Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Jan-Cees van der Meer

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Preface

The research laid down in these notes began several years ago with some questions about a particular bifurcation of periodic solutions in the restricted problem of three bodies at the equilibrium L_4 . This particular bifurcation takes place when, for the linearized system, the equilibrium L_4 changes from stable to unstable. This kind of bifurcation is called a Hamiltonian Hopf bifurcation.

During the research it became apparent that new methods had to be developed and that existing methods had to be reformulated in order to deal with the specific nature of the problem. The development of these methods together with their application to the Hamiltonian Hopf bifurcation is the main topic of these notes. As a result a complete description is obtained of the bifurcation of periodic solutions for the generic case of the Hamiltonian Hopf bifurcation.

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Jan-Cees van der Meer

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CONTENTS

Introduction

In this monograph the main topic is the study of periodic solutions of a family of Hamiltonian systems of two degrees of freedom near an equilibrium as the family passes through resonance. We concentrate on the Hamiltonian Hopf bifurcation, that is, the passage through nonsemisimple 1 : -1 resonance. The nonsemisimple 1 : -1 resonance distinguishes itself from the other resonances in two ways: first, at the resonance the linearized system is nonsemisimple having two equal pairs of purely imaginary eigenvalues; second, when passing through resonance, the equilibrium point changes from elliptic to hyperbolic type. Although we have concentrated on a specific example, many parts of the theory developed here have much wider applications, especially to other two degree of freedom resonances.

The approach we take can be divided into four main parts: (1) formal normal form theory; (2) equivariant theory of stability of maps applied to energy-momentum maps to derive standard systems; (3) geometric treatment of the standard system; and (4) Moser-Weinstein reduction to extend the results to nonintegrable systems.

The general formal normal form theory for Hamiltonian systems is treated first (chapter 2). Here we focus on the normalization of the Hamiltonian function. We do not restrict ourselves to systems of two degrees of freedom. The theory is illustrated by the classical examples of Hamiltonian systems with purely imaginary eigenvalues.

If we consider the Hamiltonian $H = H_2 + H_3 + \dots$ of a Hamiltonian system of two degrees of freedom then we may normalize H with respect to H_2 up to arbitrary order. Truncation then gives an integrable system provided that the semisimple part S of the homogeneous quadratic term H_2 is nonzero. If we consider the one parameter group $\mathcal{S}$ generated by the flow of the Hamiltonian vector field X_S corresponding to the integral S then the truncated normalized Hamiltonian $\tilde{H}$ is S-invariant. For the system corresponding to $\tilde{H}$ we consider the S-invariant energy-momentum map $\tilde{H} \times S$. To this energy-momentum map we apply the equivariant theory of stability of maps. For the case of the nonsemisimple 1 : -1 resonance we show that this energy-momentum mapping is finitely determined. The integrable system corresponding to the determining jet is called a standard system for the resonance (chapter 3).

In applying the theory of stability of maps, we drop the condition that the transformations used be symplectic. However much of the qualitative behaviour of the standard system can be translated back to the original system, especially the behaviour of periodic solutions. Using the theory of unfoldings we are able to study the behaviour of families of periodic solutions during the passage through resonance. The unfolded standard system for the nonsemisimple 1 : -1 resonance is studied in detail (chapter 4).

Finally we use some ideas of Weinstein and Moser to show how the periodic solutions of an arbitrary family of nonintegrable systems passing through resonance correspond to the periodic solutions of a family of integrable systems to which we may apply the preceding theory. This reduction from a nonintegrable to an integrable system in the search for periodic solutions is called the Moser-Weinstein reduction (chapter 5). The final result is a complete description of the behaviour of periodic solutions of short period in the generic case of the Hamiltonian Hopf bifurcation. Such a bifurcation appears in the restricted problem of three bodies at an equilateral equilibrium when the mass parameter passes through the critical value of Routh. It is this problem in the restricted problem of three bodies which inspired this study. Although combining all known results gave a fairly good description of the behaviour of periodic solutions (partially based on numerical results), a complete treatment and proof was nowhere to be found.

Because of the special properties of the nonsemisimple 1 : -1 resonance a new approach had to be followed. Many of the methods which had been succesfully used for the other resonances did not apply in this case. For the methods developed the nonsemisimple 1 : -1 resonance is the simplest example in the hierarchy of resonances, especially if one considers the computation of co-dimension and the geometric treatment of the standard system. The application of the normal form theory is a bit more complicated but the resulting normal form takes a simpler form than in the other resonances.

