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OF
DISCRETE
MATHEMATICS
14

Combinatorial and Geometric Structures and Their Applications

A. Barlotti
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MATHEMATICS STUDIES 63

Annals of Discrete Mathematics (14)
General Editor: Peter L. Hammer
University of Waterloo, Ont., Canada

Combinatorial and Geometric Structures and their Applications

edited by

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1982

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM • NEW YORK • OXFORD

ISBN: 0 444 86384 2

Publishers
NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM • NEW YORK • OXFORD

Sole distributors for the U.S.A. and Canada
ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 VANDERBILT AVENUE
NEW YORK, N.Y. 10017

Library of Congress Cataloging in Publication Data
Main entry under title:

Combinatorial and geometric structures and their applications.

1. Combinatorial designs and configurations--Addresses, essays, lectures. 2. Combinatorial geometry--Addresses, essays, lectures. I. Barlotti, A. (Adriano), 1923- . II. Series.
QA166.25.C65 511'.6 82-3515
ISBN 0-444-86384-2 (U.S.) AACR2

PRINTED IN THE NETHERLANDS

PREFACE

A Symposium on "Combinatorial Structures and their Applications" was held at Villa Madruzzo (Cognola di Trento), on October 20 - 25, 1980. The meeting was promoted and sponsored by the "Centro Interuniveritario per la Ricerca Matematica" (C.I.R.M.) of the "Università di Trento" for the purpose of offering an intense week of joint scientific activity to some outstanding scientists and a group of younger researchers. The enthusiastic co-operation of all participants made the Conference a great success.

Combinatorics is an old branch of mathematics. In recent times the advent of the electronic age and the development of computer technology has given great impetus to the study of combinatorial techniques, both by providing combinatorists with a powerful new tool and also by creating a new field of application for those techniques. As always happens in the development of science the ability to answer practical questions is greatly enhanced by the scientist making progress in the study of pure theory. This gives a clear justification for the tremendous activity and progress in this field.

Combinatorics covers too broad a range of subjects and so a further restriction had to be made on the topics to be considered. It was decided to confine the main topics to Finite Geometric Structures and in particular to Galois Geometries.

Four lecturers presented, in a series of invited addresses, the State of the Art in particular fields. Many interesting new results were also given by a number of contributed papers.

A large part of the material presented at the Symposium appears in detail in this volume. A few other papers related to the topics considered have been adjoined. We are confident that many of these papers will form an invaluable basis for further progress in this field.

I wish to thank the directors of the C.I.R.M., Professor M. Miranda and G. Zacher who organized the Symposium. Thanks must also be extended to the G.N.S.A.G.A. of the C.N.R. which gave the support which made it possible to increase the number of participants.

I am particularly grateful to Professor Peter L. Hammer for encouraging me to prepare this volume and to the referees for their invaluable assistance.

Adriano Barlotti

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ON FINITE NONLINEAR STRUCTURES

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The investigation of the geomtry of plane sections of a semi-quadric leads to the notion of generalized circle geometries. In the case of the geometry of Möbius for instance there is exactly one circle through three distinct points. This property does not carry over to the above mentioned general situation. What remains is that there exists at least one joining circle and that $a \cap b = b \cap c = c \cap a$ is a consequence of $\mathrm{card}(a \cap b \cap c) \geq 2$ for distinct circles a,b,c. We are thus led to the notion of Tracks, namely to sets $a \cap b$ ($a \neq b$ circles with $\mathrm{card}(a \cap b) \geq 2$), which we have introduced in [3],[4]. Almost nothing is known in a general setting about the inner structure of tracks, especially about their possible cardinalities in the finite case. In the first part of this surveying article we like to stress some open problems in this context, concentrating ourselves on the geometries of Möbius and Laguerre. In the second part we are furthermore interested in the theory of Lorentz transformations of the Järnefelt world. This part can be integrated in the geometry of Minkowski since Lorentz transformations are automorphisms of this geometry. Our main goal here is to present correspondences of basic theorems of space-time geometry under the assumption that the underlying field is a Galois field. We also shall include remarks about connections between Lorentz transformations and the geomtry of Laguerre.

§ 1 Generalized Möbius Planes.

