$A(t) = A_0(1 + r)^t \tag{1}$$

  that model the amount at time t in an account with annual growth rate r and initial investment A_0 (e.g., $r = 0.12$ for 12% annual growth).
- Sections 3.2 and 3.3 illustrate a wide variety of real-world problems, such as the prototype question of how long is required for an initial investment of \$1000 to double if it grows at a 12% interest rate compounded annually. This question calls for solving the equation

$$1000 \times 1.12^t = 2000. \tag{2}$$

  In a traditional course one might use logarithms, but here we can simply use a graphing calculator's intersection-finding capability to locate the intersection of the two graphs $y = 1000 \times 1.12^x$ and $y = 2000$, and thereby see that this takes about 6.12 years:



  The conceptual core of this problem is the realization that Equation (2) must be solved, rather than the particular method used to do this. Essentially all students can successfully use graphing technology for this purpose, including many who might fail to use logarithms correctly. And graphical solution is a much more widely applicable approach than the use of logarithms.
- Section 3.4 (the final section of Chapter 3) is devoted to the problem of choosing the values of the parameters A_0 and r in Equation (1) so that the resulting natural growth function best fits given data. Again, this is done using available graphing calculator facilities.

As one chapter follows another, each developing the same theme with a new class of functions, the story of mathematical modeling comes to be a familiar one. Contrast this with the traditional algebra course, which many students perceive as a sequence of unrelated topics. In pilot sections, students selected mainly because of low placement test scores have exhibited success rates not frequently seen in traditional college algebra courses.

APPLICATIONS

The use of real-world applications is vital for making mathematics more lively and interesting to students and for helping them to see connections with the world around them. This text begins with a simple but very real example of a function that is taken from the menu of the popular Waffle House Restaurant chain. Throughout the book, combinations of simulated data and real (sourced) data are used in examples and problems. Contrived data are often more effective in providing clear and straightforward explanations of new concepts. Once concepts have been introduced, more robust data-based applications help to make these concepts more concrete and to underscore their connections to real life. Each chapter concludes with a data-based Activity that may be assigned as an individual or group exercise.

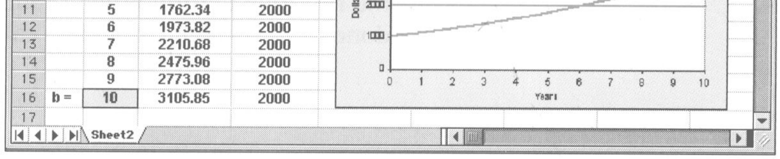
TECHNOLOGY

Graphing Calculators

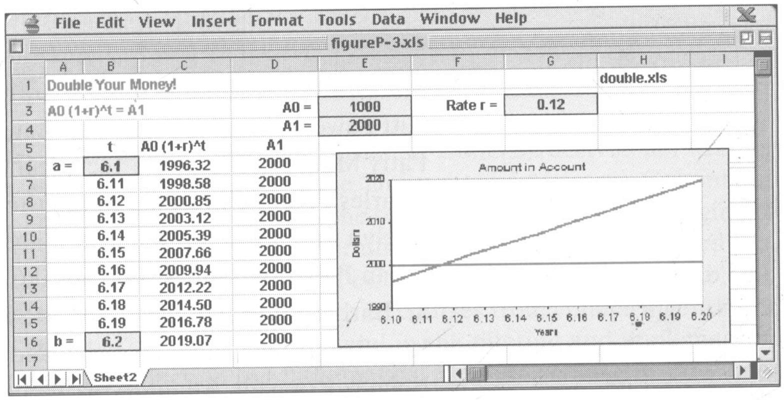
This book assumes no technology other than student use of graphing calculators. Indeed, TI-83 syntax and calculator screens are seen throughout the text. However, it is entirely possible to mix different graphing calculators in the same class—and we have done so in pilot sections—if the instructor is willing and prepared to discuss all of them when necessary.

Spreadsheets

Perhaps it is worth remarking that in each historical era, real-world practitioners (if not teachers and academics) have always assimilated rapidly the best available technology to assist their mathematical computations—whether it be a sandbox, an abacus, a slide rule, or a desktop calculating machine. The principal computational instrument used in today's workplace is the spreadsheet (rather than the graphing calculator). We have therefore explored the use of spreadsheets to augment and reinforce graphing techniques initially introduced with calculators.