The chapters are organized as follows. In the first chapter preliminaries from the theory of Hamiltonian systems are treated. In the second chapter one finds the theory of Hamiltonian normal forms. In the third chapter the equivariant theory of stability of maps is applied to energy-momentum maps invariant with respect to a symplectic S^1-action. The fourth chapter deals with the geometry of the standard integrable system for the Hamiltonian Hopf bifurcation. Chapters 2,3 and 4 can be read independently. In chapter five the Moser-Weinstein reduction is applied to the Hamiltonian Hopf bifurcation. Together with the results of chapters 2,3 and 4 this leads to the main theorem (ch. 5, sect. 3). In chapter six we show how the theory applies to the restricted problem of three bodies. We conclude with a discussion of the known results concerning the nonsemisimple 1 : -1 resonance.

Chapter I

Preliminaries

0. Introduction

In this first chapter we will give a review of some facts from Hamiltonian mechanics which are fundamental to what follows. Emphasis is laid upon the relation between the symplectic geometric and the Lie algebraic features induced by the presence of the symplectic form. Also linear Hamiltonian systems are treated because they are basic for many features of and techniques used for nonlinear systems.

Most definitions and theorems are stated without proof. For the proofs and a more detailed treatment of the theory we refer to the textbooks of Arnold [1978] and Abraham and Marsden [1978].

1. Hamiltonian systems

Consider the following system of ordinary first order differential equations on $\mathbb{R}^{2n}$

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H(q,p)}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H(q,p)}{\partial q_i} \quad ; \quad i = 1,\dots,n. \end{aligned} \tag{1.1}$$

where $H(q,p)$ is some real valued function on $\mathbb{R}^{2n}$, at least once differentiable. We call (1.1) a *Hamiltonian system of differential equations*. The function H in (1.1) is called a *Hamiltonian function*. The right hand side of (1.1) can be written as

$$X_H(q,p) = J.dH(q,p) \tag{1.2}$$

with

(1.3) $$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0_n \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. We call X_H the *Hamiltonian vector field associated to the Hamiltonian* H.

The above is the classical definition of Hamiltonian systems on $\mathbb{R}^{2n}$. This can also be obtained from the following more general differential geometric approach defining a Hamiltonian system on a manifold M.

Let ω be a two-form on M. We say that ω is *nondegenerate* if ω is a nondegenerate bilinear form on the tangent space of M at m for each $m \in M$. If there is a nondegenerate two form on M then M has even dimension. Furthermore we say that a two-form ω is *closed* if $d\omega = 0$ where d is the exterior derivative.

1.4. <u>DEFINITION</u>. A *symplectic form* ω on a manifold M is a nondegenerate closed two-form ω on M. A *symplectic manifold* (M,ω) is a manifold M together with a symplectic form ω on M.

1.5. <u>DEFINITION</u>. Let (M,ω) be a symplectic manifold and $H : M \to \mathbb{R}$ a C^k-function, $k \geq 1$. The vector field X_H determined by $\omega(X_H, Y) = dH.Y$ is called the *Hamiltonian vector field with Hamiltonian function* H. We call (M,ω,H) a *Hamiltonian system. We will suppose* H *to be* C^∞ *in the following.*

The following theorem shows that locally definition 1.5. is equivalent to the classical one.

1.6. <u>THEOREM</u> (Darboux). Let (M,ω) be a symplectic manifold then there is a chart (U,φ) at $m \in M$ such that $\varphi(m) = 0$ and with $\varphi(u) = (x_1,\ldots,x_n,y_1,\ldots,y_n)$ we have $\omega|U = \sum_{i=1}^{n} dx_i \wedge dy_i$.

The charts (U,φ) are called *symplectic charts* the coordinates x_i, y_i are called *symplectic* or *canonical coordinates*. Notice that if x_i, y_i are canonical coordinates then $X_H(x_i,y_i) = (\frac{\partial H}{\partial y_i}, -\frac{\partial H}{\partial x_i}) = J.dH$ with J given by (1.3).

We now define the notion of a flow of a Hamiltonian vector field together with some related notions. The flow in fact gives us the simultaneous motion in time of all points of M along the trajectories of the vector field.