In 1958,[3], we have introduced the notion of a Möbius plane as follows:

Consider a set X of points and a set Y of non-empty subsets of X of circles such that the following conditions are satisfied:

(M I) Through three distinct points there is at least one circle. Given distinct circles a,b,c with $\mathrm{card}(a \cap b \cap c) \geq 2$, then $a \cap b = a \cap c$.
(M II) Given a circle a and points $A \in a$, $B \notin a$. Then there is exactly one circle $b \ni A,B$ with $a \cap b = \{A\}$.
(M III) There exist four distinct points, which are non-cocircular, i.e. which are not on a common circle.

Obviously, (M I) is a consequence of

(M I') Through three distinct points there is exactly one circle.

In the past we have called structures (X,Y) satisfying (M I'), (M II), (M III) special Möbius planes (Möbiusebenen im engeren Sinn). In the meantime it has become more and more usual to denote "special Möbius planes" by "Möbius planes". Therefore we like to say "generalized Möbius planes" instead of the original Möbius planes. Generalized Möbius planes occur quite natural when studying the geometry of plane sections of a semi-quadric Q. The underlying antiautomorphism of order at most 2 is the identy iff (M I') is satisfied.

The following problem seems to be still unsolved:

A. Does there exist a finite generalized Möbius plane, which is not a Möbius plane?

Consider a generalized Möbius plane K. Given distinct circles a,b such that $\mathrm{card}(a \cap b) > 1$. Then $a \cap b$ is called a track of K (s.[3]).

In [3] we have proved the following facts:

(1) card $a \geq 3$ for every circle a.

(2) Through two distinct points there is exactly one track.

(3) Given a circle a and distinct points $P,Q \in a$. Then a contains the track (PQ) through P,Q.

(4) A circle contains at least three distinct tracks. If x is a track and $A \notin x$ a point then there is exactly one circle through A and x.

(5) Given circles a,b then one and only one of the following cases holds true:
   a) $a \cap b = \emptyset$,
   b) $\text{card}(a \cap b) = 1$,
   c) $a \cap b$ is a track,
   d) $a = b$.

(6) Given a circle a and a track x such that $\text{card}(a \cap x) = 1$. Then there is exactly one circle $b \supset x$ with $\text{card}(a \cap b) = 1$.

(7) Given a point W of K. Define the tracks through W as points and the circles through W as lines. Then this structure must be an affine plane, the socalled derivation of K in W.

Since every semi-quadric leads to a generalized Möbius plane many proper generalized Möbius planes are known. If K is such an example then it turns out that card x = card y for all tracks x,y of K. Therefore we like to pose the following problem

B. Let K be a (eventually finite) generalized Möbius plane. Is then card x = card y true for all tracks x,y of K?

Consider a semi-quadric Q such that the underlying generalized Möbius plane K is miquelian. Then it turns out that Q is a quadric and that thus K is a Möbius plane. In [3] we have proved more:
Given a miquelian generalized Möbius plane K such that the following condition holds true

(+) If there exists a track x with card $x > 2$ then card $y > 2$ for all tracks y of K.

Then K must be a Möbius plane.

In case there exist proper generalized Möbius planes, which are finite, or in case A remains unanswered it might be of interest to study known results about finite Möbius planes by replacing (M I') by (M I). In this connection s. P. Dembowski [12], [14], and J.Kahn [19].

In [16] G. Ewald has characterized generalized Möbius planes stemming from a semi-quadric by means of a configuration theorem involving incidence and orthogonality. He has moreover given, [16], a wide class of generalized Möbius planes by using his notion of weakly convex semi-surfaces.

§ 2 Generalized Laguerre Planes.

Given a set X of spears and a set Y of subsets of X of cycles. The distinct spears $S,T \in X$ are called parallel, $S//T$, iff there is no cycle a such that $a \ni S,T$. We also put $S//S$ for all $S \in X$. In case a,b are distinct cycles with $\text{card}(a \cap b) > 1$ we call $a \cap b$ a track. Two tracks x,y are called parallel, $x//y$, iff there exists to every spear S of respectively x,y a spear T of respectively y,x with $S//T$.
The structure (X,Y) is called a (now) generalized Laguerre plane ([4]) iff the following conditions are satisfied:

(L I) Through two non parallel spears S,T there is exactly one track, (ST). Given non-cocircular spears S,A,B with $A \not\!/\!/ S \not\!/\!/ B$ then $(SA)//(SB)$.

