ELEMENTS OF GEOMETRY

HAMBLIN SMITH

ELEMENTS OF GEOMETRY

CONTAINING

BOOKS I. TO VI. AND PORTIONS OF BOOKS XI. AND XII. OF EUCLID

WITH

Exercises and Notes

BY

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PREFACE.

To preserve Euclid's order, to supply omissions, to remove defects, to give short notes of explanation and simpler methods of proof in cases of acknowledged difficulty—such are the main objects of this Edition of the Elements.

The work is based on the Greek text, as it is given in the Editions of August and Peyrard. To the suggestions of the late Professor De Morgan, published in the *Companion to the British Almanack* for 1849, I have paid constant deference.

A limited use of symbolic representation, wherein the symbols stand for words and not for operations, is generally regarded as desirable, and it is certain that the symbols employed in this book are admissible in the Examinations at Oxford and Cambridge.

I have generally followed Euclid's method of proof, but not to the exclusion of other methods recommended by their simplicity, such as the demonstrations by which I propose to replace the difficult Theorems 5 and 7 in the First Book. I have also attempted to render many of the proofs, as, for instance, those of Propositions 2, 13, and 35 in Book I., and those of 7, 8, and 13 in Book II., less confusing to the learner.

In Propositions 4, 5, 6, 7, and 8 of the Second Book I have ventured to make an important change in Euclid's mode of exposition, by omitting the diagonals from the diagrams and the gnomons from the text.

In the Third Book I have deviated with even greater boldness from the precise line of Euclid's method. Thus I have given new proofs of the Propositions relating to the Contact of Circles: I have used Superposition to prove Propositions 26 to 29, so as to make each of those theorems independent of the others; and I have directed the attention of the learner to the Intersection of Loci, and to the conception of an Angle as a magnitude capable of unlimited increase.

In the Fourth Book I have made no change of importance.

My treatment of the Fifth Book was suggested by the method first proposed, explained, and defended by Professor De Morgan in his *Treatise on the Connexion of Number and Magnitude.* The method is simple and rigorous, presenting Euclid's reasoning in a clear and concise form, by means of a system of notation, to which, I think, no valid objection can be taken. I have altered the order of the Propositions in this Book, so as to give prominence to those which are of chief importance.

The only changes in the Sixth Book to which I desire to call the reader's special attention, are the applications of Superposition in the proofs of Propositions 4 and 19.

The diagrams in Book XI. form an important feature of this Edition. For them I am indebted to the kindness of Mr. Hugh Godfray, of St. John's College, Cambridge.

The Exercises have been selected with considerable care, chiefly from the University and College Examination Papers. They are intended to be progressive and easy, so that a learner may be induced from the first to work out something for himself.

A complete series of the Euclid Papers set in the Cambridge Mathematical Tripos from 1848 to 1872 will be found on pp. 198-210 and 342-349.

I have made but little allusion to Projections, because that part of the subject is fully explained by Mr. Richardson in his work on *Conic Sections treated Geometrically*, forming a part of RIVINGTON'S MATHEMATICAL SERIES.

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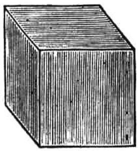
ELEMENTS OF GEOMETRY.

INTRODUCTORY REMARKS.

When a block of stone is hewn from the rock, we call it a Solid *Body*. The stone-cutter shapes it, and brings it into that which we call *regularity of form;* and then it becomes a Solid *Figure*.



Now suppose the figure to be such that the block has six flat sides, each the exact counterpart of the others; so that, to one who stands facing a corner of the block, the three sides which are visible present the appearance represented in this diagram.

Each side of the figure is called a *Surface;* and when smoothed and polished, it is called a *Plane* Surface.

The sharp and well-defined edges, in which each pair of sides meets, are called *Lines*.

The place, at which any three of the edges meet, is called a *Point*.

A *Magnitude* is anything which is made up of parts in any way like itself. Thus, a line is a magnitude; because we may regard it as made up of parts which are themselves lines.

The properties Length, Breadth (or Width), and Thickness (or Depth or Height) of a body are called its *Dimensions*.

We make the following distinction between Solids, Surfaces, Lines, and Points:

A Solid has three dimensions, Length, Breadth, Thickness.

A Surface has two dimensions, Length, Breadth.

A Line has one dimension, Length.

A point has no dimensions.

S. E.

BOOK I.

DEFINITIONS.

I. A Point is that which has no parts.

This is equivalent to saying that a Point has no magnitude, since we define it as that which cannot be divided into smaller parts.

II. A Line is length without breadth.

We cannot conceive a visible line without breadth; but we can reason about lines as if they had no breadth, and this is what Euclid requires us to do.

III. The Extremities of finite Lines are points.

A point marks *position*, as for instance, the place where a line begins or ends, or meets or crosses another line.

IV. A Straight Line is one which lies in the same direction from point to point throughout its length.

V. A Surface is that which has length and breadth only.

VI. The Extremities of a Surface are lines.

VII. A Plane Surface is one in which, if any two points be taken, the straight line between them lies wholly in that surface.