1.7. DEFINITION. Let $\gamma(t)$ be a curve in $\mathbb{R}^{2n}$. We say that γ is an *integral curve for* X_H if $\frac{d\gamma}{dt} = X_H(\gamma)$, that is, if Hamilton's equations hold. Let (M,ω,H) be a Hamiltonian system. The map $\varphi : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R} \times \mathbb{R}^{2n}$ such that $\varphi_m : t \mapsto \varphi(t,m)$ is an integral curve at m for each $m \in M$ is called the *flow of* X_H. The curve $t \mapsto \varphi(t,m)$ is called the *maximal integral curve of* X_H *at* m or *the orbit of* X_H *through* m. The picture of M decomposed into orbits is known as the *phase portrait of* X_H.

Notice that the set $\{\varphi_t | t \in \mathbb{R}\}$ is a one-parameter group of diffeomorphisms of M, if every maximal integral curve is defined for all $\mathbb{R}$.

1.8. DEFINITION. A C^∞-map $\psi: (M,\omega) \to (M,\omega)$ is *symplectic* or *canonical* if $\psi^*\omega = \omega$.

Here $\psi^*\omega$ is the *pull-back* of ω under ψ defined by $\psi^*\omega(m)(e_1,e_2,\ldots,e_{2n}) = \omega(\psi(m))(d\psi(m)e_1,\ldots,d\psi(m)e_{2n})$. For $F \in C^\infty(M,\mathbb{R})$ $\psi^*F = F\circ\psi$. We have $\psi^*X_H = X_{\psi^*H} = X_{H\circ\psi}$, if ψ is symplectic.

It is clear that φ_t, $t \in \mathbb{R}$, defined by the flow φ of the Hamiltonian vectorfield X_H is a symplectic diffeomorphism. Note that $H(\gamma(t))$ is constant in t along integral curves $\gamma(t)$ of X_H. This corresponds to conservation of energy.

The following definitions show how the presence of a symplectic form on M induces a Lie algebra structure on $C^\infty(M,\mathbb{R})$ in a natural way.

1.9. DEFINITION. Let (M,ω) be a symplectic manifold and let $F,G \in C^\infty(M,\mathbb{R})$ The *Poisson bracket of* F *and* G is

$$\{G,F\} = \omega(X_F,X_G)$$

In canonical coordinates

$$\{G,F\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial y_i} - \frac{\partial F}{\partial y_i}\frac{\partial G}{\partial x_i}\right)$$

Notice that we have

$$\{G,F\} = dF.X_G$$

It follows directly that F is constant along orbits of X_G (or G constant along orbits of X_F) if and only if $\{F,G\} = 0$.$\{F,F\} = 0$ corresponds to conservation of energy for the system (M,ω,F).

1.10. DEFINITION. $F \in C^\infty(M,\mathbb{R})$ is an *integral for the system* (M,ω,H) if $\{H,F\} = 0$.

The notion of Poisson bracket allows us to consider the real vector space $C^\infty(M,\mathbb{R})$ as a Lie algebra.

1.11. DEFINITION. A *Lie algebra* is a vector space V with a bilinear operation [,] satisfying:
$[X,X] = 0$ for all $X \in V$ and
$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (the "Jacobi identity")
for all $X,Y,Z \in V$.

It is now easily checked that $C^\infty(M,\mathbb{R})$ considered as a real vector space together with the Poisson bracket is a Lie algebra. Notice that the fact that ω is a closed two-form is essential in order to establish the Jacobi identity.

If ψ is symplectic then $\psi^*\{F,G\} = \{\psi^*F,\psi^*G\}$ for all $F,G \in C^\infty(M,\mathbb{R})$, that is, ψ^* is a Lie algebra isomomorphism. In fact the converse also holds.

On the space of Hamiltonian vector fields one has the usual Lie bracket of vector fields making this space into a Lie algebra. We have

$$(1.12) \qquad [X_F,X_G] = X_{\{F,G\}}$$

We call $[X_F,X_G]$ the *Lie bracket of* X_F *and* X_G. The Hamiltonian vector fields with Lie bracket form a Lie subalgebra of the Lie algebra of all vector fields. Notice that this Lie subalgebra is homomorphic to the Lie algebra $C^\infty(M,\mathbb{R})$ with Poisson bracket.