(L II) Given a cycle a and spears $S \in a$, $T \notin a$, $S \not\!/\!/ T$, there exists exactly one cycle b with $b \ni S,T$ and $a \cap b = \{S\}$.

(L III) Given a cycle a and a spear $S \notin a$ there exists exactly one spear $T \in a$ with $T//S$.

(L IV) There exist a cycle z and a spear V such that $V \notin z$ and card $z \geq 3$.

A generalized Laguerre plane is a Laguerre plane iff card $x = 2$ for all tracks x.

Given a (commutative or non commutative) field F, char $F \neq 2$, possessing an involutorial antiautomorphism t. Consider the 3-dimensional affine space $A^3(F)$ over F and
$$X := \{ (x,y,z) \in F^3 \mid xx^t + yy^t = 1 \} .$$
If e is a plane of $A^3(F)$ with $e \cap X \neq \emptyset$ and such that there is no line in $e \cap X$ then we call $e \cap X$ a cycle.
This structure (X,Y) turns out to be a generalized Laguerre plane ([4]) .

Problem A is easily solved in the context of generalized Laguerre planes:

Consider $F = GF(9)$ and $t: a+bi \to a-bi$, $a,b \in GF(3)$, $i^2 = -1$. In this case the above described generalized Laguerre plane consists of 216 spears, 729 cycles. There are 81 tracks through a given spear S. 45 of these tracks contain precisely 4 spears, the remaining 36 precisely 3 spears.

We now like to present properties for generalized Laguerre planes, which to some extent are similar to those given for generalized Möbius planes in §1.

(1) Two distinct spears of a track or of a cycle are non parallel.

(2) Given cycles a,b then a cannot be a proper subset of b.

(3) X is not a cycle.

(4) The parallelity relation on X is an equivalence relation. The parallelity relation on the set of tracks is an equivalence relation. $x//y$ implies card x = card y.

(5) card $a \geq 3$ for all $a \in Y$.

(6) $S,T \in a$, $S \neq T$, implies $(ST) \subset a$.

(7) Given three pairwise non parallel spears P,Q,R. Then there is exactly one cycle through P,Q,R if and only if $(PQ) \not\parallel (PR)$.

(8) Let a,b be cycles. Then one and only one of the following conditions holds true:
   a) $a \cap b = \emptyset$,
   b) card$(a \cap b) = 1$,
   c) $a \cap b$ is a track,
   d) $a = b$.

Consider a spear W and let $M(W)$ be the set of all tracks through W.
For $x \in M(W)$ define
$$x_W := \{ y \in M(W) \mid x//y \} .$$
Let $G(W)$ be the set of all classes x_W, $x \in M(W)$.
Then we get

(9) a) $M(W) \neq \emptyset$,
   b) card $x_W \geq 2$ for all $x \in M(W)$,
   c) card $G(W) \geq 2$,
   d) Given $x \in M(W)$ and a cycle $a \ni W$. Then there exists $y \in x_W$ with $y \subset a$.

Finally the following theorem holds true:

Let K be a generalized Laguerre plane and let W be a spear of K. Call the elements of $M(W)$ points and the cycles through W and the elements of $G(W)$ lines. If x is a point and g a line put "x on g" iff
$x \in g$ for $g \in G(W)$,
$x \subset g$ in case g is a cycle through W.
Then this is an affine plane, the socalled derivation $D(K,W)$ of K in W.

C. If one knows the finite order of $D(K,W)$ (and maybe other invariants of K) what are the possible cardinalities of the occuring tracks ?

Quite similar to the construction,[16], of G. Ewald of a general class of generalized

Möbius planes is the following construction concerning generalized Laguerre planes ([5]):
Let E be an affine plane. A set $C \neq \emptyset$ of points is called a pseudo-oval iff

(+) For $P \in C$ there exists exactly one line h through P with $h \cap C = \{P\}$.