For instance, the preceding spreadsheet image illustrates the solution of Equation (2). Each of the five shaded cells is a "live cell" whose numerical content can be changed by the student. The accompanying graph and table then change dynamically. We can therefore "zoom in" by table and by graph, vividly and simultaneously. Thus if we enter the new endpoint values $a = 6.1$ and $b = 6.2$, the chart instantly changes as indicated in the spreadsheet image below, where we see the approximate solution $t \approx 6.12$.



Where feasible, student use of spreadsheets can not only develop valuable familiarity with the modern world's predominant calculating technology, but also reinforce understanding of "solution by zooming" through comparison of several variants (ranging from tables to graphs). In particular, solution of the same problems using both graphing calculators and spreadsheets emphasizes that it is the general mathematical approach which we study, rather than the specific technology used to implement this approach.

The Web site *www.prenhall.com/davis* will provide a suite of spreadsheets that can be downloaded by students and instructors for use as described.

SUPPLEMENTS

Student Solutions Manual (ISBN 0-13-030771-8)

Written by Mary Ellen Davis and Henry Edwards

The Student Solutions Manual provides detailed solutions for all of the odd-numbered exercises in the text. Because these solutions were prepared by the textbook authors themselves, consistency in language and method of solution is guaranteed.

Instructor's Manual (ISBN 0-13-030760-2)

Written by Mary Ellen Davis and Henry Edwards

Solutions to all of the even-numbered problems appear in the Instructor's Solutions Manual. Sample exams and resource suggestions are also included.

Companion Web Site: www.prenhall.com/davis

In addition to the suite of spreadsheets described above, the Davis/Edwards Web Site will include additional resources including: On-line Graphing Calculator Help for various calculators, valuable resources for learning algebra, and a syllabus-building component for teachers.

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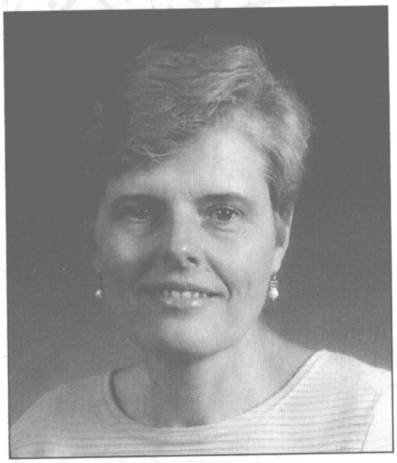

Mary Ellen Davis Georgia Perimeter College, received her Master of Arts degree in mathematics from the University of Missouri–Columbia in 1976. She has taught mathematics at the secondary level and at Georgia State University and the University of Birmingham (England). She joined the mathematics department at Georgia Perimeter College (then DeKalb College) in 1991 and has taught a wide range of courses from college algebra to calculus and statistics. She was instrumental in the piloting and implementing of the college's Introduction to Mathematical Modeling course in 1998. She was selected as a Georgia Governor's Teaching Fellow in 1996, and in 1999 received a GPC Distance Education Fellowship to develop web-based materials for applied calculus.

C. Henry Edwards (Ph. D. University of Tennessee) Emeritus professor of mathematics at the University of Georgia, Edwards recently retired after 40 years of undergraduate classroom teaching at the universities of Tennessee, Wisconsin, and Georgia. Although respected for his diverse research interests, Edwards' first love has always remained teaching. Throughout his teaching career he has received numerous college- and university-wide teaching awards, including the University of Georgia's *honoratus* medal in 1983 and its Josiah Meigs award in 1991. In 1997, Edwards was the first university-level faculty recipient of the Georgia Board of Regents newly-instituted state-wide award for teaching excellence.

A prolific author, Edwards is co-author of well-known calculus and differential equations textbooks and has written a book on the history of mathematics, in addition to several instructional computer manuals. During the 1990s, Edwards has worked on three NSF-supported projects that fostered a better integration of technology into the mathematics curriculum. The last three years of his long teaching career were devoted principally to the development of a new technology-intensive entry-level mathematics course on which this new textbook is based. Additional information is provided on his web page www.math.uga.edu/~hedwards.

