The Geometry and Dynamics of Magnetic Monopoles

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M. B. PORTER LECTURES

RICE UNIVERSITY

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PREFACE

In January 1987 I gave the Milton Brockett Porter Lectures at Rice University. This provided me with the opportunity of presenting, at some length, the results on magnetic monopoles which Nigel Hitchin and I have been investigating over the past few years. This book, written jointly, is an expanded version of the lectures and it contains a full and detailed treatment of the essentially new results. Although dependent on earlier work by many authors we have endeavoured to make it more self-contained by adding some introductory and background material.

Michael Atiyah

CONTENTS

The Geometry and Dynamics of Magnetic Monopoles

INTRODUCTION

The purpose of this book is to apply geometrical methods to investigate solutions of the non-linear system of hyperbolic equations which describe the time evolution of non-abelian magnetic monopoles. The problem we study is, in various respects, a somewhat simplified model but it retains sufficient features to be physically interesting. It gives information about the low-energy scattering of monopoles and it exhibits some new and significant phenomena.

From a purely mathematical point of view our investigation should be seen as a contribution to the area of "soliton" theory. In general a soliton is a solution of some non-linear differential equation which behaves in certain respects like a particle: it should be approximately localized in space and should be "conserved" in collisions.

There is now a very extensive theory of solitons for one-dimensional space, of which the prototype and best known example is the KdV equation [14]. These solitons have very remarkable properties and their evolution equations form an infinite-dimensional integrable Hamiltonian system. Some of the key features of the theory are:

(1) there are explicit formulae describing the interaction of k solitons (for any integer k),
(2) these formulae involve the theory of Riemann surfaces (or algebraic curves),
(3) the *inverse scattering* method (associated to linear operators) is used to construct the solutions,
(4) the scattering of solitons after a collision is essentially trivial (i.e. velocities are unaffected).

Although the theory of the KdV equation is "exact" it should be recalled that the KdV equation arises naturally as an *approximation* for shallow waves in a channel. Thus the soliton features listed in (1) through (4) are only an approximate description of the real physical situation.

The equations we shall be studying, governing the evolution of monopoles, will share some of the essential features (1), (2), (3) of the KdV theory. On the other hand the scattering of monopoles after a collision will be non-trivial in the sense that the velocities alter (i.e. there is "momentum transfer"). The monopole equations moreover take place in fully 3-dimensional space, and are correspondingly more complicated. The analogy with KdV will be examined in detail in chapter 15.

Our results will not give exact solutions but only an approximation for small relative velocities. However, as we have pointed out, the KdV theory is itself only an approximation to a more realistic model. From a physical point of view there is no essential difference between an approximate solution to an exact equation and an exact solution to an approximate equation.

The concept of a magnetic monopole, as an isolated point-source of magnetic charge, was introduced some 50 years ago by Dirac in a very influential paper [13]. In Maxwell's equations, electricity and magnetism appear on an equal footing but whereas electric charges occur naturally, magnetic charges or monopoles do not appear to do so. Nevertheless by postulating their existence Dirac was able to produce the only convincing argument leading to the quantization of electric charge, namely the fact that electric charges always appear in integer multiples of a fixed charge (that of the electron). With the advent of non-abelian gauge theories, in which the U(1) of Maxwell theory is enlarged to a non-abelian group such as SU(2), it was soon realized by ‘t Hooft and Polyakov that one could have *smooth* field configurations which behaved at large distances like a Dirac monopole. The essential point is that the non-linear equations, which generalize the linear Maxwell theory, admit "soliton" solutions in which the singular point-particle of Dirac is replaced by a smooth field approximately localized at the position of the "particle."

The 't Hooft-Polyakov monopole is only known numerically but there is a simplified model introduced by Prasad and Sommerfield (in which the coefficient λ of the Higgs potential $(1 - |\phi|^2)^2$ is put equal to 0) which has an explicit monopole solution. These are known as BPS (or self-dual) monopoles and are the ones we shall be concerned with.

The BPS-monopoles are solutions of the **Bogomolny** equations which will be introduced in chapter 1. They describe *static* monopoles in $\mathbf{R}^3$ and they have been extensively studied in recent years by many authors. They have remarkable properties which are best understood as a special case of the self-duality equations in 4-space (for solutions independent of one of the variables). In particular the Penrose twistor theory applies to these equations and this is the basis for all the methods of solution. It provides the link with complex-variable theory and ultimately explains the relevance of Riemann surfaces for the construction of monopoles. This material is explained and summarized in chapter 2 with further background given as an appendix in chapter 16.

As we explain in chapter 16 there are several approaches to the study of monopoles, each with its own advantages. We will use all these at various places in our treatment. Although it would be logically simpler to stick to one point of view there are at present some technical problems which remain to be resolved. To avoid these we have adopted a hybrid approach, essentially taking the line of least resistance.