Examples: a) Consider distinct and parallel lines a,b of E and points $U \in a$, $V \in b$. Then $(a \cup b) \setminus \{U,V\}$ is a pseudo-oval.
b) Consider two non parallel lines a,b of E. Then $(a \cup b) \setminus (a \cap b)$ is a pseudo-oval.
c) Let E be the affine plane over a field F, char $F \neq 2$, possessing an involutorial antiautomorphism t. Then

$$\left\{ (x,y) \in F^2 \mid xx^t + yy^t = 1 \right\}$$

is a pseudo-oval.

Consider now a 3-dimensional affine space A^3 and let E be a plane of A^3. If C is a fixed pseudo-oval of E and g a fixed line of A^3 with $\text{card}(g \cap E) = 1$, then define the cylinder Z to be the set of all points of A^3, which are on a line $h // g$ with $C \cap h \neq \emptyset$. Call the lines $h \subset Z$ with $h // g$ generators. Then the following theorem holds true:

Define the points of Z to be the spears and the sets $e \cap Z$ to be the cycles, where e is an arbitrary plane of A^3 such that $e \cap Z \neq \emptyset$ and such that e contains no generator. Then this is a generalized Laguerre plane, which is a Laguerre plane iff there are no three distinct and collinear points in C.

Remark 1 : Omitting characteristic 2, a semi-quadric, which leads to a proper generalized Möbius plane, can be canonically described as the set of points (x_1,x_2,x_3,x_4) of a 3-dimensional projective space such that

$$x_1x_1^t - x_2dx_2^t + x_3x_4^t + x_4x_3^t = 0 .$$

Here t denotes the underlying involutorial antiautomorphism. d is an element with $d = d^t$, which is assumed to be not of form kk^t. As one easily can check there does not exist such an element d in the finite case. So the construction of finite and proper generalized Laguerre planes does not carry over to the generalized Möbius case.

Remark 2 : Consider the 3-dimensional projective space P^3 over a field F, char $F \neq 2$, possessing an involutorial antiautomorphism t. Then the geometry of plane sections of

$$\left\{ (x,y,z,w) \in P^3 \mid xx^t + yy^t - zz^t = ww^t \right\}$$

certainly leads to a notion of generalized Minkowski planes. One should define a common basis for generalized circle planes as it was proposed for circle geometries, s. H.-R. Halder, W. Heise [18].

§ 3 Lorentz transformations.

First of all we like to describe the situation over the field R of reals.
Given the R^n, $n \geq 2$. Then

$$\overline{PQ} := (q_1-p_1)^2 + \dots + (q_{n-1} - p_{n-1})^2 - (q_n - p_n)^2$$

is called the Lorentz-Minkowski distance of the points $P(p_1,\dots,p_n), Q(q_1,\dots,q_n)$ of the R^n. In physics the notion $P \leq Q$ for $P,Q \in R^n$ is of importance: $P \leq Q$ stands for $\overline{PQ} \leq 0$ and $p_n \leq q_n$. The bijection s of the R^n is called a causal automorphism in case $P \leq Q$ iff $P^s \leq Q^s$ holds for all $P,Q \in R^n$.

The mapping $s : R^n \to R^n$ is called a Lorentz transformation iff $\overline{PQ} = \overline{P^sQ^s}$ for all $P,Q \in R^n$. It turns out that Lorentz transformations are bijective affine mappings of the R^n.

The two basic theorems of space-time geometry are the following:

Theorem 1 (A.D. Alexandrov,[1]). Let n be ≥ 3 and let s be a bijection of the

R^n such that $\overline{PQ} = 0$ iff $\overline{P^sQ^s} = 0$ for all $P,Q \in R^n$. Then s must be a Lorentz transformation up to a dilatation.

Theorem 2 (A.D. Alexandrov, V.V. Ovchinnikova,[2]). Let n be ≥ 3 and let s be a causal automorphism of the R^n. Then s must be a Lorentz tranformation up to a dilatation.

We have proved ([6],[7]) the following

Theorem 3 . Let n be ≥ 2 and let $r < 0$ be a fixed real number. In case $n = 2$ we also allow $r > 0$. Consider a mapping $s : R^n \rightarrow R^n$ such that $\overline{PQ} = r$ implies $\overline{P^sQ^s} = r$ for all $P,Q \in R^n$. Then s must be a Lorentz transformation.