Thus the ends of an uncut cedar-pencil are plane surfaces; but the rest of the surface of the pencil is not a plane surface, since two points may be taken in it such that the *straight* line joining them will not lie on the surface of the pencil.

In our introductory remarks we gave examples of a Surface, a Line, and a Point, as we know them through the evidence of the senses.

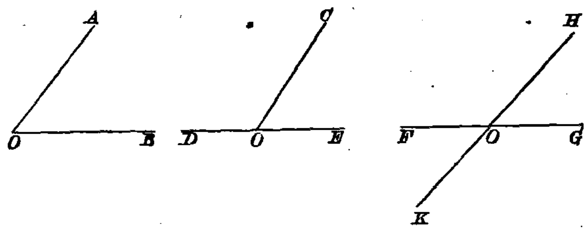
The Surfaces, Lines, and Points of Geometry may be regarded as mental pictures of the surfaces, lines, and points which we know from experience.

It is, however, to be observed that Geometry requires us to conceive the possibility of the existence

of a Surface apart from a Solid body,
of a Line apart from a Surface.
of a Point apart from a Line.

VIII. When two straight lines meet one another, the inclination of the lines to one another is called an Angle.

When *two* straight lines have one point common to both, they are said to *form* an angle (or angles) at that point. The point is called the *vertex* of the angle (or angles), and the lines are called the *arms* of the angle (or angles).

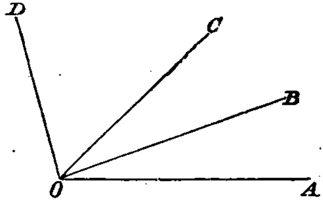


Thus, if the lines *OA*, *OB* are terminated at the same point *O*, they form an angle, which is called *the angle at O*, or *the angle AOB*, or *the angle BOA*,—the letter which marks the vertex being put between those that mark the arms.

Again, if the line *CO* meets the line *DE* at a point in the line *DE*, so that *O* is a point common to both lines, *CO* is said to make with *DE* the angles *COD*, *COE*; and these (as having one arm, *CO*, common to both) are called *adjacent* angles.

Lastly, if the lines *FG*, *HK* cut each other in the point *O*, the lines make with each other four angles *FOH*, *HOG*, *GOK*, *KOF*; and of these *GOH*, *FOK* are called *vertically opposite* angles, as also are *FOH* and *GOK*.

When *three or more* straight lines as OA, OB, OC, OD have a point O common to all, the angle formed by one of them, OD,



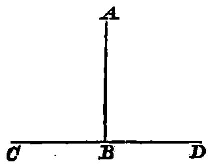
with OA may be regarded as being made up of the angles AOB, BOC, COD; that is, we may speak of the angle AOD as a whole, of which the parts are the angles AOB, BOC, and COD.

Hence we may regard an angle as a *Magnitude*, inasmuch as any angle may be regarded as being made up of parts which are themselves angles.

The size of an angle depends in no way on the length of the arms by which it is bounded.

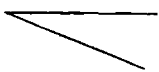
We shall explain hereafter the restriction on the magnitude of angles enforced by Euclid's definition, and the important results that follow an extension of the definition.

IX. When a straight line (as AB) meeting another straight line (as CD) makes the adjacent angles (ABC and ABD) equal to one another, each of the angles is called a RIGHT ANGLE; and each line is said to be a PERPENDICULAR to the other.



X. An OBTUSE ANGLE is one which is greater than a right angle.

XI. An ACUTE ANGLE is one which is less than a right angle.

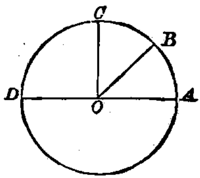


XII. A FIGURE is that which is enclosed by one or more boundaries.

XIII. A CIRCLE is a plane figure contained by one line, which is called the CIRCUMFERENCE, and is such, that all straight lines drawn to the circumference from a certain point (called the CENTRE) within the figure are equal to one another.

XIV. Any straight line drawn from the centre of a circle to the circumference is called a RADIUS.

XV. A DIAMETER of a circle is a straight line drawn through the centre and terminated both ways by the circumference.



Thus, in the diagram, *O* is the centre of the circle *ABCD*, *OA*, *OB*, *OC*, *OD* are Radii of the circle, and the straight line *AOD* is a Diameter. Hence the radius of a circle is half the diameter.

XVI. A SEMICIRCLE is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XVII. RECTILINEAR figures are those which are contained by straight lines.

The PERIMETER (or Periphery) of a rectilinear figure is the sum of its sides.

XVIII. A TRIANGLE is a plane figure contained by three straight lines.

XIX. A QUADRILATERAL is a plane figure contained by four straight lines.

XX. A POLYGON is a plane figure contained by more than four straight lines.

When a polygon has all its sides equal and all its angles equal it is called a *regular* polygon.