CONTENTS

ELEMENTARY MATHEMATICAL MODELING

FUNCTIONS AND GRAPHS

1 LINEAR FUNCTIONS AND MODELS

How hot is it *really?* Anyone who's ever visited Phoenix surely has heard the adage that the dry heat of the desert doesn't feel as hot as the temperature would suggest. Indeed, when it is 100° Fahrenheit in Phoenix, with a relative humidity of 10%, it "feels like" it is only 95°—a full five degrees cooler than the thermometer says!

The heat index is a number calculated by meteorologists to measure the effect of humidity on apparent temperature as felt by the human body. It is designed to report how hot you actually feel at a particular combination of air temperature and relative humidity. The chart in Fig. 1.0.1 shows the heat index for various levels of relative humidity at a fixed air temperature of 85°F.

The chart indicates that when the humidity is 10% the apparent temperature is only 80°F. However, if the humidity is 90% then a temperature of 85°F seems like 102°F. Thus the humidity has a big effect on how hot we actually feel.

We can see from the chart that as the relative humidity increases from 0% to 60%, the apparent temperature increases uniformly from 78°F to 90°F. Notice that this portion of the chart appears to be a straight line. But after we reach a relative humidity of 60%, the apparent temperature thereafter increases more rapidly as the relative humidity increases. That is, this portion of the chart appears to curve upward.

All around us we see quantities that seem, like apparent temperature and relative humidity, to be related to each other in some systematic way. We often describe this dependence of one quantity on another by using the word *function.* Thus, we might say that your weight is (ideally) a function of your

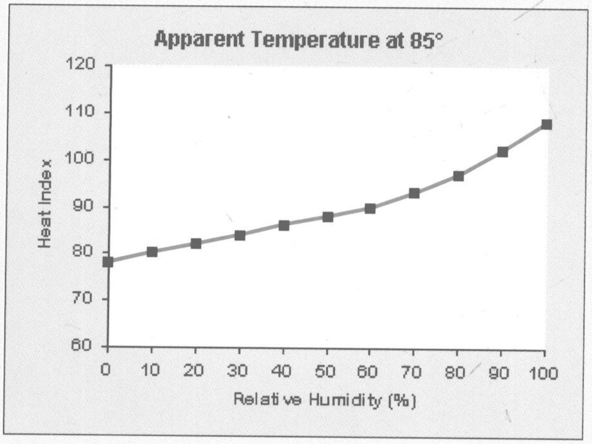

Figure 1.0.1 Source: National Weather Service, Buffalo, New York.

height; that your grade on a history test is a function of how long you studied; or that your income is a function of your education level. In each case we are noting a dependence of the first quantity on the second.

In this chapter, we will introduce the mathematical concept of a function, and we will study those functions that graph as straight lines.

1.1 FUNCTIONS AND MATHEMATICAL MODELING

The menu at a Waffle House restaurant gives the price of a breakfast of eggs, toast, jelly, and grits based on the number of eggs the customer orders, as illustrated in the following table of 1999 prices.

Number of Eggs	Price of Breakfast
1	\$1.55
2	\$2.00
3	\$2.35
4	\$2.65

Most applications of mathematics involve the use of numbers or *variables* to describe real-world quantities. In the aforementioned situation, suppose we let n represent the number of eggs ordered, and p the price of the breakfast. Then the table describes a relationship between n and p. This relationship is an example of a mathematical *function,* because for each number n of eggs ordered there is a corresponding price p charged for the breakfast.

The key concept here is that there is only **one price** associated with each number of eggs. The menu, in effect, provides a rule for determining price: If you know how many eggs were ordered, you know the price of the breakfast. If you and your friend each ordered a one-egg breakfast, then you would expect to be charged the same amount. Indeed, if one of you were charged more than \$1.55 for your breakfast, then you surely would complain!

Definition: Function
A **function** f defined on a collection D of numbers is a rule that assigns to each number x in D a specific number $f(x)$.

The number $f(x)$—we say "f of x"—is called the *value* of the function f at the number x. The "rule" mentioned in the definition can be specified by a table, by a formula or graph, or even by a verbal description that tells how the value $f(x)$ is found when the number x is given. (While we frequently use x to denote the variable and f to denote the function, we can use any other letters that we like, or that seem more natural in a particular situation.)