J. Lester has proved that Theorem 3 remains true for $n > 2$ and $r > 0$ (to appear Arch. Math.).

In [8] we have shown that Theorem 1 is equivalent to the Fundamental Theorem of the (n-1)-dimensional Laguerre geometry. By the way, Theorem 1,2 are not true in case $n = 2$. Theorem 1 has been generalized by J. Lester,[20], to the case of arbitrary metric vector spaces of index ≥ 1. H. Schaeffer,[23], has pointed out connections between the Theorem of June Lester and the geometry of Laguerre.

We now like to concentrate ourselves on the case $n = 2$. By changing the coordinate system of the R^2 we can replace the Lorentz-Minkowski form $x^2 - y^2$ by xy as we will do in the sequel. We are thus working in the sequel with the distance

$$\delta(P,Q) := (q_1 - p_1)(q_2 - p_2)$$

for the points $P(p_1,p_2)$, $Q(q_1,q_2)$.

In this new situation $P \leq Q$ simply reads as $q_1 - p_1 \geq 0$ and $q_2 - p_2 \geq 0$.

The following Theorem is due to F. Rado:

Theorem ([21]). Consider a commutative field F, char $F \neq 2,3$. Let s be a bijection of the plane F^2 such that $\delta(P,Q) = 1$ holds true iff $\delta(P^s,Q^s) = 1$ for all $P,Q \in F^2$. Then s must be a collineation of F^2.
Mappings s in this Theorem of F. Rado are obviously of form $(x,y) \rightarrow (x',y')$ such that

$$x' = a\,x^t + b$$
$$y' = \frac{1}{a}\,y^t + c$$

or

$$x' = a\,y^t + b$$
$$y' = \frac{1}{a}\,x^t + c \quad ,$$

where $t \in \text{Aut } F$ and $a \neq 0$, b, c are in F. We thus can say that s is a Lorentz transformation up to an automorphism t of F. In the Järnefelt world $GF(p)$, p a big prime number, we are thus led to Lorentz transformations. It might be remarked, that Rado's Theorem is not true for $F = GF(q)$ with $q \in \{3,4,8,9,16\}$ (s.[10],III).

In [10] we have posed the following

Problem 1 : Given a commutative field F. Determine all mappings s of F^2 into itself such that

$$\forall_{P,Q \in F^2}\ \delta(P,Q) = 1 \ \text{ implies } \ \delta(P^s,Q^s) = 1 \ .$$

As was pointed out in [10] this problem is (for char $F \neq 2$) equivalent with the following

Problem 2 : Given a fixed $k \neq 0$ in F. Determine all mappings s of F^2 into itself such that

$$\forall_{P,Q \in F^2}\ d(P,Q) = k \ \text{ implies } \ d(P^s,Q^s) = k \ .$$

(Here $d(P,Q)$ stands for $(q_1 - p_1)^2 - (q_2 - p_2)^2$.)

Let us call F regular in case that all the solutions of Problem 1 (for F) are (bijective) collineations.

The following result can be proved

Theorem : $F = GF(p^n)$ is regular for

a) $p \neq 2,3,5,7$,

b) $p \in \{5,7\}$ and n even ,

c) $p = 7$, n odd and $3|n$,

d) $p^n = 7$.

$GF(5)$ is not regular.

In proving this Theorem in [10](and in a forthcoming paper of mine) a result of G. Tallini, [25] , plays an important rôle. The cases $F = GF(5)$, $GF(7)$, $GF(11)$ in this Theorem are due to H.-J. Samaga,[22], who was applying a computer.

Remark : In an earlier paper B. Farrahi,[17], deals with injective solutions of Problem 1 for $GF(p)$ including also other metrics. The infinite dimensional case for $F = \mathbb{R}$ is included in E.M. Schröder,[24].
This last Theorem corresponds to our Theorem 3.
We now like to find a correspondence to Theorem 1 in case $n = 2$ and $F = GF(p^m)$.
(Note that $n \geq 3$ is one of the assumptions in the Theorem of June Lester.)

Three distinct points of the R^n, $n \geq 2$, are pairwise of Lorentz-Minkowski distance 0 iff they are on a common light line. Thus light lines are mapped onto light lines under bijections, which preserve Lorentz-Minkowski distance 0 in both directions.