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The Bogomolny equations have solutions for all integer values k of the magnetic charge. These k-monopoles when "well-separated" are approximately a superposition of k simple monopoles located at different points. The fact that such a configuration is static depends on the fact that the magnetic repulsion between the monopoles is balanced by an attractive force due to the Higgs field. The parameter space of all k-monopoles is a $4k$-dimensional manifold M_k which is described in detail in chapter 2. For $k = 1$, we have $M_1 = \mathbf{R}^3 \times S^1$ indicating that a 1-monopole is determined by a point in $\mathbf{R}^3$ (its "position") and a phase angle. For $k > 1$ there is a region (near ∞) in M_k which is approximately an (unordered) product of k copies of M_1, and this represents well-separated monopoles. However, this description breaks down in the interior of M_k, and this has profound implications for the interactions of monopoles which is the main thrust of this book.

Manton [35] has argued that the geodesic flow on M_k, with respect to its natural metric (induced from L^2-functions on $\mathbf{R}^3$), is the low-energy approximation to the true evolution of dynamic monopoles. His argument (reviewed in chapter 1) is based on the analogy of a particle in $\mathbf{R}^n$ moving in a potential field V. The equilibrium positions are given by the subspace $M \subset \mathbf{R}^n$ giving minima of V. For motion with energy near this minimum the trajectory of a particle whose initial velocity is tangential to M is close to the corresponding geodesic on M (with some small oscillations in the transverse directions).

The dynamics of monopoles can be put into this framework with $\mathbf{R}^n$ replaced by an infinite-dimensional manifold, and $M = M_k$. The important point is that the Bogomolny equations give the absolute minimum of the potential energy. Because we are now in an infinite-dimensional situation Manton's argument, although physically plausible, requires detailed analytic justification. We shall not undertake this analysis, but we note that a similar situation has been studied in detail by Ebin [17].

To carry out Manton's programme we therefore have to investigate the Riemannian metric on M_k. In particular, we have to show that it is *finite* and *complete*. Lack of finiteness would mean the metric was infinite (or not defined) in certain directions, so that monopole motion was constrained, while incompleteness would mean that monopoles (and so magnetic charges) could disappear in finite time. Finiteness and completeness are deduced in chapter 3 from basic analytical results of Taubes. These results also yield the asymptotic behaviour of M_k when a k-monopole separates into pieces.

The direct definition of the metric on M_k involves computing the L^2-norms of the zero-modes (solutions of the linearized equations), and this is too complicated a procedure to be useful. Fortunately, however, there are some general symmetry principles which govern the metric on M_k and these can be effectively exploited. The basic result is that each M_k is a hyperkähler manifold, which means that its holonomy group lies in the symplectic group $\mathrm{Sp}(k) \subset \mathrm{SO}(4k)$. Formally this can be seen as an infinite-dimensional case of the *hyperkähler quotient* construction of [26], but it needs to be supplemented here with the analytical results of Taubes. This is explained in chapter 4.

Hyperkähler manifolds fit well into the Penrose twistor theory as generalized by Salamon [40, 41]. This implies that to a hyperkähler manifold M one can associate a complex manifold Z (its twistor space) together with certain extra data. This data is essentially holomorphic (except for an involutary conjugation): it is analogous to having an algebraic variety defined over the real numbers. Moreover Z, together with its extra data, is equivalent to M with its hyperkähler metric. Thus to find the hyperkähler metric on the monopole parameter space M_k we have, in principle, only to find the corresponding twistor space Z_k (with its extra data). This programme is carried out in chapter 5.

In chapter 6 we reinterpret this twistor space in terms of suitable symmetric products. This ties up the "particle" picture of monopoles with the twistor description in an attractive and suggestive manner, which appears to be quite general, i.e. relevant to gauge groups other than SU(2).

The translation group of $\mathbf{R}^3$ acts naturally (and freely) on M_k and so does an overall phase factor S^1. Dividing by these we introduce in chapter 2 the reduced or relative monopole space M_k^0 of dimension $4k - 4$. In chapter 4 we show that the k-fold cover $\tilde{M}_k$ of M_k decomposes isometrically as a product

$$\tilde{M}_k \cong \mathbf{R}^3 \times S^1 \times \tilde{M}_k^0$$

where $\tilde{M}_k^0$ is the k-fold (universal) cover of M_k^0. Moreover M_k^0 is irreducible as a Riemannian manifold, and is itself hyperkähler. Geodesic motion on M_k therefore decomposes (locally) into geodesic motion on M_k^0 and geodesic motion (for the flat metric) on $\mathbf{R}^3 \times S^1$. The point in $\mathbf{R}^3$ corresponding to a k-monopole can be viewed as the centre of mass and its motion gives the linear momentum of the system (which is conserved). Motion in the S^1-factor gives rise to electric charge [35]. Note that for us this quantity (unlike magnetic charge) is not an integer, but on the contrary is taken to be very small. The residual motion in M_k^0 is the interesting part and represents motion relative to the centre of mass. The irreducibility of the metric on M_k^0 means that we cannot decompose the dynamics any further.