Example 1

For the Waffle House function indicated previously, the set D is the collection of all possible numbers of eggs—the set of numbers 1, 2, 3, and 4—which we denote here by n (rather than x). Given a number n in the first column of the table, we simply look in the second column to find the corresponding price $p(n)$. For instance, $p(3) = 2.35$ because \$2.35 is the price "assigned" to a three-egg breakfast. ◩

Example 2

Rather than keeping a running total, the Bureau of the Public Debt uses a daily accounting method to calculate the public debt of the United States. At the end of each day, approximately 50 different agencies (such as Federal Reserve Banks) report certain financial information to the Bureau. At around 11:30 A.M. EST (Eastern Standard Time) the next morning, the accounting system produces a figure for the public debt, accurate to the nearest penny, for the previous day. Thus, to each date is assigned an official public debt amount, and we therefore can say that the debt depends on the day chosen. In other words, the debt is a function of the date. In particular, if g is the public debt function, then

$$g(9/30/1987) = \$2,350,276,890,953.00,$$

while

$$g(2/2/1998) = \$5,483,592,532,096.82.$$

(Thus the national debt increased by more than 3 trillion dollars between 1987 and 1998.) The rule defining the function g is the complicated process—described verbally previously—by which the Bureau of the Public Debt determines the national debt each day. ◩

Example 3

The chart in Fig. 1.1.1 illustrates the population of St. Louis, Missouri, for the census years 1950 through 1990. This plot describes (at least approximately) a function P that is defined for each of the years 1950, 1960, 1970, 1980, and 1990—because for each of these years, there is *exactly one* population figure indi-

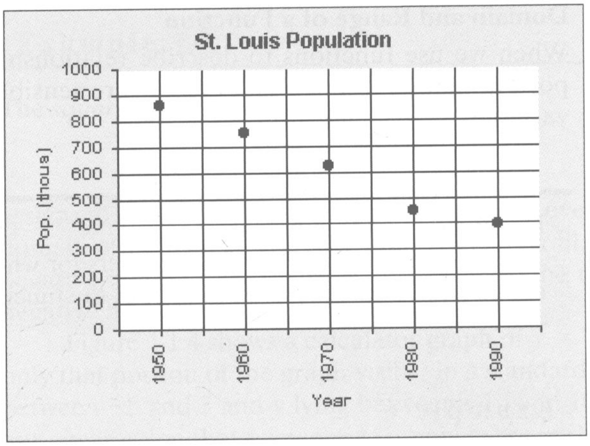

Figure 1.1.1 Source: Bureau of the Census.

cated by an appropriate dot. For instance, it appears that the 1990 population of St. Louis was approximately $P(1990) = 400$ thousand people.

Example 4

If a certain savings account earns 4% simple interest per year, then the interest I earned each year is given in terms of the amount A in the account by the formula $I = 0.04A$. This formula is a rule specifying I as a function of A. Thus, if the amount in the account is 500 dollars, then the interest earned that year is $(0.04)(500) = 20$ dollars.

Examples 1–4 illustrate functions that describe relationships between real-world variables. The key to using mathematics to analyze a real-world situation often is the recognition of such relationships among the variables that describe the situation. The following example illustrates functions defined by formulas that may be familiar from your previous studies in mathematics and science.

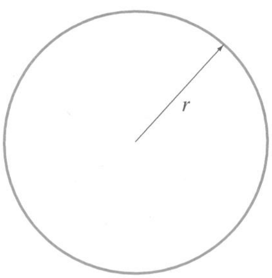

Figure 1.1.2 The area A of a circle is a function of its radius r.

Example 5

- The area A of a circle of radius r (Fig. 1.1.2) is given by

$$A = \pi r^2 \qquad \text{(where } \pi \approx 3.1416\text{)}.$$

We often write such a formula in *function notation* as $A(r) = \pi r^2$ to indicate that the area depends on the radius.

- If a rock is dropped from atop a high tower (Fig. 1.1.3) and the acceleration of gravity is 32 ft/sec^2, then its (downward) velocity v after t seconds and the distance d it has fallen are given by

$$v(t) = 32t \quad \text{and} \quad d(t) = 16t^2.$$

- If the temperature of a 3-gram sample of carbon dioxide is 27°, then its volume V in liters is given by

$$V(p) = \frac{168}{p}$$

where p is the pressure of the gas in atmospheres.

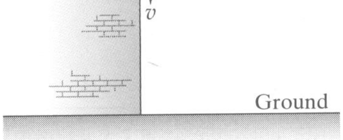

Figure 1.1.3 The distance d the rock has fallen is a function of time t.