The light lines in our general situation (observe $n = 2$) are the lines parallel to the x-axis or to the y-axis. Thus preservance of light lines means almost nothing: If, for instance, f,g are bijections of F , then $(x,y) \to (f(x),g(y))$ is a bijection of F^2 preserving light lines and thus distance 0 in both directions. Of course Theorem 1 cannot be true for $n = 2$ in Järnefelt's world $GF(p)$, p a big prime number. But having in the R-world two coordinate systems of a line moving against each other with constant speed v ,one knows that the four world lines of both origins in both line-time coordinate systems must be lines. Taking this into account we arrive at the following mathmatical situation ([9]) in case $F = GF(q)$, $q = p^m$.

Given two copies C_1, C_2 of the affine plane over $F = GF(q)$ and moreover two pencils of lines in each copy, say $L_1^1 \neq L_1^2$ in C_1 and $L_2^1 \neq L_2^2$ in C_2 . Consider a bijection

$$s : X_1 \to X_2 \quad ,$$

where X_i denotes the set of points of C_i . Assume that every line of L_1^i is mapped under s onto a line of L_2^i , $i = 1,2$. Consider two distinct lines h_1,h_2 of C_1 not in $L_1^1 \cup L_1^2$ and two distinct lines k_1,k_2 of C_2 not in $L_2^1 \cup L_2^2$. By

$$a = \begin{bmatrix} L_1^1 & L_1^2 \\ h_1 & h_2 \end{bmatrix} \quad \text{(similarly,} \quad A = \begin{bmatrix} L_2^1 & L_2^2 \\ k_1 & k_2 \end{bmatrix}) \quad ,$$

we denote the cross ratio of the points of intersection of L_1^1 , L_1^2 , h_1, h_2 with the infinite line of C_1 . Assume finally, $h_i^s = k_i$, $i = 1,2$. Then the following holds true

Theorem ([9]) . If a is a primitive (q-1)th root of unity and if there is an automorphism t of $GF(q)$ such that $A = a^t$, then s is an affine mapping of C_1 onto C_2 .

The assumption $A = a^t$ is of course necessary since the cross ratio a has to be transformed under an affine mapping in $A = a^t$, $t \in \text{Aut } GF(q)$. But also the other assumptions of this Theorem are essentially necessary as we have shown in [9], where the other occuring cases are discussed (s. the two theorems below). In a real situation of the R-world it must be $A = a$ corresponding to the fact that $t = 1$ is the only automorphism of R. That a needs to be a primitive $(q-1)$th root of unity expresses the fact that only for certain speeds v the connecting transformation is of Lorentz type. For other speeds v complicated connecting transformations may occur according to the

Theorem ([9]). If a is not a primitive $(q-1)$th root of unity, if in case $a = 1$ the number q is $\neq 4$ and not a prime number, and if furthermore there exists $t \in \mathrm{Aut}\,GF(q)$ such that $A = a^t$, then there exists a bijection $s : X_1 \to X_2$ such that the following conditions hold:

(i) s is not an affine mapping,

(ii) $h_i^s = k_i$, $i = 1,2$,

(iii) Every line of L_1^i is mapped under s onto a line of L_2^i, $i = 1,2$.

In this connection we finally mention the

Theorem ([9]). If $A = a = 1$ and if q is a prime number or equal to 4, then any bijection $s : X_1 \to X_2$ having the properties (ii), (iii) of the previous theorem must be an affine mapping of C_1 onto C_2.

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Annals of Discrete Mathematics 14 (1982) 9–38

THE GEOMETRY ON GRASSMANN MANIFOLDS REPRESENTING SUBSPACES IN A GALOIS SPACE

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1. ON THE GRASSMANN MANIFOLD $G_{r,d,q}$

Let PG(r,q) be an r-dimensional ($r \geq 3$) Galois space of order q ($q = p^h$, p a prime, h a non-negative integer). The Grassmann manifold representing the d-flats (d-dimensional subspaces), $1 \leq d \leq r - 2$, in PG(r,q) will be denoted by $G_{r,d,q}$. Such a manifold - as it is well known - is an algebraic manifold, intersection of quadrics, in PG($\binom{r+1}{d+1} - 1$,q) and has