This is as far as we take the case of general k and the remaining chapters concentrate entirely on the case $k = 2$, and the study of the 4-dimensional manifold M_2^0. For this we shall replace the general theory of chapter 5 by a more concrete and direct approach.

For $k = 2$ the spectral curve of a k-monopole is an elliptic curve and Hurtubise [27] has used this to give an explicit description of M_2^0. He also shows that a generic 2-monopole has 3 principal axes. These results are reviewed in chapter 7 and are compared with the Donaldson parametrization of M_k^0 by rational functions described in chapter 2. Then in chapter 8 we compute the three "spectral radii" associated to the three principal axes of a 2-monopole and use these to find the *conformal* structure of M_2^0.

In chapter 9 we start from another point of view. In dimension 4, since $\mathrm{Sp}(1) = \mathrm{SU}(2)$, a hyperkähler manifold is equivalently an anti-self-dual Einstein manifold (with zero scalar curvature). Now M_2^0 has a natural SO(3)-action, induced by rotations about the centre of mass in $\mathbf{R}^3$. Hence the metric on M_2^0 is an SO(3)-invariant anti-self-dual Einstein metric. The SO(3)-invariance reduces these conditions to a system of ordinary differential equations described by Gibbons and Pope [20]. In chapter 9 we make a qualitative study of the equations and in particular of the induced equations for the conformal structure. We show that there is an essentially unique solution representing a complete manifold and this must therefore be M_2^0. In particular the results of chapter 8 give the conformal structure.

Although the formula of chapter 8 is explicit it involves elliptic integrals and so is not very tractable analytically. Numerical computation indicates the shape of the solution but in chapter 10 we derive a number of useful inequalities on the solution directly from the differential equation. Then in chapter 11 we complete the story by going from the conformal structure formula of chapter 8 to an explicit form for the metric on M_2^0.

In chapter 12 we use this explicit formula for the metric to make asymptotic expansions for the behaviour near the "collision states" and when the two monopoles are far apart. We also investigate the geometry of the totally geodesic surfaces in M_2^0 representing 2-monopoles with one principal axis fixed. The most interesting one has the shape of a funnel. The geodesic motion on these surfaces is then studied in chapter 13. In particular geodesics on the funnel fall into two types, those which fall through the hole and those which return.

Finally in chapter 14 we interpret the geodesic motion of chapter 13 in terms of the scattering of monopoles. We first show that a direct collision of monopoles (without electric charge) produces a 90-degree scattering in a plane determined by the initial relative phase. We then consider displacing the initial velocities of the monopoles in one of the two principal planes (through the Higgs axis). We describe the outcome as a function of the displacement (or impact) parameter μ. In one of the principal planes the scattering angle is monotonic in μ and decreases from 90 degrees to 0. For the other principal plane the situation is more complex and it corresponds to the funnel of chapter 13. In particular, for small μ, the emerging monopoles acquire (equal and opposite) electric charges, so that they become dyons. Thus electric charge (for each monopole separately) is not conserved. This corresponds to the fact that M_2^0 has an asymptotic U(1)-symmetry which is not present in the interior.

There are many open problems left in this area. In the first place although we have given an explicit description of the metric on M_2^0 we have only described some very special geodesics, namely those fixed by an involution on $\mathbf{R}^3$. In general it would be interesting to know if there are any closed or "trapped" geodesics.

It would also be interesting to investigate the geodesics on M_k^0 for $k > 2$. This would describe k-monopole scattering. We could also replace the group SU(2) by other Lie groups, say SU(n). The corresponding monopole spaces are still hyperkähler and our methods still apply but some interesting new features seem likely to emerge particularly in the degenerate case when the Higgs field ϕ acquires equal eigenvalues at infinity.

Manton and Gibbons [19] have used the results on M_2^0 to analyse the classical scattering in greater detail. They have also gone on to consider the quantum scattering. In particular they have observed that the exchange of electric charge noted above has a natural interpretation arising from the emission and absorption by the dyons of massive charged mesons.

Finally there is of course the analytical problem of justifying Manton's claim that the low energy dynamics of monopoles is approximately given by geodesic flow on M_k. One would also like to know the length of time for which this approximation is valid as a function of the velocity. Moreover it would be desirable to prove that this approximation continues to hold (for significant time-intervals) for the 't Hooft-Polyakov monopoles, provided the parameter λ is close to 0.