Returning to the Lie algebra $C^\infty(M,\mathbb{R})$ we may define for each $F \in C^\infty(M,\mathbb{R})$ the map $ad(F) : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ by $ad(F)(G) = \{F,G\}$. The map $ad : F \to ad(F)$ is called the adjoint representation of $C^\infty(M,\mathbb{R})$. Because of the Jacobi identity $ad(F)\{G,H\} = \{ad(F)(G), H\} + \{G,ad(F)(H)\}$ for each $G,H \in C^\infty(M,\mathbb{R})$, $ad(F)$ is an inner derivation of $C^\infty(M,\mathbb{R})$ for each $F \in C^\infty(M,\mathbb{R})$.

1.13. REMARK. In the special case when M is a vector space we speak of a *symplectic vector space*. As before we may introduce the notions of Hamiltonian function, Hamiltonian vector field and Poisson bracket. Here we have global coordinates so these notions can be defined in terms of coordinates.

1.14. REMARK. Notice that our definition of Poisson bracket (definition 1.9.) differs from the one in Abraham and Marsden [1978] by a minus sign. This is done in order to obtain formula (1.12.) which gives rise to the Lie algebra homomorphism between Hamiltonian functions and Hamiltonian vector fields. Our definition agrees with Arnold [1978] if one takes into account that his standard symplectic form differs from ours by a minus sign.

According to Dugas [1950] our conventions agree with those of Poisson. Studying other literature it becomes clear that historically both conventions for Poisson bracket have been used.

2. Symmetry, integrability and reduction

In this section we will restrict ourselves to $\mathbb{R}^{2n}$ with coordinates $(x,y) = (x_1,\dots,x_n,y_1,\dots,y_n)$ and standard symplectic form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. Then $(\mathbb{R}^{2n},\omega)$ is a symplectic vector space as well as a symplectic manifold and $C^\infty(\mathbb{R}^{2n},\mathbb{R})$ with Poisson bracket as given by definition 1.9. is a Lie algebra.

In the following proposition some statements about Lie series are collected. The proofs are straight forward and left to the reader as an exercise. We define the *Lie series* $\exp \operatorname{ad}(H) = \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{ad}^n(H)$.

1.15. PROPOSITION. (i) $\operatorname{ad}(H)(x,y) = X_H(x,y)$ where $\operatorname{ad}(H)(x,y)$ is defined as $(\operatorname{ad}(H)x_1,\dots,\operatorname{ad}(H)y_n)$.

(ii) $\exp(t\,\operatorname{ad}(H))(x,y)$ is the flow of X_H

(iii) $(F\circ\exp(\operatorname{ad}(H)))(x,y) = \exp(\operatorname{ad}(H))(F(x,y))$

(iv) $\exp(\operatorname{ad}(H))$ and $\exp(\operatorname{ad}(F))$ commute if and only if $\{H,F\}$ is constant.

In the last statement of the above proposition one might replace the condition $\{H,F\}$ is constant by $[X_H,X_F] = 0$ where $[\,,]$ is the Lie bracket given by (1.12). Proposition 1.15.(iv) is then equivalent to the statement that two Hamiltonian vector fields commute in the Lie algebra of vector fields if and only if their flows commute.

Now recall that the space of all maps $\operatorname{ad}(F)$, $F \in C^\infty(\mathbb{R}^{2n},\mathbb{R})$ is a Lie algebra with bracket $[\operatorname{ad}(F),\operatorname{ad}(G)] = \operatorname{ad}(\{F,G\})$. Therefore we have a group A generated by the $\exp(\mathrm{ad}(F))$, $F \in C^\infty(\mathbb{R}^{2n},\mathbb{R})$. Each one-parameter group $\exp(t\,\mathrm{ad}(F))$, $t \in \mathbb{R}$ forms a one-parameter subgroup of A. On the symplectic space $\mathbb{R}^{2n}$ each one-parameter group of diffeomorphisms is the flow of a Hamiltonian vector field. Thus we have found all one-parameter subgroups of A because each generator of A is a symplectic diffeomorphism which is the time one flow of a Hamiltonian vector field by prop. 1.15.(ii).