(1.1) $$|G_{r,d,q}| = \prod_{i=0}^{d} \vartheta_{r-i}/\vartheta_{d-i}$$

points, where

(1.2) $$\vartheta_s = \sum_{j=0}^{s} q^j .$$

Let S_d be a d-flat in PG(r,q); its Grassmann coordinates will be the coordinates of the point representing it in PG($\binom{r+1}{d+1} - 1$,q); therefore, a pencil of d-flats (i.e. all the d-flats through a (d-1)-flat contained in the same (d+1)-flat) will be represented by a line; conversely, any line on $G_{r,d,q}$ is the image of a pencil of d-flats. Thus, two d-flats in PG(r,q) meeting in a (d-1)-flat are represented by two points on $G_{r,d,q}$ such that the line through them is completely contained in $G_{r,d,q}$. Now, recall that in PG(r,q) a collection of d-flats which pairwise meet in a (d-1)-flat consists of either d-flats through the same (d-1)-flat or d-flats belonging to the same (d+1)-flat; next, the following definitions will be made.

A <u>(d,s)-star</u>, $1 \leq s \leq r - d$, in PG(r,q) is the set of all d-flats through a (d-1)-flat belonging to a fixed (d+s)-flat. (Such a set is also called an s-dimensional star of d-flats through a (d-1)-flat).

A <u>(d,s)-dual-star</u>, $1 \le s \le d+1$, is the set of all d-flats belonging to a (d+1)-flat and passing through a fixed (d-s)-flat. (Such a set is also called an s-dimensional star of d-flats in a (d+1)-flat).

Therefore, any s-dimensional subspace (s-space) on $G_{r,d,q}$ represents either a (d,s)-star or a (d,s)-dual-star in PG(r,q), and conversely.

For any a, $1 \le s \le r-d$, Σ_s will denote the collection of s-spaces on $G_{r,d,q}$ each of which represents a (d,s)-star in PG(r,q). For any t, $1 \le t \le d+1$, Σ'_t will denote the collection of those t-spaces on $G_{r,d,q}$ each of which represents a (d,t)-dual-star in PG(r,q). Obviously, $\Sigma_1 = \Sigma'_1$ is the collection, R, of lines on $G_{r,d,q}$. Furthermore, $G_{r,d,q}$ contains exactly two collections of maximal spaces, namely Σ_{r-d} and Σ'_{d+1}, an element T in Σ_{r-d} representing all the d-flats in PG(r,q) through a fixed (d-1)-flat and an element T' in Σ'_{d+1} representing all d-flats in PG(r,q) belonging to a fixed (d+1)-flat.

It is easy to prove:

(1.3) $$T_1, T_2 \in \Sigma_{r-d}, \quad T_1 \neq T_2 \Rightarrow |T_1 \cap T_2| \le 1 ,$$

(1.4) $$T'_1, T'_2 \in \Sigma'_{d+1}, \quad T'_1 \neq T'_2 \Rightarrow |T'_1 \cap T'_2| \le 1 ,$$

(1.5) $$T \in \Sigma_{r-d}, \quad T' \in \Sigma'_{d+1} \Rightarrow \text{either} \quad T \cap T' = \emptyset \quad \text{or} \quad T \cap T' \in R,$$

(1.6) $$\forall U \in \Sigma_s \Rightarrow \exists!\ T \in \Sigma_{r-d} : U \subseteq T,$$

(1.7) $$\forall U' \in \Sigma'_t \Rightarrow \exists!\ T' \in \Sigma'_{d+1} : U' \subseteq T' ,$$

(1.8) $$\forall \ell \in R \Rightarrow \exists!\ T \in \Sigma_{r-d}, \quad \exists! T' \in \Sigma'_{d+1} : \ell \subseteq T \cap T'.$$

The geometry on $G_{r,d,q}$ is the study (which the author started in [2]) of point k-sets on $G_{r,d,q}$ with respect to Σ_s (s = 1,2,...,r-d) and Σ'_t (t = 1,2,..., d+1), that is the study of k-sets of d-flats with respect to (linear) families of d-flats in PG(r,q). Such an investigation is the subject of this paper.

<u>Remark</u>. On $G_{r,1,q}$ the two collections of maximal spaces are Σ_{r-1} (whose elements represent the stars of lines in PG(r,q), such a star consisting of all lines through a point) and $\Sigma'_2 = P$ (whose elements represent the ruled planes