We are indebted to many people for help and advice. Nick Manton proposed the problem, Cliff Taubes and Simon Donaldson assisted with the analysis and Garry Gibbons helped with the Einstein equations. In dealing with the ordinary differential equations of chapter 9 we benefited from discussion with John Ockendon, while David Atiyah dealt with the computer calculations.

The main results have previously been briefly summarized in [3] and [4]. For general background on the analysis of monopoles, [30] is the standard reference, while more recent work on explicit solutions via twistor methods can be found in [22], [47], [9], [36].

CHAPTER 1

The Monopole Equations

We give here an outline of the physical background out of which the monopoles we analyse, described by solutions of the Bogomolny equations, arise. The reader is directed to [30], [12], and [11] for further information on the links between the mathematical and physical theory.

We are concerned here with a **gauge theory**, the prototype of which is electromagnetic theory. In differential geometric terms the electric field E in Maxwell theory is considered as a 1-form on $\mathbf{R}^3$ and the magnetic field B as a 2-form. The Maxwell field tensor $F = B + c\,dt \wedge E$ is then, as a consequence of the Maxwell equations, a *closed* 2-form in Minkowski space $\mathbf{R}^4$ and can therefore be expressed as $F = dA$ for a 2-form A, the electromagnetic potential. To A is associated a vector particle—the photon.

There is an ambiguity $A \to A + d\lambda$ in the choice of A. From the point of view of gauge theory this has an interpretation as the ambiguity $\psi \to e^{i\lambda}\psi$ in the choice of phase of a wave function. Geometrically this requires us to consider A as a connection form on a (trivial) principal bundle with structure group U(1) and F as its curvature. The circle group U(1) measures the difference in phase. Even at this stage there is a noticeable difference in the role of the electric and magnetic fields, for at a fixed time $t = t_0$, the *magnetic field* B is the curvature of a connection on $\mathbf{R}^3$, whereas E has no such geometrical interpretation.

The equation of motion of the field A is determined by an action density which is required to be *gauge-invariant*. For electromagnetism the density is given by (F, F), using the Minkowski space inner product. This gives rise to the source-free Maxwell equations

$$dF = 0 = d * F.$$

Pure Yang-Mills theory is a gauge theory directly modelled on electromagnetism, but using a non-abelian Lie group G instead of the abelian group U(1). The vector potential A, whose geometrical interpretation is that of a connection on a principal G-bundle, has the physical interpretation of giving rise to a vector particle for each generator of the Lie algebra of G. The action density (F, F), analogous to that of electromagnetism, satisfies the gauge-invariance condition and gives rise to the Yang-Mills equations

$$D_A F = 0 = D_A * F$$

where $D_A = d + A$ is the covariant exterior derivative, depending on the connection A.

One of the restrictions of Yang-Mills theory, which was recognized early on, is the fact that like a photon the Yang-Mills particle is constrained to have zero mass. To incorporate mass, for physical reasons one requires a term like $m^2 Q(A)$ in the action, where Q is quadratic in the vector potential. It is impossible, however to construct such terms in a gauge-invariant manner. One way around this problem is to incorporate a new field—the **Higgs field** ϕ—into the gauge theory.

Mathematically speaking, ϕ is a section of a vector bundle associated to the principal G-bundle, on which A is defined, by a representation. Physically, ϕ is a scalar field transforming under some representation of G, often the adjoint representation. The action density is taken to be of the form

$$a = (F, F) + (D\phi, D\phi) + V(\phi)$$

where V is a gauge-invariant potential function and $D\phi$ the covariant derivative of ϕ. An action of the above form is called a **Yang-Mills-Higgs** action.

The way in which the mass of A enters may be seen for example by considering the Higgs field to be in a ground state—that is at a minimum of the potential function V. Since V is gauge-invariant, the minimum is not attained at a unique field ϕ, but in general at an *orbit* of ϕ under the group of gauge transformations. Any ϕ in a ground state can then be gauge transformed to a constant Higgs field ϕ_0. In this gauge, since ϕ_0 is constant,

$$D\phi_0 = d\phi_0 + A\phi_0 = A\phi_0.$$

Thus

$$(D\phi_0, D\phi_0) = (A\phi_0, A\phi_0)$$

giving the required quadratic term in A. The actual value of the mass is determined by the constant ϕ_0. When ϕ transforms under the adjoint representation, the most common form for the potential is the quartic expression $V = \lambda(1 - |\phi|^2)^2$. This corresponds to the action density

$$a = (F, F) + (D\phi, D\phi) + \lambda(1 - |\phi|^2)^2 \tag{1.1}$$

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