On $(\mathbb{R}^{2n}, \omega)$ let $\Phi : G \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a *symplectic action* of the Lie group G on $\mathbb{R}^{2n}$, that is, for each $\varphi \in G$ the map $\Phi_\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} : x \mapsto \Phi(\varphi x)$ is symplectic. In a natural way the action Φ induces an action $\Psi : G \times C^\infty(\mathbb{R}^{2n},\mathbb{R}) \to C^\infty(\mathbb{R}^{2n},\mathbb{R}) : (\varphi,H) \mapsto H\circ\Phi_\varphi$ of G on $C^\infty(\mathbb{R}^{2n},\mathbb{R})$. In the following we will write $\varphi.H$ for $\Psi(\varphi,H)$.

1.16. <u>DEFINITION</u>. A Lie group G acting symplectically on $\mathbb{R}^{2n}$ is a *symmetry group for the system* $(\mathbb{R}^{2n},\omega,H)$ if $\varphi.H = H$ for all $\varphi \in G$.

Proposition 1.15. gives

1.17. <u>PROPOSITION</u>. If F is an integral for the system $(\mathbb{R}^{2n},\omega,H)$ then the one-parameter group $\exp(t\,\mathrm{ad}(F))$, $t \in \mathbb{R}$, given by the flow of F, is a symmetry group for $(\mathbb{R}^{2n},\omega,H)$.

The converse of proposition 1.17. also holds in the sense that each symmetry group of a Hamiltonian system gives rise to an integral. To make this precise we first introduce the notion of momentum mapping.

1.18. <u>DEFINITION</u>. On $(\mathbb{R}^{2n},\omega)$ let Φ be a symplectic action of the Lie group G with Lie algebra $\mathfrak{g}$. We say that a mapping $J : \mathbb{R}^{2n} \to \mathfrak{g}^*$ is a *momentum mapping for the action* Φ if for every $\xi \in \mathfrak{g}$ we have

$$X_{\hat{J}(\xi)} = \frac{d}{dt}\,\Phi(\exp t\xi,x)\Big|_{t=0}$$

where the right hand side is called the *infinitesimal generator* of the action corresponding to ξ. $\hat{J}(\xi) : \mathbb{R}^{2n} \to \mathbb{R}$ is defined by $\hat{J}(\xi)(x) = J(x).\xi$.

1.19. PROPOSITION. Let Φ be a symplectic action on $(\mathbb{R}^{2n},\omega)$ of the Lie group G having momentum mapping J. If G is a symmetry group for $(\mathbb{R}^{2n},\omega,H)$ then $\{\hat{J}(\xi),H\} = 0$.

If one considers a one-parameter symmetry group $\exp(t\ \mathrm{ad}(F))$, $t \in \mathbb{R}$ for $(\mathbb{R}^{2n},\omega,H)$ then one obtains a momentum mapping J such that $\hat{J}(\xi) : \mathbb{R}^{2n} \to \mathbb{R}\ ;\ x \mapsto F(x)$. Consequently F is an integral for $(\mathbb{R}^{2n},\omega,H)$.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. If $g \in G$ then $I(g) : h \to ghg^{-1}$ is a isomorphism of G onto itself. Put $\mathrm{Ad}(g) = dI(g)_e$ then $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$. We have $\mathrm{Ad}(\exp X) = \exp \mathrm{ad}(X)$ for $X \in \mathfrak{g}$. $\mathrm{Ad}^*(g)$ is the corresponding automorphism of $\mathfrak{g}^*$. Also $\mathrm{Ad}^*(g^{-1})$ is an automorphism of $\mathfrak{g}^*$, its action is called the co-adjoint action of G.

1.20. DEFINITION. We say that a momentum mapping J is Ad*-*equivariant* if $J(\Phi_g(x)) = \mathrm{Ad}^*(g^{-1})(J(x))$ for every $g \in G$.

It is clear that the momentum mapping for a one-parameter group $\exp(t\ \mathrm{ad}(F))$, $t \in \mathbb{R}$, $F \in C^\infty(\mathbb{R}^{2n},\mathbb{R})$ is trivially Ad*-equivariant.

Under certain conditions the presence of a symmetry group for a Hamiltonian system allows us to reduce our system to a system of lower dimension. With some abuse of language one might say that the reduced system is obtained by factoring out the symmetry group. We will state the classical reduction theorem as it can be found in Abraham and Marsden [1978] and Arnold [1978]. Our own construction of reduced systems in chapter 4 will be somewhat different.

1.21. THEOREM. Let G_X denote the isotropy subgroup of G under the coadjoint action Ad*, that is, $G_X = \{g \in G \mid \mathrm{Ad}^*(g^{-1})X=X\}$. Furthermore